A Price for Everything?

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Abstract and Keywords
This chapter reviews old and recent arguments for and against monetary aggregates as social welfare indicators. It is organized as follows. Sections 3.1 and 3.2 examine two different revealed-preference arguments that can be used to link the sign of welfare variations with the evolution of total consumption at market prices. Section 3.3 reviews the theory of index numbers, which has the more ambitious goal of providing cardinal measures of welfare. Section 3.4 focuses on the more modest, but perhaps more promising idea of decomposing social welfare into factors separately representing efficiency and equity. Section 3.5 introduces the problem of imputing prices for nonmarketed goods and computing a “full income.” The chapter concludes that economic theory does not provide much support for relying on the market value of total consumption as a proxy for social welfare.

Keywords: monetary aggregates, social welfare indicators, index numbers theory, efficiency, equity, market value, total consumption
The project of correcting GDP has been often understood as adding or subtracting terms that have the same structure as GDP, that is, monetary aggregates computed as quantities valued at market prices or at imputed prices in case market prices are not available. For instance, in their seminal work, Nordhaus and Tobin set out to compute “a comprehensive measure of the annual real consumption of households. Consumption is intended to include all goods and services, marketed or not, valued at market prices or at their equivalent in opportunity costs to consumers” (1972, p. 24).

As we will see in this chapter, economic theory is much less supportive of this approach than usually thought by most users of national accounts. Many official reports swiftly gloss over the fact that economic theory has established total income as a good index of social welfare under some assumptions (which are usually left unspecified). To be sure, there is a venerable tradition of economic theory that seeks to relate social welfare to the value of total income or total consumption. Surveys of this tradition can be found in Sen, (1979) and Slesnick, (1998). As a matter of fact, the assumptions under which one can say something about social welfare from total income or total consumption are extremely restrictive. Most of the theory, moreover, deals with the limited issue of determining the sign of the welfare change for a given population. The theory is silent about the magnitude of the change, not to mention the level of welfare itself. It is also silent about comparisons across populations with different preferences. In this perspective, the widespread use of GDP per capita, corrected or uncorrected, as a cardinal measure allowing percentage scaling of differences and variations across countries and generations should be seen as lacking foundations in economic theory.

(p.77) At this point it is worth asking what is needed actually. Certainly, policy debates need an ordinal ranking of the social and economic consequences of various policy options. This is the minimum. There are, however, contexts in which different populations are considered. First, in the long run different policies may generate different populations with predictably different habits and preferences. Is it better, for instance, to encourage rural or urban lifestyles by urban development policies? Second, the evaluation of growth over the long run involves different generations, while international comparisons of living standards involve very different populations across the world. Is the population better off in
the United Kingdom or in Japan? This question is meaningful but cannot be answered without a measure that covers variations in preferences. There are, moreover, contexts in which rankings are not sufficient and one would like a cardinal scale. The measurement of growth and the comparison of living standards naturally require the ability to evaluate some differences as larger than other differences. It seems important, for instance, to know if the improvement in social welfare has slowed down (if it has not decreased) after the first oil crisis.

In this chapter we review old and recent arguments for and against monetary aggregates as social welfare indicators. Although the topic of this chapter is the approach that consists in “correcting” national income by adding or subtracting elements of nonmarketed consumption, valued at imputed prices, it will not be a surprise that most of the attention will be on the core problem of using national income as a measure of social welfare, even when no corrections are needed. The problems and difficulties in this “easy” case are obviously at least as severe in the difficult case in which corrections are needed. Sections 3.1 and 3.2 examine two different revealed-preference arguments that can be used to link the sign of welfare variations with the evolution of total consumption at market prices. Section 3.3 reviews the theory of index numbers, which has the more ambitious goal of providing cardinal measures of welfare. Section 3.4 is about the more modest, but perhaps more promising, idea of decomposing social welfare into factors separately representing efficiency and equity. Section 3.5 introduces the problem of imputing prices for nonmarketed goods and computing a “full income.”

Because this chapter is, apart from section 3.4 on the decomposition of social welfare, mostly critical, it can be skipped by the readers who are already convinced that a notion of total income cannot be a good proxy for social welfare. This is also the most demanding chapter in terms of economic theory background.

(p.78) 3.1 A revealed preference argument
The cornerstone of the theory of national income as a proxy for social welfare is the $p\Delta x \geq 0$ criterion, where $p$ is the initial price vector and $\Delta x$ the change in total consumption. This is equivalent to checking if the Laspeyres index of total consumption, $p(x + \Delta x)/px$, is greater than 1. This approach is inspired by a similar criterion for individual welfare. We first review the idea at the individual level, before examining how it can be extended to the social level.

### 3.1.1 The argument for an individual consumer

Consider a consumer who consumes a bundle $x \in \mathbb{R}^n$ at market prices $p \in \mathbb{R}^n$, and is free to choose any bundle $x'$ such that $px' \leq px$. Obviously, $x$ is at least as good as any such $x'$ for the consumer, because if one of them were better, the consumer could have picked it.

Moreover, if local nonsatiation is satisfied (which means in every vicinity of every bundle there is a strictly preferred bundle), $x$ must be strictly preferred to any bundle $x'$ such that $px' \geq px$. Indeed, if $x$ was as good as some $x'$ such that $px' \geq px$, one could find a bundle $x''$, strictly better than $x$ and $x'$ in a vicinity of $x$ that is small enough so that $px'' \cdot (px$.

Now consider a change from $x$ to and from $p$ to $p'$. Under what conditions can one say that the change represents an increase in welfare, if one assumes that the consumer’s preferences have not changed? If $p'x \geq px$, $x$ is at least as good as $x$ by the first observation, and, under local nonsatiation, it is strictly better if $p'x \succ px$. Conversely, the inequalities $px \geq px'$ and $px \succ px'$ betray a (weak or strict) decrease in welfare. There is a gray area in which the change in welfare is not determined by this methodology, namely, when $p'x \cdot (px$ and $px \cdot (px$.

This is illustrated in figure 3.1, which depicts the typical consumer choice of the most preferred bundle in a budget set. In the figure, one has, and indeed $x$ is revealed preferred to $x'$ but, as $px \cdot (px$, $x'$ can be better or worse than $x$ depending on the precise location of the consumer’s indifference curve.
When the changes are infinitesimal (i.e., \( x = x + dx, \)
\( p' = p + dp \), the gray area vanishes if the consumer has preferences that are representable by a differentiable utility function \( u(x) \) and if \( x \) is interior, that is, \( x \in \mathbb{R}_+^n \). Indeed, in this case, the price vector \( p \) is proportional to the gradient of \( u \) at \( x \),

\[
\frac{\partial u}{\partial x_k} \frac{p_k}{p} \]

or, in economic terms, relative prices \( \frac{p_k}{p} \) are equal to marginal rates of substitution \( \left( \frac{\partial u}{\partial x_k} \right) / \left( \frac{\partial u}{\partial x_j} \right) \). This implies that the differential \( du \) is proportional to \( pdx \) and therefore has the same sign as \( pdx \):

\[
du \geq 0 \iff pdx \geq 0.
\]

Note the importance of the interiority assumption here. If the consumer did not consume some of the goods, \( pdx \) \( 0 \) would be compatible with a decrease in satisfaction.

Before examining how this revealed preference argument has been extended to social welfare, let us ponder its limitations. These limitations come primarily from the assumptions. The assumption that the consumer freely chooses in a budget set excludes rationing and nonmarketed (in particular, nonexcludable) public goods. The assumption of local nonsatiation excludes the situation of people who donate part of their income. Their welfare cannot be examined in terms of personal consumption, and one must (p.80) include the donation itself as a consumption (of generosity), which raises all sorts of ethical problems because it is debatable whether other-regarding preferences deserve as much attention in social evaluation as self-centered preferences. The assumption that preferences do not change is essential and reveals that this approach has nothing to say about comparisons involving different consumers with different preferences. Even

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**Figure 3.1** The revealed preference argument for a consumer
consumers who have the same preferences about consumption commodities may not be comparable in such simple terms, because there may be other determinants of welfare (e.g., health) that are missing in the framework. The assumption of an interior consumption is obviously unrealistic, as every consumer buys only a very small fraction of the available commodities.

The limitations also come from the result itself, which has to do with the sign of welfare change. No indication of the magnitude of the change is obtained, although the observation that $du$ is proportional to $pdx$ suggests that slightly more could be squeezed out of the reasoning—for example, for small changes $2dx$ has twice the effect of $dx$ on utility.\(^5\) No indication about how to measure welfare in a cardinal way is given. The market value of consumption cannot be declared a measure of welfare from this reasoning. Finally, let us notice that extending this result to a setting with time and uncertainty is far from obvious. With a set of complete markets and a fully rational consumer, one can evaluate his ex ante lifetime welfare with this methodology. But with or without complete markets, the connection between welfare and the market value of per period consumption (or income) is not immediate, as it has been discussed in the previous chapter.

3.1.2 Extending the argument to social welfare through a representative agent

How can one extend this revealed preference argument from individual income or consumption to national income? Assume that social welfare for a society of $n$ members is defined as the value of a social welfare function applied to individual utilities $W(u_1(x_1), \ldots, u_n(x_n))$. The aim is to link this magnitude to the market value of total consumption $X = x_1 + \cdots + x_n$, more precisely, to the criterion $pX \cdot pX'$.\(^6\)

Two ways of establishing such a link have been proposed. One approach consists in considering a “representative agent” who consumes $X$ and whose demand behavior mimics the aggregate demand of the society under (p.81) consideration. It is customary in macroeconomics to identify the preferences of the representative agent with social welfare. The transposition of the revealed preference argument is then immediate because society is simply identified with this
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macroeconomic individual. This approach, however, is quite discredited, as explained in Kirman, (1992).

To begin with, the existence of a representative agent is rare. In particular, when income is redistributed across individuals and prices remain unaltered, it is generally unlikely that aggregate demand remains unchanged (individual Engel curves, which describe the relation between demand and income for given prices, should be linear and parallel), but a representative agent must keep the same demand. As it turns out, the condition of linear and parallel Engel curves is actually necessary and sufficient for the existence of a representative agent when prices are also allowed to vary.\(^6\)

The existence of a representative agent is less unlikely when the variations of incomes and prices are restricted. For instance, Dow and Werlang, (1988) and Jerison, (2006) assume that income shares are determined by a function (“distribution rule”) that depends only on the price vector and total income: letting \(M\) denote total income, the distribution rule is a vector-valued function \(f(p, M)\) such that \(\sum_{i=1}^n f_i(p, M) = M\), and \(f_i(p, M)\) is individual \(i\)’s income. The representative agent is then studied under the assumption that income shares never deviate from the distribution rule. Even under this restriction, the existence of the representative agent\(^7\) is not guaranteed in general. Studying cases in which a representative agent exists, and assuming a fixed distribution rule, these authors show the following interesting facts.

First, it may happen that the representative agent prefers \(X\) to \(X'\) while all the individuals weakly prefer their bundle in \(X\) to their bundle in \(X'\), or that the representative agent weakly prefers \(X\) to \(X'\) while all the individuals weakly prefer their bundle in \(X'\) to their bundle in \(X\) and at least one individual has a strict preference. This is called a “Pareto-inconsistent” representative agent. The examples showing such a possibility are not very simple (see the quoted references). But the problem is not limited to pathological preferences because almost any preferences over \(X\) can be those of a Pareto-inconsistent representative agent.

Second, however, if \(pX \succ pX'\), that is, if \(X\) is revealed preferred to \(X'\) by the choice made under \(p\) by the representative agent, then at least one (p.82) individual prefers his bundle in \(X\) to
his bundle in $X_i$. Indeed, if $pX \succ pX'$, necessarily $pX_i \succ pX_i'$ for at least one $i \in \{1, \ldots, n\}$ because $pX = \sum_{i=1}^{n} pX_i$ and $pX' = \sum_{i=1}^{n} pX_i'$. The fact that $pX_i \succ pX_i'$ implies that $i$ prefers $x_i$ to $x_i'$ (assuming local nonsatiation). But obviously, it may be that only one individual agrees with the representative agent.

Third, if the representative agent is Pareto consistent (i.e., is not Pareto inconsistent), then its preferences can be represented by a social welfare function $W(u_1(x_1), \ldots, u_n(x_n))$. In order to see this point, it is more convenient to take the pairs $(p, M)$ as the objects of preferences. For the representative agent, preferences over $(p, M)$ are like ordinary indirect preferences of an agent who has to choose under the budget constraint $pX \leq M$: $(p, M)$ is preferred to $(p', M')$ if the bundle chosen under the former budget is preferred to the bundle chosen under the latter. For individuals $i = 1, \ldots, n$, preferences over $(p, M)$ can also be defined. They are deduced from the distribution rule $f(p, M)$, which determines the income $m_i$ of each $i$, and from their indirect preferences over $(p, m_i)$: $(p, M)$ is preferred to $(p', M')$ by $i$ if choosing from the budget $(p, f_i(p, M))$ is better than choosing from the budget $(p, f_i(p', M'))$. Let $v_i(p, M)$ denote a utility function representing $i$’s preferences over $(p, M)$, and let $V(p, M)$ denote the representative agent’s preferences over $(p, M)$.

If the representative agent is Pareto consistent, one has $V(p, M) = V(p', M')$ whenever $v_i(p, M) = v_i(p', M')$ for all $i$. One can then define a social welfare function $W$ as follows: for every vector $(\hat{u}_1, \ldots, \hat{u}_n)$ such that there is $(p, M)$ for which $v_i(p, M) = \hat{u}_i$ for all $i$, let $W(\hat{u}_1, \ldots, \hat{u}_n) = V(p, M)$. This function is well defined because for any other $(p', M')$ for which $v_i(p', M') = \hat{u}_i$ for all $i$, necessarily $V(p', M') = V(p, M)$. In addition, this function must be strictly increasing over the domain of vectors $(\hat{u}_1, \ldots, \hat{u}_n)$ such that there is $(p, M)$ for which $v_i(p, M) = \hat{u}_i$ for all $i$.

This result connecting the representative agent to a Paretian social welfare function is, however, limited because the function is defined only for a given distribution rule. For another distribution rule the representative agent would typically have different preferences. Moreover, the function
need not satisfy good ethical properties apart from satisfying the Pareto principle.

This last remark can be illustrated with an example. Consider an economy with two goods in which individual 1 only likes good 1 and the other individuals only like good 2. The distribution rule is such that \( m_1 = .9M \), while \( m_i = .1M/(n - 1) \) for \( i = 2, \ldots, n \). The aggregate demand of good 1 is \( x_1 = .9M/p_1 \) and can be obtained with Cobb-Douglas preferences, for the representative agent, that are represented by the function \((X_1)^{\theta}(X_2)^{1-\theta}\). Let us measure individual \( i \)'s utility \( u_i \) by the quantity consumed of the good \((p.83)\) he likes. This is the natural measure here. One then sees that the representative agent’s preferences, formulated as a social welfare function over individual utilities, can be written

\[
(\hat{u}_1)^\theta(\hat{u}_2)^{1-\theta} \cdots (\hat{u}_n)^{1-\theta}.
\]

Obviously, the social welfare function is directly influenced by the distribution rule and gives more weight to the “rich” individual. In this example, the distribution rule is optimal for the social welfare function (3.1). But other social welfare functions could rationalize the representative agent’s preferences, because the variation of \((\hat{u}_1, \ldots, \hat{u}_n)\) is restricted by the distribution rule. For instance, the function \((\hat{u}_1)^\theta(\hat{u}_2)^{1-\theta}\), which ignores all agents \( i = 3, \ldots, n \), works just as well under the given distribution rule, which clearly is not optimal for this social welfare function. Observe that this social welfare function is not even Paretoian. Dow and Werlang, (1988) and Jerison, (2006), with different arguments, show that a Pareto-consistent representative consumer may exist even when the distribution rule is not optimal for any social welfare function.

3.1.3 Extending the argument to social welfare with an optimality assumption

The bottom line of the previous subsection is that unless the distribution rule is assumed to be optimal for a reasonable social welfare function, there is no reason to give much ethical credit to a Pareto-consistent representative agent. As it turns out, optimality of the distribution underlies the second approach to be examined here. This approach provides a different justification to the “\( pX \geq p\bar{X} \)” criterion.
Due to Samuelson, (1956), it consists in assuming that \( X \) is optimally distributed, in the sense that \( (x_1, \ldots, x_n) \) maximizes \( W(u_i(x_i), \ldots, u_n(x_n)) \) subject to the constraint \( x_1 + \cdots + x_n = X \). Under this assumption there is a welfare function that depends on total consumption and coincides with social welfare:

\[
W^*(X) = \max \, W(u_i(x_i), \ldots, u_n(x_n)) \mid x_1 + \cdots + x_n = X.
\]

It is much like a utility function and can serve to apply the revealed preference argument to total consumption for the evaluation of market allocations, if one assumes that \( W \) is increasing in each argument (Pareto principle).

This is shown by connecting this approach to the previous one, as the social optimality of the distribution is a sufficient (but not necessary) condition for the existence of a Pareto-consistent representative agent. For a given \((p, M)\), find some \( X \) that maximizes \( W^*(X) \) under the constraint \( pX \leq M \). This is equivalent to maximizing \( W(u_i(x_i), \ldots, u_n(x_n)) \) under the constraint that \( p(x_1 + \cdots + x_n) \leq M \) and letting \( X = x_1 + \cdots + x_n \). Pick a corresponding optimal distribution of consumption \((x_1, \ldots, x_n)\).

One may then define the distribution rule by setting \( f(p, M) = px_i \). When the prevailing market price vector is \( p \) and when individual income is \( f(p, M) \), \( x_i \) must be optimal for \( i \). If not, there would be a better \( x'_i \) for \( i \) satisfying \( px'_i \leq f(p, M) \), and one could thereby find an alternative allocation \((x'_1, \ldots, x'_n)\) satisfying the constraint \( p(x'_1 + \cdots + x'_n) \leq M \) and yielding greater social welfare than \((x_1, \ldots, x_n)\) by the Pareto principle. But this would be in contradiction with the optimality property that served to choose \((x_1, \ldots, x_n)\).

Under this distribution rule, therefore, there is a representative agent because the sum of individual demands \( x_i \) coincides with the maximization of \( W^*(X) \) under the constraint \( pX \leq M \). The representative agent’s preferences are then represented by the function \( W^* \). Moreover, it is obviously Pareto consistent because of the connection between \( W^* \) and the Paretian \( W \), and its corresponding social welfare function is ordinally equivalent to \( W \) over the relevant range of vectors.
(\vec{u}_i, \ldots, \vec{u}_n),\) that is, the vectors such that there is an optimal \((x_i, \ldots, x_n)\) for which \(u_i(x_i) = \vec{u}_i\) for all \(i\).

These preliminary considerations make it possible to formulate and prove the message of this second approach. If \((x_i, \ldots, x_n)\) is, under prices \(p\), a market allocation that optimally distributes \(X\) according to the social welfare function \(W\), and if \(pX \geq p'X\), then necessarily social welfare measured by \(W\) is at least as great at \((x_i, \ldots, x_n)\) as at any distribution of \(X\). This is because \(X\) is the representative agent’s choice under the budget constraint defined by \(p\) and \(M = pX\), and the inequality \(pX \geq p'X\), by the standard revealed preference argument, implies \(W^*(X) \geq W^*(X')\), which itself implies

\[ W(u_1(x_1), \ldots, u_n(x_n)) = W^*(X) \geq W^*(X') \geq W(u_1(x'_1), \ldots, u_n(x'_n)) \]

for all \((x'_i, \ldots, x'_n)\) such that \(X' = x'_1 + \cdots + x'_n\).

Under the same assumptions, plus the assumption that at least one individual’s preferences satisfy local nonsatiation, it holds that if \(pX \geq p'X\), then social welfare is greater at \((x_i, \ldots, x_n)\) than at any distribution of \(X\). This is due to the fact that \(W^*\) then also satisfies local nonsatiation.

(p.85) The limitation of this second approach is that it is not realistic to assume that consumption or wealth can be redistributed at will by lump-sum transfers (as is implicit in positing that the only constraint is \(x_1 + \cdots + x_n = X\)), and that the status quo (in the application of the revealed preference argument) is always socially optimal. Dropping either of these two assumptions invalidates the approach. When the status quo is not optimal, an improvement in social welfare is even compatible, obviously, with a decrease in total consumption, not just a decrease in the value of total consumption. A smaller cake better distributed may be socially preferable to a larger cake badly distributed.

When redistribution is made by distortionary taxes, even if the distribution is optimal under the constraint of using such taxes (this constraint may itself be due to imperfect information about the individuals’ characteristics), the revealed preference argument fails as well. Indeed, at a second-best optimum the social marginal value of income\(^8\) for different individuals typically remains unequal, so that a change of allocation that favors individuals with greater social marginal value may
improve social welfare even if the market value of total consumption is reduced. Concretely, what usually happens is that the poor retain a greater social marginal value of income at the social optimum because redistribution is less successful with such taxes than with lump-sum transfers. Then, a change of allocation that increases the value of their consumption less than it decreases that of richer people may improve social welfare and nonetheless reduce the market value of total consumption at the initial prices.

The conclusion should be clear. Under reasonable assumptions, total income, or the market value of total consumption, has no direct connection with social welfare. The revealed preference argument is already quite limited at the individual level, as has been emphasized above. At the social level, distribution simply cannot be ignored and severely restricts the possibility of using income as a proxy for welfare.
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3.2 A variant of the revealed preference argument
Sen, (1976) examines a variant of the revealed preference argument. If consumer preferences are convex, at any bundle $x$ there is a price vector $p$ such that for all bundles $x'$, $px' \prec px$ implies that $x$ is preferred to $x$. In figure 3.1, for instance, $p$ can be chosen endogenously so as to obtain a line of equation $px' = c_s$ that is tangent to the indifference curve at bundle $x$.

This argument does not assume that $x$ maximizes utility in the budget set, which means that the vector $p$ need not coincide with market prices. In fact some components of $x$ may not even be marketed.

As explained by Sen, this argument can be immediately extended to social welfare if social welfare is measured by a function $\bar{W}(x_1, \ldots, x_n)$ that is defined directly on quantities and if this function is quasi-concave. But the corresponding price vector $\bar{p}$ must then have $n$ components (like the arguments of $\bar{W}$), each component $\bar{p}_k$ weighting the consumption of a particular commodity $k$ consumed by a particular individual $i$.

If $(x_1, \ldots, x_n)$ is a market allocation in which all individuals face the same 1-dimensional price vector $p \in \mathbb{R}^1$, and freely choose in their budget set, and if the social welfare function is Pareto-optimal, then the $n$-vector $p$ can be chosen such that the ratios $\frac{p_k}{p_l}$ (individual $i$, commodities $k$ and $l$) are the same for all $i$ and are simply equal to the relative market price $\frac{p_k}{p_l}$. The computation of the inner product $p \cdot (x_1, \ldots, x_n)$ then simply boils down to $a_1 px_1 + \cdots + a_n px_n$, for a vector of weights $(a_1, \ldots, a_n)$ which embodies distributional preferences.

The criterion obtained with this approach can then be formulated as meaning that for any quasi-concave function $\bar{W}(x_1, \ldots, x_n)$, a market allocation $(x_1, \ldots, x_n)$ is at least as good as another allocation $(\bar{x}_1, \ldots, \bar{x}_n)$ whenever $\sum_{i=1}^{n} a_i \bar{x}_i \geq \sum_{i=1}^{n} a_i x_i$, for a suitable vector of weights $(a_1, \ldots, a_n)$ that may depend on $(x_1, \ldots, x_n)$.

The main limitation of this approach is that the weights $(a_1, \ldots, a_n)$ generally depend on the allocation in a complex way. For instance, if the function $\bar{W}(x_1, \ldots, x_n)$ is derived from a social welfare function $W(u_1(x_1), \ldots, u_n(x_n))$, the weight $a_i$ must be
proportional to the social marginal value of \(i\)'s income, \(\frac{\partial W}{\partial v_i} \) (where \(v(p, m)\) is the indirect utility function, and \(m_i = px_1\)).

Another limitation, in the context of seeking theoretical foundations for national accounts, is that national income bears no direct relation with \(\sum_{i=1}^{n} a_i px_i\), The weights seem to complicate the picture considerably.

Sen examines a way to address both issues. First, total income can appear in a decomposition formula. If the weights are scaled so that \(a_1 + \cdots + a_n = n\), the sum can be decomposed into two terms: \(p.87\)

\[
\sum_{i=1}^{n} a_i px_i = pX \times \frac{a_1 px_1 + \cdots + a_n px_n}{pX},
\]

the second term being less than unity if \((a_1, \ldots, a_n)\) and \((px_1, \ldots, px_n)\) are inversely ranked, that is, in case of inequality aversion over expenditures.

As an example of such a weighting system, Sen suggests taking weights that are simply proportional to the ranks (giving rank 1 to the richest, and so on) of the individuals in \((px_1, \ldots, px_n)\). The second term of the decomposition then takes a familiar form. Indeed, if one assumes that \(px_1 \leq \cdots \leq px_n\), one obtains

\[
\frac{a_1 px_1 + \cdots + a_n px_n}{pX} = \frac{2}{n+1} \frac{np x_1 + \cdots + np x_n + 1}{pX}.
\]

For large numbers the ratio \(\frac{(2i - 1)}{(2j - 1)}\) is not very different from \(i/j\), so that for a large \(n\),

\[
\frac{a_1 px_1 + \cdots + a_n px_n}{pX} = 1 - G(px_1, \ldots, px_n),
\]

where \(G(px_1, \ldots, px_n)\) is the Gini coefficient, which can be defined as follows:

\[
G(px_1, \ldots, px_n) = 1 - \frac{(2n - 1) px_1 + \cdots + (2i - 1) px_i + \cdots + px_n}{npX}.
\]

More precisely, one computes that

\[
\frac{\frac{a_1 px_1 + \cdots + a_n px_n}{pX} - (1 - G(px_1, \ldots, px_n))}{\frac{1}{n} \left(1 - \frac{a_1 px_1 + \cdots + a_n px_n}{pX}\right)},
\]

an expression that is in effect small when \(n\) is large.
This proposal provides simple and intuitive weights and includes total income in a decomposition of the social welfare criterion. Such rank-order weights, unfortunately, are difficult to reconcile with a Paretian social welfare function approach. Hammond, (1978) shows that the partial ordering of allocations generated by this approach (i.e., fixing the proportional rank-order weights assigned to various ranks and applying the revealed preference argument for all possible market prices) is compatible with a Paretian social welfare function only if all individual Engel curves are linear and identical. As Hammond’s result holds for a larger set of fixed rank-order weights than the proportional weights, and his proof contains imperfections, a proof of the generalized result is provided in appendix B.

It is striking that the restriction that is obtained is almost the same as (actually, is stronger than) the condition needed to have a representative agent (namely, parallel linear Engel curves). The conclusion is that this approach, which appears initially promising because it is applicable to nonmarket goods, does not go very far. The difficulty is to find the relevant weights, and the “easy” case of a market allocation reveals that there is no simple way to define the distributional weights on individual incomes if one wants the evaluation to be compatible with the Pareto principle, that is, to respect individual preferences.

Another limitation of revealed preference approaches is that, not only is the criterion incomplete and therefore silent in some comparisons, but when it is conclusive, it only indicates the sign of welfare change. One does not get a quantitative evaluation of welfare change, as would be obtained with an index number such as GDP in volume terms. It is only in a decomposition like equation (3.3) that one can give a quantitative meaning to the evolution of total consumption at certain well-defined prices. More direct attempts to obtain cardinal indices can be found in the theory of index numbers, to which we turn in the next section. The idea of making total income appear in a decomposition of social welfare will be explored further in a later section.

3.3 The theory of index numbers
Initiated by Fisher, (1922), the theory of index numbers seeks to define price and quantity indices in the hope of being able to compare monetary values in a meaningful way when prices and quantities change across the situations to be compared. Compared to the revealed preference approach, it offers the perspective of measuring the size of changes, not just their sign.

After Fisher’s seminal contribution, it has burgeoned in three directions. The first is a direct axiomatic study of index numbers, which does not directly make a link with welfare concepts but is nevertheless inspired by the revealed preference argument reviewed above; the second seeks approximations of true welfare criteria by simple price-quantity statistics; the third is the conceptually and ethically more ambitious, but data-greedy, money-metric utility. Here we examine the first two approaches, as the money metric is the topic of the next chapter.

3.3.1 An axiomatic approach

The axiomatic approach consists in defining desirable properties for the price and quantity indexes \( P(p^1, p^0, x^1, x^0) \) and \( Q(p^1, p^0, x^1, x^0) \) (where \( p^t \) and \( x^t \) denote the price and quantity vectors at \( t = 0, 1 \)). The problem can be formulated as follows.

The monetary value index \( p^1 x^1 / p^0 x^0 \) is not meaningful for the evaluation of changes in quantities or welfare when prices change, and therefore one seeks to decompose it into separate summary indexes for prices and for quantities.

Here are three examples of desirable properties. The first refers to the decomposition idea and says that the product of the price index and the quantity index must equal the monetary value index. The second refers to the revealed preference argument and says that when the price vector is unchanged, the quantity index must equal the value index. The third says that reversing time should yield inverse values of indexes.

\[
\begin{align*}
\text{(3.5)} & \quad P(p^1, p^0, x^1, x^0)Q(p^1, p^0, x^1, x^0) = \frac{p^1 x^1}{p^0 x^0}, \\
\text{(3.6)} & \quad Q(p, p, x^1, x^0) = \frac{p x^1}{p x^0} \\
\text{(3.7)} & \quad Q(p^1, p^0, x^1, x^0)Q(p^0, p^1, x^0, x^1) = 1.
\end{align*}
\]
Diewert, (1992b) provides a list of 21 axioms of this sort and shows that the pair of Fisher indexes

\[(3.8)\]

\[P^f(p^1, p^0, x^1, x^0) = \left(\frac{p^{x^0}_0}{p^{x^1}_0} \frac{p^{x^1}_1}{p^{x^0}_1}\right)^{1/2},\]

\[(3.9)\]

\[Q^f(p^1, p^0, x^1, x^0) = \left(\frac{p^{x^1}_0}{p^{x^0}_0} \frac{p^{x^0}_1}{p^{x^1}_1}\right)^{1/2},\]

satisfies all of them and is the only one to do so. Observe for instance that (3.7) is satisfied neither by the Laspeyres indexes, defined as

\[(3.10)\]

\[P^l(p^1, p^0, x^1, x^0) = \frac{p^{x^0}_1}{p^{x^0}_0} Q^l(p^1, p^0, x^1, x^0) = \frac{p^{x^1}_0}{p^{x^0}_0},\]

nor by the Paasche indexes

\[(3.11)\]

\[P^p(p^1, p^0, x^1, x^0) = \frac{p^{x^1}_1}{p^{x^1}_0}, Q^p(p^1, p^0, x^1, x^0) = \frac{p^{x^1}_0}{p^{x^1}_0}.\]

(p.90) Diewert’s characterization of the Fisher index is quite simple. In addition to (3.5) and (3.7) and the requirement that the indexes be strictly positive, the characterization relies on the following properties, which impose symmetry among the weights of periods 0 and 1:

\[(3.12)\]

\[P(p^1, p^0, x^1, x^0) = P(p^1, p^0, x^0, x^1),\]

\[(3.13)\]

\[Q(p, p^1, x^1, x^0) = Q(p, p^0, x^1, x^0)\]

Here is a variant of his argument adapted for the quantity index.

\[\frac{p^{x^1}_1}{p^{x^1}_0} \frac{p^{x^0}_1}{p^{x^0}_0} \quad \text{by rearranging the terms at the denominator}\]

\[= \frac{p^{x^0}_1}{p^{x^1}_0} \frac{p^{x^1}_1}{p^{x^0}_0} \quad \text{by (3.5)}\]

\[= \frac{Q(p^1, p^0, x^1, x^0)}{Q(p^1, p^0, x^1, x^0)} \quad \text{by (3.12)}\]

\[= \frac{Q(p^1, p^0, x^1, x^0)}{Q(p^1, p^0, x^1, x^0)} \quad \text{by (3.13)}\]

\[= Q(p, p^1, x^1, x^0) \quad \text{by (3.7)}\]

Taking the square root on both sides, one obtains the Fisher quantity index. From (3.5) one then derives the Fisher price index.

A great advantage of this approach is that it can be applied to aggregate quantities just as well as to individual consumption. The counterpart is that the connection to welfare is hard to make. One would like the quantity index to reflect welfare variations. In order to appreciate the required properties, it is convenient to try them on a true hedonic index in the case of
individual welfare. In order to make things as easy as possible, suppose that the consumer’s preferences are homothetic, with an indirect utility function equal to $m/\beta(p)$, for a linearly homogeneous function $\beta(p)$.\(^9\) This indirect utility can then serve as a quantity index by taking

(3.14)

$$Q(p, p^i, x^i, x^o) = \frac{p^i x^i / \beta(p)}{p^o x^o / \beta(p)}$$

(p.91) There seems to be little harm in having a price index that derives from (3.5). With the true index (3.14) one would obtain

$$P(p, p^i, x^i, x^o) = \frac{\beta(p)}{\beta(p^i)},$$

which makes a lot of sense. Let us now examine if the other properties are satisfied. First, observe that this price index satisfies (3.12), although (3.12) is problematic for a reason to be explained shortly.

The quantity index (3.14) satisfies (3.6) and (3.7) without any problem, which seems to confirm that these properties are sensible. The difficulties, however, come with (3.13), which cannot be satisfied by this quantity index.

One might try to defend (3.13) on the grounds that if it reflects welfare, the quantity index should primarily depend on quantities and not so much on prices. But this property reveals a problem. If the index is applied to market situations reflecting maximization in a budget set, it may not be possible to obtain the configurations $(p^o, x^o)$ and $(p^i, x^i)$. This problem concerns both (3.12) and (3.13). The true index (3.14) is applicable only to situations in which $p$ is the supporting price of $x^i$, that is, in which relative prices in $p$ correspond to the marginal rate of substitutions at $x^i$.

One way out of the difficulty consists in redefining the true index, for any arbitrary situation, by taking the supporting price vector $\beta(x)$. One then obtains a pair of indexes that satisfy (3.12) and (3.13) readily because they only depend on quantities and prices, respectively:

$$Q(x^i, x^o) = \frac{p(x^i) x^i / \beta(p(x^i))}{p(x^o) x^o / \beta(p(x^o))}, \quad P(p, p^i) = \frac{\beta(p)}{\beta(p^i)}.$$
This pair satisfies (3.5) whenever $p'$ is the supporting price of $x'$ for $t = 0, 1$. The other axioms of the characterization are fully satisfied. Yet this does not correspond to the Fisher index. The difference comes only from the fact that the characterization requires (3.5) to be satisfied in all situations.

If one wants (3.5) to be satisfied, one can use it to define the price index, which then gives the pair of indexes

$$Q(x', x^0) = \frac{p(x') x^0}{x^0} \beta(p(x^0)), P(p^1, p^0, x', x^0) = \frac{p^1 x}{p^0 Q(x', x^0)}.$$

Obviously, one then loses (3.12).

(p.92) In conclusion, the fact that the Fisher index does not coincide in general with a true index of welfare implies that the seemingly nice axioms of its characterization are problematic. A similar analysis would be obtained with social welfare instead of consumer welfare.

3.3.2 Approximating welfare changes

The second branch of the theory of index numbers seeks indexes that depend only on price and quantity data but are good proxies for welfare change at the individual level. From the revealed preference argument invoked in section 3.1, it is well known that $x'$ is revealed preferred to $x^0$ if the Paasche index satisfies $Q^P(p^1, p^0, x', x^0)$ and that $x^0$ is revealed preferred to $x'$ if the Laspeyres index satisfies $Q^L(p^1, p^0, x', x^0)$. In market situations reflecting maximization in a budget set, it is impossible to have $p^0 x^0$ and $p' x^0$ at the same time. Therefore $p^0 x^0$ implies $p^0 x^0 \leq p' x^0$, so that the Fischer index satisfies $Q^F(p^1, p^0, x', x^0)$. Similarly, $p^0 x^0$ and $p^0 x^0$ implies $p^0 x^0 \leq p^0 x^0$, so that $Q^F(p^1, p^0, x', x^0)$.

In other words, the Fisher index never fails to track situations of revealed preferences that are identified with the Laspeyres and the Paasche indexes. So far, however, no progress has been made beyond the revealed preference argument.

Along the same vein, Diewert, (1976, 1992a), following Hicks, (1941) and discussing Weitzman, (1988), develops a more ambitious approach. Let $\epsilon(p, u)$ denote the individual expenditure function (i.e., the minimum expenditure needed to obtain utility level $u$ when market prices $p$ prevail). The
equivalent variation and compensating variation are respectively equal to

\[ EV(p^1, p^0, x^1, x^0) = e(p^0, u(x^0)) - e(p^0, u(x^0)) \]

\[ CV(p^1, p^0, x^1, x^0) = e(p^1, u(x^1)) - e(p^1, u(x^0)) \]

Each of these magnitudes correctly records if individual utility has increased or decreased when moving from \( x^0 \) to \( x^1 \) — this holds true for the expression \( e(p, u(x^1)) - e(p, u(x^0)) \) for any arbitrary price vector \( p \), because the function \( e(p, u) \) is increasing in \( u \).

If these situations are obtained by maximization in a budget set, then for \( t = 0, 1 \), \( p^t x^t = e(p^t, u(x^t)) \), and by Shephard’s Lemma, for all goods \( k \),

\[ (3.15) \]

\[ x_k^t = \frac{\partial e}{\partial p_k^t}(p^t, u(x^t)) \]

Let \( S' \) denote the \( 1 \times 1 \) Slutsky matrix, whose \( kk \) term is defined as

\[ (3.16) \]

\[ S'_{kk} = \frac{\partial e}{\partial p_k \partial p_k^t}(p^t, u(x^t)) \]

Let us now study how to approximate the EV and CV expressions. When prices are not too different, a second-order Taylor expansion of \( e(p^0, u(x^1)) \), for instance, yields:

\[ e(p^0, u(x^1)) = e(p^1, u(x^1)) + \sum_k (p_k^0 - p_k^1) \frac{\partial e}{\partial p_k^t}(p^t, u(x^1)) + \frac{1}{2} \sum_{k,k'} (p_k^0 - p_k^1) (p_{k'}^0 - p_{k'}^1) \frac{\partial^2 e}{\partial p_k \partial p_{k'}^t}(p^t, u(x^1)) \]

\[ = p^0 x^1 + (p^0 - p^1) x^1 + \frac{1}{2} (p^0 - p^1)' S'(p^0 - p^1), \]

where the second line uses \((3.15)\) and \((3.16)\), and \((p^0 - p^1)' \) denotes the row vector transposed from the column vector \( p^0 - p^1 \).

A second order approximation of the equivalent and compensating variations can then be computed as follows:

\[ (3.17) \]

\[ EV(p^1, p^0, x^1, x^0) = p^0 x^1 + (p^0 - p^1) x^1 + \frac{1}{2} (p^0 - p^1)' S'(p^0 - p^1) \]

\[ CV(p^1, p^0, x^1, x^0) = p^0 x^1 - p^0 x^0 - (p^1 - p^0) x^0 - \frac{1}{2} (p^1 - p^0)' S'(p^1 - p^0) \]

Note that the first-order terms do not depend on preferences and have signs consistent with the comparison to one of the Laspeyres and Paasche quantity indexes, respectively. The
second-order terms, however, cannot be derived from price and quantity information and depend on information about preferences. Obviously, one should always bear in mind that such approximations are valid only if the prices \( p^0 \) and \( p^1 \) are not too different (large differences in quantities and utilities are allowed).

Diewert’s approach consists in seeking functional forms for the expenditure functions that are (i) flexible enough so that they provide good approximations to the second order of any twice differentiable expenditure function, and (ii) simple enough so that the corresponding (exact) equivalent variation depends only on price and quantity data. The obtained measure of equivalent variation is called “superlative” by Diewert.

\[(p.94)\] For instance, he shows that the expenditure function defined as

\[
\epsilon(p, u) = [p^0 a u^2 + 2(p a) (p b) (u - \alpha)(u - \beta)]^{1/2},
\]

where \( A \) is a \( 1 \times 1 \) symmetric matrix, \( a \) and \( b \) are \( 1 \)-vectors, and \( \alpha, \beta \) are real numbers, is flexible enough to approximate any twice continuously differentiable expenditure function to the second order at any particular point \( p, u \) such that \( u \neq 0 \) and \( p \in \mathbb{R}^1,^{10} \)

Now assume that \( a = u^0, \beta = u^1 \neq u^0 \), that \( A \) is normalized so that \( p^0 A p^0 = 1 \), and that \( a \) is normalized so that \( p^0 a = 0 \). (As shown by Diewert, such assumptions are compatible with (3.19) approximating any twice continuously differentiable expenditure function to the second order at the point \((p^0, u^0)\), provided that \( u^0 \neq 0 \) and \( p^0 \in \mathbb{R}^1 \).) Under these assumptions, the equivalent variation for (3.19) is equal to an expression that relies only on prices and quantities and involves the Fisher quantity index:

\[
\epsilon(p, u) = [p^0 a u^2 + 2(p a) (p b) (u - \alpha)(u - \beta)]^{1/2},
\]

This last point is easy to show. Under the assumed normalizations, one has \( \epsilon(p^0, u) = u \), so that

\[
EV(p^1, p^0, x^1, x^0) = u^1 - u^0 = p^0 \chi^0 \left( \frac{u^1}{u^0} - 1 \right).
\]
From (3.15), and using \( e(p', u(x')) = p'x' \), one computes

\[ x' = \frac{A p'(u')^2}{p'x'}, \]

implying

(3.22)

\[
\frac{p'x'}{p_0x_0} \frac{p^0x^1}{p^0x_0} = \frac{p'Ap'(u')^2/p'x'}{p^0Ap^0(u^0)^2/p^0x_0} = \left( \frac{u_1}{u_0} \right)^2,
\]

where the last equality relies on the fact that by symmetry of \( A \), one has \( p'Ap' = p'Ap^0 \). Putting (3.22) into (3.21), one obtains (3.20).

It is important to emphasize that there are many flexible functions yielding different formulas for the equivalent variation. For instance, with (p.95) another function one can obtain a formula that involves the arithmetic mean of the Laspeyres and Paasche quantity indexes, instead of their geometric mean as in the Fisher index:

(3.23)

\[
p'x_0 \left[ \frac{1}{2} \frac{p'x^1}{p_0x^0} + \frac{1}{2} \frac{p'x^1}{p_0x_0} - 1 \right].
\]

It is puzzling that different expressions, which equally pretend to be good approximations of the equivalent variation, may have different signs in certain cases. Even when the Laspeyres and Paasche indexes are arbitrarily close to unity, their geometric mean can be lower than unity while their arithmetic mean is greater. It is equally puzzling that the true second-order approximations in (3.17) and (3.18) involve terms that vary with individual preferences and do not just depend on quantities and prices. How can (3.20) and (3.23) then be claimed to approximate welfare changes?

The explanation of this apparent contradiction in the arguments is that the approximation is made by Diewert at the initial situation \((p^0, u^0)\) only. Each formula for the equivalent variation is correct only when the true expenditure function is exactly equal to the contemplated flexible function at the situation \((p^0, u^0)\), for which there is no guarantee of approximation. Otherwise there is an error term, which converges to zero when \( u^t \) tends to \( u^0 \) but may remain larger than the welfare change throughout. Therefore, no matter how close the two situations are, a formula like (3.20) or (3.23) may
not give the correct sign, let alone the magnitude, of the welfare change.

Note that in the derivation of (3.20) it is apparently “assumed” that \( \beta = u^1 \), but this is misleading because it does not mean that any value of \( u^1 \) is admissible. Quite to the contrary, the various normalizations imply that \( u^1 \) has a specific value, namely, \( p^0 \phi Q^f (p^1, p^0, x^0) \), as can be seen from (3.22). While these normalizations are compatible with the flexible function approximating the true expenditure function at \( (p^0, u^0) \), they may impose a serious discrepancy over the estimation of \( e(p^0, u^1) \). This is the cost for obtaining an expression of \( e(p^0, u^1) \) that depends only on prices and quantities.

All in all, no approximation result can circumvent the hard fact that price and quantity data cannot completely determine the sign of welfare changes, even locally. One must also note that this approach applies only to individual consumption welfare, not to total consumption and social welfare.

\( \textbf{(p.96)} \) In conclusion, the theory of index numbers, in the two approaches described here, does deliver interesting concepts, but it fails to provide measures that adequately reflect individual well-being and social welfare. It certainly does not provide a justification for the use of total income as a measure of social welfare.

3.4 Decomposing welfare

If total income cannot be used as a reasonable proxy for social welfare, one can still hope to give it a role as a factor of social welfare along other aspects. In particular, we have seen that distribution is bound to matter and is not represented in total income. But could it be that variations in total income reflect variations in social welfare when the distribution factor stays constant? This idea inspires the project of finding a convenient decomposition of social welfare in which total income appears as one term, as in Sen’s decomposition (3.3). It would be nice to be able to say that when a certain monetary measure of social welfare is worth, say, 73% of total income, this means that inequity in the distribution dampens the contribution of total income to welfare by 27%.

3.4.1 A first decomposition, with the social expenditure function
Pollak, (1981) introduced the notion of social expenditure function. The usual definition of this function takes the standard form of an expenditure function and computes the minimum amount of (total) expenditure needed under a given price vector $p$ to attain a given level of (social) welfare $w$:

\begin{equation}
V(p, w) = \min \{x_1 + \cdots + x_n \mid s.t. W(u(x_1), \ldots, u(x_n)) \geq w\}.
\end{equation}

One can then define the corresponding “money-metric” social welfare function, for which, in the arguments, the physical allocation is substituted for the level of social welfare:

\begin{equation}
V(p, x_1, \ldots, x_n) = V(p, W(u(x_1), \ldots, u(x_n)))
\end{equation}

Equivalently,

\begin{equation}
V(p, x_1, \ldots, x_n) = \min \{x_1 + \cdots + x_n \mid s.t. W(u(x_1), \ldots, u(x_n)) \geq W(u(x_1), \ldots, u(x_n))\}
\end{equation}

Under mild regularity conditions, these functions are well defined and, for a given $p$, the $V$ function is ordinally equivalent to $W(u(x_1), \ldots, u(x_n))$.

Being expressed in monetary units, its evolution in time can be compared to that of GDP, no matter what social welfare function $W(u(x_1), \ldots, u(x_n))$ is adopted. But this does not provide a justification for GDP itself. When $(x_1, \ldots, x_n)$ is a market allocation with prices $p$, one always has

\begin{equation}
V(p, x_1, \ldots, x_n) \leq pX,
\end{equation}

because equality between the two terms is obtained when the allocation is socially optimal (given $X$), and strict inequality is obtained when the distribution is not optimal. When $(x_1, \ldots, x_n)$ is not a market allocation with prices $p$, one still has

\begin{equation}
V(p, x_1, \ldots, x_n) \leq pX,
\end{equation}

but a strict inequality remains possible even if $(x_1, \ldots, x_n)$ is an optimal distribution of $X$.

For a market allocation, Jorgenson, (1990) therefore proposes the following decomposition:

\begin{equation}
V(p, x_1, \ldots, x_n) = pX \times \frac{V(p, x_1, \ldots, x_n)}{pX}.
\end{equation}

The additive variant of this decomposition,

\begin{equation}
V(p, x_1, \ldots, x_n) = pX + (V(p, x_1, \ldots, x_n) - pX),
\end{equation}

\begin{figure}[h]
has also been proposed by Jorgenson and Slesnick (see Slesnick, 1998, p. 2152). This decomposition can accommodate a wide variety of social welfare functions \( W(u_i(x_1), \ldots, u_k(x_n)) \).

Using decomposition (3.26) for the analysis of variations (across time or space) raises a complication when prices change. If one wants to compare a market allocation described by \( p, (x_p, \ldots, x_o) \) with another allocation described by \( p', (x'_p, \ldots, x'_o) \), one can then write

\[
\frac{V(p, x_p, \ldots, x_o)}{V(p, x_p, \ldots, x_o)} \times \frac{p'X}{pX} = \frac{V(p, x_p, \ldots, x_o)}{V(p, x_p, \ldots, x_o)} \times \frac{p'X}{pX}.
\]

The first term nicely features the Laspeyres quantity index. The second term, unfortunately, may be less than 1 even if the distribution is optimal in both allocations, because one typically has \( V(p, x_p, \ldots, x_o) / pX \) if \( p \neq p' \). A related problem is that this formula is not symmetric. If one took \( p, (x_p, \ldots, x_o) \) as the benchmark, one would obtain the decomposition (p.98)

\[
\frac{V(p, x_p, \ldots, x_o)}{V(p, x_p, \ldots, x_o)} \times \frac{p'X}{pX} = \frac{V(p, x_p, \ldots, x_o)}{V(p, x_p, \ldots, x_o)} \times \frac{p'X}{pX},
\]

in which the two terms are not the inverses of the terms of the previous decomposition.

This observation reveals a limitation of (3.26). The function \( V(p, x_p, \ldots, x_o) \) is ordinally equivalent to \( W \) when \( p \) is fixed. For the comparison of a wide array of allocations, one should therefore fix a reference price vector \( p \) and use it throughout. This is not a problem for the first term of the decomposition, which would then behave like a constant-price index of quantity. But, as observed above, the second term of the decomposition accurately measures inequity only if \( p \) is the prevailing price at the allocation \( (x_p, \ldots, x_o) \).

A further limitation of the decomposition must be mentioned. Jorgenson and Slesnick call the first term in (3.26) the “efficiency” term and the second one the “equity” term. This terminology is slightly misleading because \( pX \) could be quite inefficient, for instance if the production sector did not make use of the technology efficiently. It might also happen that the allocation \( (x_p, \ldots, x_o) \) is an inefficient distribution of \( x_i \) for instance if rationing occurred on the market. In this case, the second term would partly measure the degree of distributive
inefficiency of the allocation and not just inequity in the distribution.

This remark makes it interesting to consider a related kind of decomposition, which has been proposed by Graaff.

3.4.2 A second decomposition, in terms of efficiency and equity

Graaff’s decomposition does not involve total income as an element, but it provides interesting ideas that will be exploited in the next subsection. Graaff, (1977) computes an efficiency index as the smallest fraction of any producible bundle that maintains everyone at current satisfaction, normalizes social welfare as the smallest fraction of any producible bundle that maintains social welfare at its current level, and measures an equity index as the ratio of normalized social welfare over efficiency. Then normalized social welfare is the product of the efficiency and equity indexes.

Let us examine these notions in more detail. We retain the assumption that the social welfare function is increasing in individual utilities, and that individual preferences satisfy local nonsatiation. A key element of the decomposition is the Scitovsky set, which contains the vectors of total consumption that can be distributed so as to maintain all individuals’ current satisfaction. It is formally defined as (p.99)

$$S(x_p, ..., x_o) = \{X \in \mathbb{R}^l | \exists (x'_p, ..., x'_o, x_1 + \cdots + x_o) = X, \quad u(x'_i) \geq u(x_i) \forall i \}$$

which is equivalent to computing the Minkowski sum of the individual upper contour sets \( \{x'_i \in \mathbb{R}^l | u(x'_i) \geq u(x_i) \} \).

Let the production possibilities be described by a production set \( P \subset \mathbb{R}^l \). Grafaff’s index of efficiency is defined as

$$\sigma(x_p, ..., x_o) = \min \lambda \text{ s.t. } \exists X \in P, \lambda X \in S(x_p, ..., x_o),$$

$$= \min \lambda \text{ s.t. } \lambda P \cap S(x_p, ..., x_o) \neq \emptyset.$$  

This index is equal to 1 when it is impossible, under the prevailing production possibilities, to raise the satisfaction of one individual without decreasing the satisfaction of anyone. It therefore tracks inefficiency in production, when \( X \) is not on the upper boundary of \( P \). It also tracks inefficiency in the distribution, when \( (x_p, ..., x_o) \) could be redistributed so as to increase everyone’s satisfaction.
This index bears some similarity with Debreu’s (1951) coefficient of resource utilization, which is the smallest fraction of the resources available before production that would have made it possible, given the technology and the preferences, to maintain everyone at current satisfaction. Unlike Graaff’s index, Debreu’s coefficient depends on the role of commodities as net inputs or net outputs in production.

The measure of social welfare used by Graaff is not exactly the social expenditure function. It involves the Bergson set, which contains the vectors of total consumption that can be distributed so as to maintain social welfare. It is formally defined as

\[
B(x_1, \ldots, x_n) = \left\{ X \in \mathbb{R}^n \mid \exists x_1, \ldots, x_n \text{ s.t. } x_1 + \cdots + x_n = X, W(u(x_1), \ldots, u(x_n)) \geq W(u(x_1), \ldots, u(x_n)) \right\}
\]

The lower boundaries of Bergson sets correspond to the indifference map of the function \(W^\ast\). Interestingly, the Bergson set is also the union of the Scitovsky sets corresponding to a given level of social welfare:  

\[
B(x_1, \ldots, x_n) = \bigcup_{(x_1', \ldots, x_n') \in S(x_1', \ldots, x_n')} S(x_1', \ldots, x_n')
\]

The lower boundaries of Scitovsky sets, also called Scitovsky curves, do not form an indifference map because they cross. But if one restricts attention to a set of utility vectors \((u_1, \ldots, u_n)\) that all dominate each other for all components, the corresponding Scitovsky curves do not cross.

Graaff’s index of social welfare is defined in the same way as the efficiency index, with the Bergson set taking up the role of the Scitovsky set:

\[
\Gamma(x_1, \ldots, x_n) = \min \lambda \text{ s.t. } \exists X \in P, \lambda X \in B(x_1, \ldots, x_n), \lambda \in \mathbb{R}
\]

This index is equal to 1 when it would be impossible to increase social welfare under the current production possibilities.

The ingredients of the decomposition are illustrated in figure 3.2. The \(P\) curve is the upper frontier of the production set (i.e., any aggregate bundle on or below this curve can be produced), the \(S\) curve is the lower boundary of the Scitovsky set for allocation \((x_1, \ldots, x_n)\) (i.e., any aggregate bundle on or above this curve can be distributed so as to give every \(i = 1, \ldots, n\))
his satisfaction at \( x_i \), and the \( B \) curve is the lower boundary of the Bergson set (i.e., any aggregate bundle on or above this curve can be \( \textbf{(p.101)} \) distributed so as to yield as much social welfare as at \((x_p, \ldots, x_q)\). Note that the \( B \) curve is also the indifference curve at the level \( W(u_i(x_p), \ldots, u_i(x_q)) \) for the function \( W^* \) defined in \((3.2)\).

One always has

\[
S(x_p, \ldots, x_q) \subseteq B(x_p, \ldots, x_q),
\]

because maintaining everyone’s satisfaction obviously maintains social welfare for a Paretian social welfare function, while it may be possible to maintain social welfare with other allocations in which some individuals endure a loss of satisfaction. When \((x_p, \ldots, x_q)\) is efficiently distributed (in the sense that no other distribution of \( X \) would be better for someone and worse for no one), \( X \) belongs to the lower boundary of \( S(x_p, \ldots, x_q) \). When it is optimally distributed for \( W, X \) belongs to the lower boundary of \( B(x_p, \ldots, x_q) \).

Graaff’s equity index is the ratio of the social welfare and equity indexes:

\[
r(x_p, \ldots, x_q) = \frac{\Gamma(x_p, \ldots, x_q)}{\Phi(x_p, \ldots, x_q)},
\]

which is always less or equal to 1 by virtue of the inclusion \( S(x_p, \ldots, x_q) \subseteq B(x_p, \ldots, x_q) \). It is equal to 1 when \((x_p, \ldots, x_q)\) is an optimal allocation for \( W \), in which case all three indices are actually equal to 1, due to local nonsatiation that forces any Pareto-efficient production plan to be on the upper boundary of the production set. In summary, the decomposition is then

\[
(3.27)
\]
The equity index may be less than 1 even if \((x_p, \ldots, x_n)\) is an optimal distribution of \(X\) for \(W\), which is somewhat disturbing. This will occur when the allocation is not fully efficient, so that social welfare would be maximized with another production plan in the production set. The lower boundaries of the Scitovsky and the Bergson set typically coincide only at \(X\), when \((x_p, \ldots, x_n)\) is an optimal distribution of \(X\) for \(W\). If the reduced set \(I(x_p, \ldots, x_n)\) intersects \(I^*(x_p, \ldots, x_n)\) at a different point than \(X\), it is typically the case that \(I(x_p, \ldots, x_n)\) does not intersect \(s(x_p, \ldots, x_n)\), so that \(I(x_p, \ldots, x_n)\) may remain unchanged. The size of the pie is just not part of the picture. In particular, this decomposition does not include total expenditure and the vector \(X\) itself plays little role in it. This approach therefore cannot serve the purpose of this section, but it provides the inspiration for the next subsection.

Another unpalatable feature of Graaff’s decomposition is that his measure of normalized social welfare depends on the production set \(P\), unlike Pollak’s social expenditure function. This implies that situations involving different production sets cannot be compared. For instance, if technical progress makes it possible to increase social welfare, and social welfare is increased accordingly, the index \(I(x_p, \ldots, x_n)\) may remain unchanged. The size of the pie is just not part of the picture. In particular, this decomposition does not include total expenditure and the vector \(X\) itself plays little role in it. This approach therefore cannot serve the purpose of this section, but it provides the inspiration for the next subsection.

This is an area where research can still make progress. In the next two subsections we propose two new decompositions.

### 3.4.3 A new decomposition, based on Bergson curves

Taking inspiration from the decompositions proposed by Graaff, Jorgenson, and Slesnick, let us seek a better decomposition that would avoid the drawbacks pinpointed for these decompositions. Figure 3.3, similar to figure 3.2, illustrates a typical configuration (with a somewhat pessimistic outlook for efficiency and equity in order to better visualize the different curves). The lines with negative slopes represent bundles of equal value at reference prices \(\hat{p}\).
Let us take the Pollak function $V(p, x_1, \ldots, x_n)$ as the measure of social welfare. Apart from the fact that it has a convenient monetary scale, it also has the advantage of being independent of the technology, which makes it easy to compare allocations obtained under different technologies. The magnitude $V(p, x_1, \ldots, x_n)$ is represented on the figure, taking good 1 as numeraire.

The benchmark, ideal situation to which actual welfare should be compared can be constructed as the maximum level of social welfare obtainable with the production set \( (p.103) \)

$$V^{\text{max}}(p, P) = \max \{ V(p, W^*(Y)) | s.t. \ Y \in P \}.$$  

This quantity is also represented in figure 3.3. The curve $B^*$ is the highest indifference curve for $W^*$ that intersects the production set.

The starting point of the decomposition is the product

$$V(p, x_1, \ldots, x_n) = V^{\text{max}}(p, P) \times \frac{V(p, x_1, \ldots, x_n)}{V^{\text{max}}(p, P)},$$

in which the second term, typically less than 1, encapsulates inefficiency and inequity.

In order to see why $V^{\text{max}}(p, P)$ is the appropriate benchmark, note that the ideal situation is obtained when the production set and the Bergson set are tangent. This is a situation in which all indices of efficiency and equity should equal 1, so that social welfare must then be equal to the benchmark magnitude from which, in the decomposition, inefficiency and inequity would deduct some amount in a nonideal situation. Conversely, this means that the benchmark magnitude must equal the highest Bergson curve that is tangent to the production set. \( \text{Indeed, fix a Bergson curve (an indifference} \)
curve for \( W^* \)) and consider various possible optimal allocations belonging to different production sets that would each be tangent to this curve. The benchmark magnitude must then be the same for all these situations, as it must be equal to social welfare at this Bergson curve.

Now let us examine how to decompose the second term of (3.28) into efficiency and equity terms. A first term that can play a role is \( V(\tilde{p}, W^*(X)) \), that is, the maximum social welfare that could be obtained with an optimal distribution of \( X \). The indifference curve of \( W^*(X) \) is the curve \( B^X \) on the figure. The ratio

\[
\chi(x_1, \ldots, x_n) = \frac{V(\tilde{p}, W^*(X))}{V^{\text{max}}(\tilde{p}, P)}
\]

then measures the welfare loss due to the suboptimality of the production plan \( \chi \). It does not involve the way \( X \) is distributed, but only the fact that a better plan could be found in the production set. It is not just a measure of productive efficiency, because it also accounts for the suboptimality of the direction of \( \chi \). This ratio may be less than 1 even if \( X \) lies on the production frontier (i.e., the upper boundary of \( P \)).

If \( X \) is inefficiently distributed, it is as if part of \( X \) was wasted before the distribution, so that the actual quantity distributed lies on the part of curve \( S \) that lies below \( \chi \). Let this part of the curve, identifiable in the figure (p.104) by dotted lines below \( \chi \), be denoted \( S^X \). The lowest level of welfare that could be obtained with an optimal distribution of resources taken from \( S^X \), namely,

\[
V^{\text{min}}(\tilde{p}, S^X) = \min V(\tilde{p}, W^*(Y)) \text{ s.t. } Y \in S^X,
\]

yields an interesting quantity to be used in a measure of distributive efficiency. Let

\[
\delta(x_1, \ldots, x_n) = \frac{V^{\text{max}}(\tilde{p}, S^X)}{V(\tilde{p}, W^*(X))}.
\]

Such an index measures the gap between the level of welfare attainable with \( X \) and the level that is attainable with what is actually used of \( \chi \), given distributive inefficiencies.

It is equal to 1 when \( X \) is efficiently distributed and therefore lies on the curve \( S \), so that \( S^X \) reduces to the singleton \( \{X\} \). For this property of \( \delta(x_1, \ldots, x_n) \) to be satisfied, it is important to take \( V^{\text{min}}(\tilde{p}, S^X) \) rather than \( V^{\text{min}}(\tilde{p}, S) \) in the definition of the ratio.
Taking the latter would induce an index \( \delta(x_p, ..., x_n) \) that is generally lower than 1 when \( X \) is efficiently but not socially optimally distributed.

Finally, equity can be measured by the ratio
\[
\xi(x_p, ..., x_n) = \frac{V(\hat{\rho}, x_p, ..., x_n)}{V^\text{max}(\hat{\rho}, S^\delta)}.
\]
This ratio is equal to 1 when the distribution of \( X \) is optimal. It is also equal to 1 when the distribution is inefficient but the individual shares are optimal, so that there is \( Y \in S^X \) on the Bergson curve for \((x_p, ..., x_n)\). This is a case in which one could throw away part of \( X \) but keep the same distribution of utilities and obtain a socially optimal distribution of the remaining quantities.

In summary, one obtains the following decomposition:

\[
(3.29) \\
V(\hat{\rho}, x_p, ..., x_n) = V^\text{max}(\hat{\rho}, P) \times \chi(x_p, ..., x_n) \times \delta(x_p, ..., x_n) \times \xi(x_p, ..., x_n).
\]
This decomposition appears to avoid the main drawbacks listed for the previous decompositions. It relies on a notion of social welfare that respects preferences, that clearly separates distributive equity from distributive efficiency and productive optimality, unlike (3.26), and that does not depend on the technology, unlike (3.27). Its equity index is always equal to 1 when \( X \) is optimally distributed, a property that neither (3.26) nor (3.27) (p.105) satisfied. Moreover, its index of distributive efficiency is always equal to 1 when \( X \) is efficiently distributed.

One nice property of Graaff’s decomposition (3.27) can also be retrieved. When the production set and the Scitovsky set are tangent, its index of efficiency \( \epsilon(x_p, ..., x_n) \) is equal to 1. As a matter of fact, our new index of productive optimality \( \chi(x_p, ..., x_n) \) can be decomposed into an index of productive efficiency and an index of directional optimality. The former may be defined as \( \pi(x_p, ..., x_n) = \frac{V(\hat{\rho}, W^*(X))}{V(\hat{\rho}, W^*(X^*))} \), where \( X^* \) is the expansion of \( X \) that lies on the production frontier (see the figure). Taking the product of the two indexes of productive and distributive efficiency, \( \pi(x_p, ..., x_n) \delta(x_p, ..., x_n) \), the desired property is then retrieved.

A limitation of this decomposition is that it is conceived for a very special model of private goods and is not easily
extendable to a setting with nonmarket goods (such as public goods or personal nontransferable characteristics) and unequal productive skills.

In the context of a search for a decomposition featuring total income as a particular term, however, the decomposition (3.29) is not successful. It does not display the market value of $X$ as a specific term. However, the product of the first two terms is equal to

$$V^{\text{max}}(\bar{p}, \mathbf{p}) \times_\mathcal{X}(x, \ldots, x_n) = \bar{V}(\bar{p}, W'(X)),$$

which is not too different from $\bar{p}X$ when the supporting price vector of the curve $B^X$ at $X$ is not too different from $\bar{p}$.

Rigorously, however, this is not the same thing, and the difference may be substantial in some cases.

The fact that the money-metric social value of $X$ rather than a simple market value of $X$ at suitable prices appears in this decomposition may, perhaps, reveal a deep difficulty. It appears intrinsically hard to make $pX$ an interesting term in the analysis of social welfare.

3.4.4 Another decomposition, for small variations

In this last decomposition, we consider a richer setting in which individual utility $u_i(x, y_i)$ is determined by market commodities $x_i$ and nonmarket goods $y_i$. The latter can also include leisure when individuals have unequal skills and labor time cannot be transferred between individuals without affecting the production. Let $v_i(p, y_i, m_i)$ denote the indirect utility function, defined as the maximum utility $u_i(x, y_i)$ that $i$ can obtain when buying $x_i$ with income $m_i$ under prices $p$, for a fixed $y_i$:

$$v(p, y_i, m_i) = \max \{ u(x, y_i) \mid px_i \leq m_i \}.$$  

(p.106) For small variations, assuming continuous differentiability of $u_i$, a rather simple decomposition of social welfare can be obtained. It relies on the assumption that market commodities $x_i$ are always bought on a market in which all individuals face the same price vector $p$.

The starting point is to notice that, at a market allocation, for a small change $(dp, dx_i, dy_i, dm_i)$, the variation in utility is equal to
\[ dW = \frac{\partial v}{\partial m} (pdX + w), \]

where

\[ w = \sum_l \frac{\partial v_l}{\partial y_l} dy_l \]

is the willingness-to-pay to incur the change in the vector \( y_i \) (\( l \) is the label of dimensions of \( y_i \)).

Letting \( \beta_i = \frac{\partial v}{\partial m_i} \) denote the social marginal utility of \( m_i \), one can write

\[ dW(v_1, v_2, \ldots, v_n) = \sum_{i=1}^n \frac{\partial v_i}{\partial x_i} dx_i = \sum_{i=1}^n \beta_i (pdX + w). \]

This makes it possible to compute the following decomposition (letting \( \bar{\beta} \) denote the mean \( \frac{1}{n} \sum_{i=1}^n \beta_i \) and \( \bar{w} \) the mean \( \frac{1}{n} \sum_{i=1}^n w_i \)):

\[ dW(u_1, u_2, \ldots, u_n) = \bar{\beta} pdX + \bar{\beta} \bar{w} \]

\[ + \sum_{i=1}^n (\beta_i - \bar{\beta})(pdX - pdX/n) + \sum_{i=1}^n (\beta_i - \bar{\beta})(w_i - \bar{w}). \]

The formula uses the fact that the mean of a product equals the product of the means plus the covariance.

The last two terms in the decomposition (3.31) depend on the correlation between the social priority of individuals and the variation in their situation and is quite intuitive for a distributional term. Each term is positive if the individuals whose situation improves more than average tend to have a greater \( \beta_i \) coefficient.

This decomposition is very simple, additive, and features \( pdX \) as a component. It nicely distinguishes the effects of changes in the pie and changes in the distribution, both for market and for nonmarket goods. One could argue that it makes it legitimate to focus on the evolution of total expenditure at constant prices, provided it is explicitly stated that distributive issues and nonmarket dimensions are ignored. Obviously, it suggests that the most interesting approach is to compute the four terms rather than just one, because a positive evolution in
pdX may hide ominous developments in nonmarket dimensions and in the distribution.

The difficulty in implementing this decomposition is that one needs not only data on the joint distribution of \((x, y)\), but also the weights \(\beta_i\) which depend on the calibration of utilities. This is where the equivalent income concept discussed in the next chapter will appear particularly useful because it provides a calibration of utilities that delivers interpersonal comparisons across individuals with different preferences.

To conclude this section, the quest for a good decomposition of social welfare turns out to be a rather complex and intriguing task. Among the decompositions introduced in this section, (3.29) and (3.31) appear the most interesting. The former, which can be applied to any conception of social welfare, contains aggregate terms that are similar but not identical to total income. The latter, for small variations, does feature total income (at fixed prices) and is also compatible with any social welfare function.

The field of social welfare decomposition deserves more research. It would be nice to obtain axiomatic justifications of specific decompositions, and to be able to identify terms that embody particular efficiency notions and especially particular equity notions. Another decomposition will be introduced in the next chapter, involving a specific calibration of utility and with equity defined as inequality aversion with respect to such calibrated utilities.

One can dream that in the near future, implementing these decompositions with real-life statistics would become a routine and would inform the evaluation of social progress by showing the contribution to social welfare made by efficiency and equity components of the social situation.

3.5 Specific problems with imputed prices and full income
In most of this chapter (i.e., except in the last subsection and in a small part of section 3.2), we have assumed away the presence of nonmarketed goods or similar considerations that would obviously call for a correction to income before looking for a link between income and welfare, at the individual level. The fact that in absence of these difficulties, the link has already proved hard to establish, especially at the social level, suggests that the idea of refining the concept of total income
by making such corrections is not very promising. Refinements are usually elaborated for concepts that are successful in the simplest case, not for concepts that are already flawed in the simplest conditions. It is, however, worth examining how such corrections affect the relation between income and welfare.

The addition of the shadow price of nonmarketed goods is usually said to yield a measure of “full income.” The principle of computation of full income is very simple and can be illustrated in the case in which there are only two goods, a generic marketed good called “income” (taken as the numeraire) and a nonmarketed good \( y \) (e.g., health, leisure, safety, environmental quality). Consider a consumer whose income is \( m \) and who consumes the quantity \( y \) of the other good. He has a marginal rate of substitution \( r \) that measures how much of \( m \) he is willing to pay in order to have an additional unit of \( y \).

Full income is then defined as the magnitude \( m + ry \). The concept, illustrated in figure 3.4, is natural if one seeks to find the market-like hypothetical situation that is the closest to the actual situation of this consumer. Assuming that preferences are convex, one sees that if this consumer could buy both goods on the market at the price vector \((1, r)\), with total income \( m + ry \), he would indeed take the bundle \((m, y)\) as a best choice.

The notion of full income is often invoked in relation to leisure. Although there is a labor market, it requires some imagination to consider that a worker “buys” his leisure on a market at a posted price of leisure. In particular, the net wage varies with

![Figure 3.4 Full income](image-url)
the amount of leisure, and many workers are not really free to choose their preferred quantity of labor in their budget set. It is therefore interesting to seek to estimate the true shadow value of leisure.

Once the analysis is cast in the framework of consumption bundles \((m, y)\) at prices \((1, r)\), the analysis of the previous sections can be unfolded again. Not much is changed at the individual level. For instance, the revealed preference argument is still valid. If \(m + ry \geq m' + ry'\), the individual is at least as well off with \((m, y)\) as with \((m', y)\).

The only nuance one could think of is that there might be a difference, for the evaluation of welfare, between a free choice in a budget set of a full bundle \((m, y)\) and being constrained on \(y\), or choosing \((m, y)\) from an opportunity set with a shape different from a typical budget set. This issue is ignored in the typical consumer preferences, which only care about the final bundle consumed. If individual preferences were enriched so as to care about the opportunities and not just the final bundle, the analysis could be developed again in a richer space.

\((p.110)\) At the social level, however, there is an important new problem. The shadow price \(r\) is specific to the particular individual situation that is examined. In particular, different individuals will have different shadow prices. One cannot therefore talk about “the” shadow price of leisure or health. This considerably complicates the implementation of monetary measures based on prices.

Given such practical complexity, it is not a big step to move beyond the use of the “pricing” methodology and to move to a “willingness-to-pay” methodology, as will be done in the next chapter. The difference between the two methodologies is illustrated in figure 3.4, where \(m_i + e_i\) denotes the level of income that renders the individual indifferent between enjoying the current bundle \((m, y)\) and obtaining instead the income increment \(e_i\) without any quantity of \(y\).\(^{15}\) It turns out that most of the problems implied by measuring welfare with prices are avoided when one relies instead on willingness-to-pay. From figure 3.4, it is intuitive that a measure like \(m_i + e_i\), which is based on the indifference curve, is more precise than
a measure like \( m_i + r_i'y_i' \), based on a local marginal rate of substitution. This alternative approach will be explored further in chapter 4.

We conclude this section by briefly noting that there is an extension of Samuelson’s optimality result to the setting with nonmarket goods. Suppose that \( (x_1, \ldots, x_n) \in \mathbb{R}_+^n \) is a market allocation at prices \( p \), with \( px_i \geq 0 \) for all \( i \), while \( (y_1', \ldots, y_n') \in \mathbb{R}_+^n \) is a nonmarket allocation of other goods, with shadow price vectors \( (r_1, \ldots, r_n) \). Individual preferences are assumed to be locally nonsatiated and representable by differentiable utility functions. If \( (x_1, \ldots, x_n) \) is an optimal distribution of \( X \) given \( (y_1', \ldots, y_n') \) for a social welfare function \( W(u_1(x_1, y_1'), \ldots, u_n(x_n, y_n')) \) that is quasi-concave and differentiable in the quantities of commodities, then for all allocations \( (x_1', \ldots, x_n'), (y_1', \ldots, y_n') \) such that

\[
\sum_{i=1}^n px_i + r_i' y_i' \geq \sum_{i=1}^n px_i' + r_i' y_i'
\]

one has

\[
W(u_1(x_1, y_1'), \ldots, u_n(x_n, y_n')) \geq W(u_1(x_1', y_1'), \ldots, u_n(x_n', y_n')) .
\]

(p.111) This is proved by showing that, in a social budget set defined by the condition

\[
\sum_{i=1}^n px_i + r_i' y_i' \leq M,
\]

with \( M = \sum_{i=1}^n px_i + r_i' y_i' \), the allocation \( (x_1, \ldots, x_n), (y_1', \ldots, y_n') \) is optimal for the objective \( W(u_1(x_1, y_1'), \ldots, u_n(x_n, y_n')) \) because of quasi-concavity with respect to quantities.

Indeed, by differentiability the fact that \( (x_1, \ldots, x_n) \) is an optimal distribution of \( X \) implies that for some \( \lambda > 0 \), for all individuals \( i \), all goods \( k \), either \( \frac{\partial W}{\partial u_i} \frac{\partial u_i}{\partial x_k} \bigg| p_k = \lambda \) or \( \frac{\partial W}{\partial u_i} \frac{\partial u_i}{\partial x_k} \bigg| p_k = \lambda \) and \( x_{ik} = 0 \).

By definition of shadow prices, one has for all \( k, l \) such that \( x_{ik} \neq 0 \),

\[
\frac{\partial u_i}{\partial x_k} \frac{\partial x_k}{\partial p_k} \geq \frac{\partial u_i}{\partial x_l} \frac{\partial x_l}{\partial p_l} ,
\]

with equality when \( y_{il} = 0 \).

Now consider an infinitesimal change to the allocation \( (x_1, \ldots, x_n), (y_1', \ldots, y_n') \). It induces
where the last inequality comes from the fact that for all $i$, $\frac{\partial W}{\partial u_i} p_i \leq \lambda$ and $\frac{\partial W}{\partial u_i} r_i y_i \leq \lambda$. This proves that $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is optimal in the defined budget $M$, because $\sum (pdx_i + r_idy_i) = 0$ when the equality $M = \sum_{i=1}^{n} pX_i + r_iY_i$ is preserved, and quasi-concavity with respect to quantities has been assumed.

(p.112) The argument is then completed by observing that any alternative allocation $(x_1', \ldots, x_n'), (y_1', \ldots, y_n')$ that belongs to the same budget set $\sum_{i=1}^{n} pX_i + r_iY_i \leq M$ yields at most as good an outcome as the optimal $(x_1, \ldots, x_n), (y_1', \ldots, y_n)$.

In summary, the fact that $(x_1, \ldots, x_n)$ is optimally distributed (with nonnegative income for every individual) given $(y_1', \ldots, y_n)$ is enough to make sure that the trade-offs embodied in the price vector $(p, r_1, \ldots, r_n)$ reflect the true trade-offs for the social objective.

3.6 Conclusion

In conclusion, economic theory does not provide much support for relying on the market value of total consumption as a proxy for social welfare. The revealed preference argument justifies an incomplete criterion based on the Laspeyres and Paasche indexes (i.e., based on the inequality $pX \leq pX'$) only under stringent assumptions of social optimality that have no chance of being realized in a world of imperfect information. The decomposition approach, however, does deliver interesting insights about how to separate efficiency from equity in the analysis of social welfare, and there are interesting decompositions featuring the market value of consumption. In particular, (3.31) seems promising and makes it possible to disentangle the welfare effect of total expenditures, nonmarketed goods, and inequalities.

A limitation of the revealed preference approaches is that they are conceived for the evaluation of changes affecting a given population. They do not help much in comparisons across different populations across countries or across generations.
Sen, (1976) does raise the issue explicitly and shows that, under the additional principle that only the statistical distribution of individual situations matters, not the size of the population, his approach can be extended to comparisons between populations in a limited way. Specifically, one can then check if the population in one country is better off than if it were served the distribution of consumption of another country. Concretely this is done by checking if

\[ \alpha_1 p_x^1 + \cdots + \alpha_n p_x^n \]

where the subscripts 1, ..., n refer to n-quantiles instead of concrete individuals. The same methodology can be adapted to the per capita criterion, which then simply has to be formulated in per capita terms, as well as to the decompositions listed in a previous section.

But this sort of comparison does not address the challenge of comparing populations with different preferences. It is for instance possible to find in (p.113) some cases that country A is better off (in its eyes) than country B and that country B is also better off (in its eyes) than country A.

This problem is related to another general limitation of most of the approaches described in this chapter, namely, their silence about how to make interpersonal comparisons. In particular, the various approaches assuming that the distribution is optimal, or that it is described in a separate term in a decomposition such as (3.26), make it possible to focus on total consumption without even specifying further the distributive judgments. The function \( W(x_1, \ldots, x_n) \) or \( W(u(x_1), \ldots, u(x_n)) \) that is referred to in these constructions could be based on many different sorts of evaluation of individual well-being.

That can be viewed as an advantage because it makes a focus on total expenditure compatible with many different distributive principles. But as soon as one wants to also evaluate the distribution, it creates a danger. Indeed, formulas such as (3.26) and (3.31) naturally suggest taking the monetary value of individual consumption as the measure of individual well-being, and this is indeed done explicitly in (3.4). It is, however, important to be aware that the theory presented so far does not give any reason to adopt this.
measure of individual well-being. Independent arguments are needed.

Here is an example of such an argument. In the theories of justice proposed by Rawls, (1971) or Dworkin, (2000), social justice consists in allocating resources in a fair way, letting individuals make use of the resources at their disposal according to their own conception of the good life. If one ignores differences in internal resources (talent, disabilities), the fair distribution is that which maximizes the share of resources of those who have the least—the market valuation appearing convenient for the comparison of resource shares. Recall, however, that Rawls’s list of primary goods includes nonmarketed items such as basic freedoms, the powers and prerogatives of positions of responsibility, and the social bases of self-respect. Wealth is only one item, but an important one. In sum, provided that the other aspects of resources such as basic freedoms and status are well distributed, individual wealth is then a suitable measure for interpersonal comparisons. The market value of individual consumption can then be defended as a reasonable proxy for comparisons of wealth.

If personal abilities and needs are unequal, however, this line of argument no longer supports the market value of consumption as a proper metric. A practically important example concerns the valuation of leisure time, which is part of consumption broadly construed and is unequally accessible to individuals with different wage rates on the labor market. As shown in the previous section, the revealed preference argument suggests (p.114) using individuals’ net wage rate as the proper valuation of leisure time. But one should resist the temptation to compare full incomes (i.e., earnings plus the value of leisure) across individuals without a suitable deflator that corrects for their unequal opportunities. One cannot simply consider that those with a greater full income are better off, and that it would be nice to reduce inequalities in full incomes. In fact, equalizing full incomes across individuals would imply that those with greater productivity would have strictly smaller budget sets than those with lower productivity (because their leisure is more expensive), in a strange reversal from the laissez-faire situation. Moreover, they would typically be forced to work at their highest wage rate in order to pay their taxes, implying what Dworkin, (2000) called a “slavery of the talented.” The literature on fairness has proposed various
reasonable ways of comparing the resource shares of individuals with unequal productivity. More generally, there are interesting ways to extend the notion of wealth to the case of unequal needs and abilities. These issues will be addressed in more detail in chapters 4 and 6.

The conclusion of this one is that putting uniform prices on the different dimensions of individuals’ lives is too simple an approach if one wants to respect individual preferences.

Notes:

(1) The expression \( p\Delta x \) is the inner product \( \sum_k p_k \Delta x_k \), where \( k \) is the commodity label.

(2) The Laspeyres index of quantities uses the prices of the initial period; the Laspeyres index of prices uses the quantities of the initial period. The corresponding Paasche indices use the final values instead. See subsection 3.3.1, equations (3.10) and (3.11).

(3) The curve is an indifference curve; the line delineates the budget. The consumer can afford all bundles below or on the line, but not the bundles above.

(4) In simple terms, as can be seen on figure 3.1, at \( x \) the slope of the indifference curve is identical to the slope of the budget line. The gradient of \( u \), \( (\partial u / \partial x_1, \partial u / \partial x_2) \), is by definition a vector that is orthogonal to the indifference curve, while the price vector \( (p_1, p_2) \) is orthogonal to the budget line. Therefore, these two vectors must be proportional.

(5) One can perhaps view the theory of index numbers (discussed in section 3.3) as exploiting this idea.


(7) The existence of the representative agent requires the Slutsky matrix of the aggregate demand function to be symmetric and negative semidefinite (see Varian, 1992, chap. 08, for explanations, and (3.16) for the definition of the Slutsky matrix).

(8) Consider the social welfare function defined on individual indirect utilities:

\[
W(v_1(p, m_1), \ldots, v_n(p, m_n)).
\]
The social marginal value of income for \( i \) is defined as

\[
\frac{\partial w}{\partial v_i} \frac{\partial v_i}{\partial m_i},
\]

i.e., it measures the change in social welfare induced by giving one more dollar to individual \( i \).

(9). A linearly homogeneous function satisfies \( \beta(\lambda p) = \lambda \beta(p) \) for \( \lambda \in \mathbb{R}^+ \). Homothetic preferences are such that \( x \) is at least as good as \( y \) if and only if \( \lambda x \) is at least as good as \( \lambda y \), for all \( \lambda \in \mathbb{R}^+ \). For instance, Cobb-Douglas preferences represented by \( x^\alpha y^{1-\alpha} \) are homothetic and induce the indirect utility function \( m_i/\beta(p) \) with \( \beta(p) = p_1^\alpha p_2^{1-\alpha} \).


(12). The Minkowski sum of two sets \( A \) and \( B \) is the set of elements \( a + b \), where \( a \in A \) and \( b \in B \).

(13). Equation (3.30) is proved as follows. By definition,

\[
dv_i = \sum_k \frac{\partial v_i}{\partial p_k} dp_k + \sum \frac{\partial v_i}{\partial y_{il}} dy_{il} + \frac{\partial v_i}{\partial m_k} dm_k.
\]

By Roy’s identity, which remains valid in the presence of a given \( y_i \),

\[
x_k = \frac{\partial v_i}{\partial p_k}\frac{\partial p_k}{\partial v_i},
\]

and by definition, \( dm_i = pdx_i + x_i dp \). One therefore has

\[
dv_i = \frac{\partial v_i}{\partial m} \sum_k x_k dp_k + \sum \frac{\partial v_i}{\partial y_{il}} dy_{il} + \frac{\partial v_i}{\partial m_k} dm_k
\]

from which it is immediate to derive (3.30) by factorizing \( \partial v_i/\partial m \).

(14). Such terms should not be confused with the “distributional characteristic” (Feldstein, 1972a, 1972b; Atkinson and Stiglitz, 1980) that plays an important role in the analysis of optimal indirect tax or public pricing. The distributional characteristic depends on the correlation between \( \beta \) and the level rather than the variation of consumption of a commodity.
The distinction between shadow prices and willingness-
to-pay vanishes for infinitesimal variations, as in the 
computation of decomposition (3.31), where willingness-to-pay 
w_i relies on shadow prices computed with marginal rates of 
substitution between m_i and y_{il}.

More precisely, if x_{ik} = 0 and \( \frac{\partial w}{\partial x_{ik}} \frac{\partial x_{ik}}{\partial y_{il}} \) (\( \lambda \)), then \( dx_{ik} \geq 0 \) and 
\( \frac{\partial w}{\partial y_{il}} \frac{\partial x_{ik}}{\partial \lambda} dx_{ik} \) (\( \lambda dx_{ik} \)). If \( x_{ik} \neq 0 \), then 
\( \frac{\partial w}{\partial y_{il}} \frac{\partial x_{ik}}{\partial \lambda} = \lambda \) and 
\( \frac{\partial w}{\partial y_{il}} \frac{\partial x_{ik}}{\partial \lambda} dx_{ik} = \lambda dx_{ik} \). The 
same holds for \( y_{il} \). Note that the argument extends easily to the 
case in which \( y_{il} \) must belong to a compact interval. When it is 
equal to the higher bound, one can have 
\( \frac{\partial w}{\partial y_{il}} \frac{\partial x_{ik}}{\partial \lambda} \) (\( \lambda \)), but then 
\( dy_{il} \leq 0 \), which implies 
\( \frac{\partial w}{\partial y_{il}} \frac{\partial x_{ik}}{\partial \lambda} dy_{il} \leq \lambda dy_{il} \).