BOND PRICING, INTEREST-RATE PROCESSES, AND THE LIBOR MARKET MODEL

Ser-Huang Poon (Contributor Webpage)  
Richard Stapleton (Contributor Webpage)

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Abstract and Keywords

‘Bond Pricing, Interest-rate Processes, and the LIBOR Market Model’ uses the complete market, pricing kernel approach to value bonds, given stochastic interest rates. To value interest-rate derivatives, one important and practical problem is to model bond prices and interest rates with the correct drifts. The authors derive here the drift of the bond prices and interest rates under the period-by-period risk-neutral measure. As a special case, they derive the drift of the forward London Interbank Offer Rate (LIBOR) in what is generally known as the LIBOR Market Model.

Keywords: Bond Pricing, drift of the bond prices, drift of the forward London Interbank Offer Rate (LIBOR), interest-rate derivatives, LIBOR Market Model, period-by-period risk-neutral measure, risk-neutral measure

Bond pricing is an important application of the rational expectations approach to valuation. Long-term bond prices and bond forward prices can be obtained by taking
appropriate expected values of future spot prices, under the risk-neutral measure. In this chapter we use the complete market, pricing kernel approach to value bonds, given stochastic interest rates. One important, practical problem is to model bond prices and interest rates with the correct drifts. This is required in order to value interest-rate derivatives. We derive here the drift of the bond prices and interest rates under the period-by-period risk-neutral measure. As a special case, we then take the case of interest rates defined on a London Interbank Offer Rate (LIBOR) basis. We derive the drift of forward rates in what is generally known as the LIBOR market model.

7.1 Bond Pricing under Rational Expectations
In the previous chapter we used the rational expectations approach to value a cash flow $x_{t+T}$ which occurs at time $t+T$. Its value at time $t$ is given by

$$S_t = E_t\left\{E_{t+T}^Q[P_{t+T}X_{t+T}^{t+T}]\right\}$$

In order to value a bond paying certain cash flows, we first take the case where $x_{t+n} = \$1$ in every state and $t+n$ is the maturity (p. 114) date of the bond. The value of the cash flow in this case is hence

$$B_{t+T} = E_t\left\{E_{t+T}^Q[P_{t+T}X_{t+T}^{t+T}]\right\}$$

This is the value of a zero-coupon bond which has $n$ periods to maturity. As in the previous chapter it is often convenient to write this as an expectation under the ‘risk-neutral’ measure:

$$B_{t+T} = E_t^Q\left\{E_{t+T}^Q[P_{t+T}X_{t+T}^{t+T}]\right\}$$

or, simply, using the property of expectations,

$$B_{t+T} = E_t^Q\left\{\prod_{k=0}^{n-1} B_{t+T+k} \right\}$$

Equation (7.3) expresses the value of a long-term bond as an expectation of future stochastic short-term bond prices. We can now use equation (7.4) to obtain a further important result. First note that (7.4) can be written as:

$$B_{t+T} = B_{t+T}^Q E_t^Q\left\{E_{t+T}^Q[\prod_{k=0}^{n-1} B_{t+T+k}]\right\}$$

$$= B_{t+T}^Q E_t^Q(B_{t+T+k})$$

since $B_{t+T+k}$ is non-stochastic. Hence dividing through by $B_{t+T+1}$ we have, using spot-forward parity,
This states that the (one-period-ahead) forward price of a bond is the expectation, under the $Q$ measure, of the (one-period-ahead) spot price of the bond.\textsuperscript{49} (p.115) These pricing equations will now be used to analyse the forward prices of bonds, and in particular the drift, or expected increase, of forward prices.

### 7.1.1 Bond Forward Prices

In the table below, we present the spot prices and forward prices of a zero-coupon bond which pays $1 at time $t + n$. The forward contract matures at time $t + T < t + n$. In the first part of the table we show the spot and forward prices using the $\hat{B}$ and $\hat{F}$ notation to emphasise the fact that these prices are stochastic. The zero-coupon bond prices converge to $1$ at time $t + n$. The forward prices for delivery of the bond at time $t + T$ converge to the spot price $\hat{B}_{t,T+n}$ at time $t + T$. 

\[
F_{t,T+n} = \frac{\hat{B}_{t+n}}{\hat{F}_{t,T}} = E^Q_{t,T+n}\left[\hat{B}_{t+n}\right]
\]
<table>
<thead>
<tr>
<th>( t )</th>
<th>( t+1 )</th>
<th>( t+T )</th>
<th>...</th>
<th>( t+n )</th>
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<tr>
<td>( B_{t,t+n} )</td>
<td>( \hat{B}_{t+1,t+n} )</td>
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</table>

\[
F_{t,t+T,t+n} \rightarrow \hat{F}_{t+1,t+T,t+n} \rightarrow B_{t+T,t+n}
\]

\[
F_{t+T,t+n} = \frac{E^Q(\hat{F}_{t+1,t+T,t+n}) + \text{cov}^Q(F_{t+T,t+n}, \hat{F}_{t+T,t+n}) \hat{B}_{t+T,t+n}}{\hat{B}_{t+T,t+n}}
\]
In the second part of the table we state a result that the $T$-period forward price of the bond at time $t$ equals the expected forward price at $t + 1$ plus a term whose sign depends on the covariance (under the $Q$ measure) of the time $t + 1$ bond price and bond forward price. Since this covariance is likely to be positive, this result shows that the drift of the forward bond price under the $Q$ measure:

\[(7.5)\]

\[E^Q_t[F_{t+1,t+T,n}] - F_{t+1,T,n} = -\frac{B_{t+1}}{B_{t+T}} \text{cov}^Q \left(F_{t+1,t+T,n}, B_{t+1}\right)\]

is likely to be negative. In the following argument we establish this fundamental result.

First, we use forward parity to write the bond price at time $t + 1$:

\[B_{t+1} = F_{t+1,T,n} B_{t+T}.\]

Next, we take the expected value and use the definition of covariance to obtain

\[e^Q_t[B_{t+1} = E^Q_t[F_{t+1,T,n}] E^Q_t[B_{t+T}] + \text{cov}^Q \left(F_{t+1,T,n}, B_{t+1}\right)\]

Now, using the result derived above that the one-period-ahead forward bond price is the expected value, under the $Q$ measure of the one-period-ahead spot bond price, we can write

\[\frac{B_{t+1}}{B_{t+T}} = E^Q_t[F_{t+1,T,n}] B_{t+1}\]

\[+ \text{cov}^Q \left(F_{t+1,T,n}, B_{t+1}\right)\]

Then, equation (7.5) follows immediately by multiplying by $B_{t,t+1}$ and dividing by $B_{t,t+T}$.

7.1.2 Some Further Implications of Forward Parity and Rational Expectations

One special case of the drift in (7.5) above is the case where $T = 1$. Here equation (7.5) simplifies to

\[(7.6)\]

\[F_{t+1,T,n} = E^Q_t[B_{t+1}n] \]

since $F_{t+1,t+1,t+n} = B_{t+1,t+1}$ and $B_{t+1,t+1} = 1$, a constant. Hence, the one-period-ahead forward price of the $n$-period bond is just the expected value of the subsequent period spot price of the bond. We now apply forward parity arguments to expand the right- and left-hand sides of equation (7.6).

First, the spot price at $t + 1$ on the right-hand side of (7.6) can be written as:

\[(7.7)\]

\[B_{t+1} = B_{t+2} F_{t+2,t+3} F_{t+3,t+4} \cdots F_{t+n+1,t+n}\]
i.e., the spot price is the product of successive forward prices. Also, using a similar argument the left-hand side of (7.6) can be written as:

\[
F_{t,T} = F_{t,T-1} \cdot F_{t,T-2} \cdot \ldots \cdot F_{t,1}
\]

This follows from the fact that a long-term forward contract can be replicated by a series of short-term contracts.

**7.2 The Drift of Forward Rates**

One of the challenges of financial theory has been to construct dynamic models of the term structure of interest rates which are consistent with no-arbitrage. In order to achieve this, models have to satisfy the condition derived above that bond forward prices for all maturities equal the expectations of the one-period-ahead spot bond prices for those maturities, under the risk-neutral measure. However, in order to price interest-rate dependent derivatives, often what is required is a model of the evolution of forward interest rates rather than forward prices. In this section, we therefore derive the correct drift of the forward interest rate.

Interest rates can be expressed in many different ways. In this chapter we use the conventional annual rate. For simplicity, assume that the period length, from \( t \) to \( t + 1 \) is one year. The annual yield rate at time \( t \), \( y_t \), is then defined by the equation

\[
B_{t+1} = \frac{1}{1 + y_t}
\]

Consistent with this spot yield rate we define the forward rate at time \( t \), as \( f_{t,t+T} \), in the equation

\[
F_{t,t+T} = \frac{1}{1 + f_{t+1}}
\]

Note that we will always refer to forward rates for one-year loans and hence we have dropped the final subscript on the forward rate.

Using these definitions and substituting (7.7) and (7.8) in the equation (7.6), we find the following:

\[
E\left[ \frac{1}{1 + y_{t+1}} \cdot \frac{1}{1 + y_{t+2}} \cdot \ldots \cdot \frac{1}{1 + y_{t+T}} \right] = \frac{1}{1 + f_{t+1}}
\]

Note that equation (7.9) holds for all values of \( T \). Hence, in particular, with \( T = 1 \),

\[
E\left[ \frac{1}{1 + y_{t+1}} \right] = \frac{1}{1 + f_{t+1}}
\]

These relationships will play a key role in our subsequent derivation of the forward rate drift.
We now ask the question: what is the drift of the forward rate, $f_{t,t+T}$, under the risk-neutral measure? That is, what is $E_{t}^{q}[f_{t,T+1}]-f_{t,T}$? To answer the question, we introduce the following contract definition.

**Definition 6** A forward rate agreement (FRA) is an agreement made at time $t$ to exchange fixed-rate interest payments at a rate $k$ for variable rate payments, on a principal amount $A$, for the loan period $t + T$ to $t + T + 1$.

The time scale for payments on a $T$-maturity FRA is shown below:

Here, $t$ is the contract agreement date, $t + T$ is the settlement date, and $t + T + 1$ is the date on which the notional loan underlying the FRA is repaid.

The contract is usually settled in cash at $t + T$ on a discounted basis. The settlement amount at time $t + T$ on a long FRA is

$$FRA_{t,T} = \frac{A(y_{t+T} - k)}{1 + y_{t+T}}.$$  

From here onwards, for convenience we assume that the principal $A = 1$. Note that the FRA settlement takes place at $t + T$, and is discounted by the spot rate discount factor $1/(1 + y_{t+T})$. In contrast, a futures contract on an interest rate payoff would not be discounted, although it is also settled at $t + T$. At the time of the contract inception, an FRA is normally structured so that it has zero value. To guarantee this, the strike rate $k$ is set equal to the market forward rate $f_{t,t+T}$. We denote the value of the FRA at time $t$ as $FRA_{t,T}$. Hence, if the FRA is correctly priced we must have:

$$E_{t}^{q}[\frac{y_{t+T} - f_{t,T}}{1 + y_{t+T}}] = 0.$$  

**Note again that equation (7.11) holds for all forward maturities $T$. Given a probability distribution of possible outcomes of the interest rate $y_{t+T}$, the equation determines the market forward rate, $f_{t,t+T}$.**

7.2.1 FRA Pricing and the Drift of the Forward Rate: One-period Case
The determination of $f_{t,t+T}$ is complicated by the fact that the FRA payment is discounted over successive periods at the stochastic rate of interest. The argument is easier to understand, if we start initially with a one-period FRA and then proceed to price a two-period FRA, before moving to the general $T$-period case.

We first consider a one-year FRA. Since a one-period FRA struck at $f_{t,t+1}$ has zero value, employing equation (7.11), with $T = 1$, we have

$$E_t \left( \frac{y_{t+1} - f_{t+1}}{1 + y_{t+1}} \right) = 0,$$

and it follows that

$$E_t \left( \frac{f_{t+1}}{1 + y_{t+1}} \right) = E_t \left( \frac{y_{t+1}}{1 + y_{t+1}} \right),$$

and hence that

$$f_{t+1} E_t \left( \frac{1}{1 + y_{t+1}} \right) = E_t \left( y_{t+1} \right) E_t \left( \frac{1}{1 + y_{t+1}} \right) + \text{cov}_t \left( y_{t+1}, \frac{1}{1 + y_{t+1}} \right).$$

One problem in evaluating this equation is the term $E_t \left[ 1/(1 + y_{t+1}) \right]$. However, we can now use the result from equation (7.9) with $T = 1$ to write

$$E_t \left( \frac{1}{1 + y_{t+1}} \right) = \frac{1}{1 + f_{t+1}},$$

and it follows, multiplying by $(1 + f_{t,t+1})$,

(7.12) $E_t \left( y_{t+1} \right) - f_{t+1} = -\left(1 + f_{t+1}\right) \text{cov}_t \left( y_{t+1}, \frac{1}{1 + y_{t+1}} \right)$.

Equation (7.12) shows that the drift of the forward rate is determined by a covariance term involving $y_{t+1}$ and a function of $y_{t+1}$. As $y_{t+1}$ increases, $1/(1 + y_{t+1})$ decreases. Hence, the covariance term is negative and the drift of the forward rate $E_t \left( y_{t+1} \right) - f_{t+1}$ is always positive.

7.2.2 FRA Pricing and the Drift of the Forward Rate: Two-period Case

We now proceed to calculate the drift of a two-period forward rate over the first period. We use a similar argument, but this time consider a two-period FRA. At time $t$, assume that we enter a long two-period maturity FRA contract with a strike price $k_1$. The expected payoff at the maturity date, $t + 2$ is

$$\frac{y_{t+2} - k_1}{1 + y_{t+2}}.$$
Under no-arbitrage, the strike rate must equal the two-year forward rate, i.e., \( k_1 = f_{t,t+2} \). At the end of the first period, we enter a short FRA contract (i.e., this is known as a reversal strategy) with the following payoff
\[
\frac{k_2 - y_{t+2}}{1 + y_{t+2}},
\]
again at time \( t + 2 \). Under no arbitrage, the strike rate on this second FRA must equal the one-period-ahead forward rate at \( t + 1 \), i.e., \( k_2 = f_{t+1,t+2} \). Now we evaluate the portfolio of the original FRA plus the short FRA entered into at \( t + 1 \). Substituting for \( k_1 \) and \( k_2 \), the payoff on the portfolio at time \( t + 2 \) is given by
\[
\frac{f_{t+1,t+2} - f_{t+2}}{1 + y_{t+2}},
\]
since the uncertain interest rate, \( y_{t+2} \), in the numerator cancels out.

The value of the portfolio at time \( t + 1 \) is found by taking the expected value at \( t + 1 \), under the \( Q \) measure and discounting by \( (p.121) \) the interest rate \( y_{t+1} \). This is

\[
(7.13)
E_t^Q \left( \frac{f_{t+1,t+2} - f_{t+2}}{1 + y_{t+2}} \right) = 0,
\]
and hence
\[
E_t^Q \left( \frac{f_{t+1,t+2}}{1 + y_{t+1}} \right) = f_{t+2} E_t^Q \left( \frac{1}{1 + y_{t+2}} \right).
\]
It then follows that the drift of the two-period forward rate is given by

\[
(7.14)
E_t^Q \left( f_{t+1,t+2} \right) - f_{t+2} = - \text{cov}_t \left( f_{t+1,t+2}, \frac{1}{1 + y_{t+2}} \right) \times \left( 1 + f_{t+1} \right) \left( 1 + f_{t+2} \right).
\]
To obtain the last term in (7.14) we have used (7.9) with \( T = 2 \). In general the drift of the \( T \)-period forward rate (7.15)
\[
E_t^Q \left( f_{t+1,t+T} \right) - f_{t+T} = - \text{cov}_t \left( f_{t+1,t+T}, \frac{1}{1 + y_{t+T}} \right) \times \left( 1 + f_{t+1} \right) \left( 1 + f_{t+T} \right).
\]
To obtain the last term in (7.15) we have again used (7.9). In general, the covariance term in equation (7.15) is difficult to evaluate. However, if the one-period ahead spot rates and forward rates are assumed to be lognormal, the covariance can be easily evaluated (p.122) in terms of logarithmic covariances. This leads to a model that can be implemented easily and practically.

7.2.3 The Drift of the Forward Rate under Lognormality

We now assume that the forward rate $f_{t+1,t+T}$ is lognormal for all forward maturities, $T$. We can then evaluate the covariance term, using an approximation. In the appendix, we show that the following approximation holds for the covariance of two variables $X$ and $Y$, by expanding $X$ around a constant $a$ and $Y$ around a constant $b$:

$$\text{cov}(X, Y) = ab\text{cov}(\ln X, \ln Y)$$

We now evaluate the drift of the yield rate in (7.12), assuming that $y_{t+1}$ is lognormal. Here we take $a = f_{t+1}$ and $b = 1/(1 + f_{t+1})$. We then have

$$\text{cov}\left(\frac{1}{y_{t+1}}, \ln y_{t+1}\right)$$

and substituting in (7.12), the drift of the one-year forward rate is (7.16)

$$E_t(f_{t+1}) - f_{t+1} = -f_{t+1}\text{cov}\left(\ln y_{t+1}, \ln \frac{1}{1+y_{t+1}}\right)$$

We now proceed to evaluate the drift of the two-year forward rate in equation (7.14). Here we have the term

$$\text{cov}\left(\frac{1}{y_{t+2}}, \ln y_{t+2}\right)$$

and substituting this in the first part of equation (7.14) and using the property of logarithms we find that (7.17)

$$E_t(f_{t+2}) - f_{t+2} = -f_{t+2}\text{cov}\left(\ln f_{t+2}, \ln \frac{1}{1+y_{t+2}}\right)$$

In general, the drift of the $T$-maturity forward rate depends on the sum of a series of covariance terms. The drift in the general case is (7.18)

$$E_t(f_{t+T}) - f_{t+T} = -f_{t+T}\text{cov}\left(\ln f_{t+T}, \ln \frac{1}{1+y_{t+T}}\right)$$
Finally, in order to state the covariance terms in a more recognisable form, we use Stein’s lemma to evaluate the terms with a form

$$\text{cov}_t \left[ \ln f_{t+1,T}, \ln \left( 1 + \frac{1}{1 + Y} \right) \right]$$

for example. In the appendix (at the end of this chapter), we show that if X and Y are joint-lognormal variables then

$$\text{cov}[\ln X, \ln \left( 1 + Y \right)] = E[Y] \text{cov}(\ln X, \ln Y).$$

This follows from an extension of Stein’s lemma, which was used in chapter 1 to derive the CAPM. Using $X = f_{t+1,t+T}$ and $Y = f_{t+1,t+2}$, we have

$$\text{cov}_t \left[ \ln f_{t+1,T}, \ln \left( 1 + f_{t+1,t+2} \right) \right]$$

Finally, substituting similar expressions in the drift equation (7.18) and using the relation:

$$E_t \left[ \frac{f_{t+1,T}}{1 + f_{t+1,1}} \right] = \frac{f_{t+T}}{1 + f_{t,T}}$$

gives

$$E_t \left[ f_{t+1,T} \right] - f_{t,T}$$

$$= f_{t,T} \left[ f_{t+1,T} \right] \text{cov}_t \left[ \ln f_{t+1,T}, \ln f_{t+1,1} \right]$$

$$+ \cdots + f_{t,T} \left[ f_{t+1,T} \right] \text{cov}_t \left[ \ln f_{t+1,T}, \ln f_{t+1,1} \right]$$

In this model, the drift of the forward rate over the first period depends on the logarithmic covariances of the forward rates.

7.3 An Application of the Forward Rate Drift: The LIBOR Market Model

The LIBOR is a short-term interest rate quoted for a period less than or equal to one year. The most important rate is the three-month US Dollar LIBOR. Most interest-rate derivative contracts, FRAs for example, are contracts on LIBOR. Let $f_{t,T}$ denote the T-period forward LIBOR at time t. Following market convention, $f_{t,T}$ is quoted as a simple annual rate.

The relationship of the forward price of a zero-coupon bond to the quoted rate is given by

$$F_{t,T} = \frac{1}{1 + \delta},$$

where $\delta$ is length of the loan period, or re-set interval. If $T = 0$, $f_{t,t}$ is the spot LIBOR, where

$$F_{t,t} = \frac{1}{1 + \delta}. $$
In this section, we derive the drift of the forward rate: \( f_{t,t+T} \), when it is quoted on a LIBOR basis. We make the assumption that forward rates, in one period's time, are joint lognormally distributed, for all maturities \( T \). With this assumption, we can use the results (p.125) of the previous section, merely substituting the LIBOR for the annual yield rate. Also, since time is now measured in \( \delta \) intervals, the settlement payment for an FRA on LIBOR is given by

\[
\text{FRA}_{t,T} = \frac{A(f_{t,T,T} - k)\delta}{1 + df_{t,T,T}}.
\]

The effect of the LIBOR convention is to modify the drift of the forward rate. The following is a straightforward generalisation of equation (7.15). The drift of the forward rate is given in the two-period case by

\[
E_t^f(f_{t,1+\delta}) - f_{t+\delta} = -\frac{1}{2} \text{cov}_{t}^f(\delta f_{t,1+\delta}, \delta f_{t,1+\delta}) \times (1 + \delta f_{t,1+\delta})(1 + \delta f_{t,1+\delta})
\]

and in general for the \( T \)-maturity forward rate:

\[
E_t^f(f_{t,1+\delta^T}) - f_{t+\delta^T} = -\frac{1}{2} \text{cov}_{t}^f(\delta f_{t,1+\delta^T}, \delta f_{t,1+\delta^T}) \times (1 + \delta f_{t,1+\delta^T})(1 + \delta f_{t,1+\delta^T})
\]

Hence it follows that

\[
E_t^f(f_{t,1+\delta^T}) - f_{t+\delta^T} = \delta f_{t,1+\delta^T} \text{cov}_{t}^f(\ln f_{t,1+\delta^T}, \ln f_{t,1+\delta^T}) + \cdots + \delta f_{t,1+\delta^T} \text{cov}_{t}^f(\ln f_{t,1+\delta^T}, \ln f_{t,1+\delta^T})
\]

Finally, to obtain the drift of LIBOR in the Brace et al. (1996) model, we assume that the covariance structure is inter-temporally (p.126) stable. That is, we assume that \( \text{cov}_{t}^f(\ln f_{t+\tau,1+\delta^T}, \ln f_{t+\tau,1+\delta^T}) \) is a function of the forward maturities and is not dependent on \( t \). Then we can write

\[
\text{cov}_{t}^f(\ln f_{t+\tau,1+\delta^T}, \ln f_{t+\tau,1+\delta^T}) = \sigma_{t,T}
\]

where \( \sigma_{t,T} \) is the covariance of the log \( \tau \)-period forward LIBOR and the log \( T \)-period forward LIBOR. We can then write:

(7.19)

\[
E_t^f(f_{t+\tau,1+\delta^T}) - f_{t+\delta^T} = \delta f_{t+\tau,1+\delta^T} \sigma_{0,T-1} + \delta f_{t+\tau,1+\delta^T} \sigma_{1,T-1} + \cdots + \delta f_{t+\tau,1+\delta^T} \sigma_{T-1,T-1}
\]

For example, when \( t = 0 \) and \( T = 2 \)

\[
E_t^f(f_{2,1+\delta^2}) - f_{2+\delta^2} = \frac{\delta f_{2,1+\delta^2}}{1 + \delta f_{2,1+\delta^2}} \sigma_{0,1} + \frac{\delta f_{2,1+\delta^2}}{1 + \delta f_{2,1+\delta^2}} \sigma_{1,1}
\]
Equation (7.19) shows how to calculate the drift of the forward LIBORs under the period-by-period risk-neutral measure. It shows that the drift depends on a series of discounted covariances. One difficulty highlighted by the equation is that the drift is stochastic, since it depends on the future state-dependent forward LIBORs. This implies that in any implementation, it is difficult to produce a simple re-combining tree of rates. Also note that the spot LIBOR is not unconditionally lognormal, given the stochastic drift. For these reasons most implementations of the LIBOR market model use Monte Carlo simulation to compute interest-rate derivatives prices.

7.4 Conclusions
In this chapter we have introduced the topic of bond pricing in a rational expectations, pricing kernel model. We have used the no-arbitrage model to derive bond forward prices, the drift of bond forward prices, and the drift of interest rates. The LIBOR market model suggested by Brace et al. (1996) and Milterson et al. (1997), is an application of these basic ideas to the case where interest rates are defined on a LIBOR basis. We have derived the drift of LIBOR under the risk-neutral measure but have not investigated the pricing of interest-rate derivatives using the model. Readers (p.127) interested in pursuing this topic could look at Hull (2003) and Bjork (2004).

7.5 Appendix
In this appendix, we present two technical results which are required in the proof of the drift of forward rates. The first follows from Taylor's theorem. The second is an implication of Stein's lemma.

Lemma 7 (Covariances of Logarithms) Taylor's series expansion involves approximating the value of $g(x)$ around the value $x = a$

$$g(x) = g(a) + g'(a)(x - a) + \frac{1}{2}g''(a)(x - a)^2 + \cdots$$

Now define $g(x) = \ln X$, then from Taylor's theorem, we can write

$$\ln X = \ln a + \frac{1}{2}(X - a) + \cdots$$

and similarly

$$\ln Y = \ln b + \frac{1}{2}(Y - b) + \cdots$$

Hence

$$\text{cov}(\ln X, \ln Y) = \frac{1}{2}\text{cov}(X, Y)$$

$$\text{cov}(X, Y) = abc\text{cov}(\ln X, \ln Y)$$
with the first-order approximation.

**Lemma 8 (Stein's Lemma for lognormal variables)**

For joint-normal variables $x$ and $y$

$$\text{cov}(x, y) = \text{E}[	ext{g}(y) \text{cov}(x, y)].$$

Hence, if $x = \ln X$ and $y = \ln Y$, then

$$\text{cov}[\ln X, \ln \left(\frac{1}{1+Y}\right)] = \text{E}[-\frac{Y}{1+Y}] \text{cov}(\ln X, \ln Y).$$

\[(p.128)\]

**Proof** Let $g(\ln Y) = \ln \left(\frac{1}{1+Y}\right)$

$$g(\ln Y) = \frac{d}{dY} \ln \left(\frac{1}{1+Y}\right) = \frac{-1}{(1+Y)^2} \times Y$$

From Stein's lemma,

$$\text{cov}[\ln X, \ln \left(\frac{1}{1+Y}\right)] = \text{E}[-\frac{Y}{1+Y}] \text{cov}(\ln X, \ln Y).$$

\[\blacksquare\]

\[(p.129)\] **Exercises**

7.1. What is the drift of the futures price of a bond under the risk-neutral measure? How does this differ from the drift of the forward price of a bond under the risk-neutral measure?

7.2. Show that

$$F_{t+1|2} = F_{t+1|2} F_{t+2|3}.$$

and hence that

$$E_{t}^{\mathbb{Q}_{2|3}} \left[ \frac{1}{1+y_{t+1}} \frac{1}{1+f_{t+1|2}} \right] = \frac{1}{1+f_{t+1}} \frac{1}{1+f_{t+2|2}}.$$

7.3. In the case of the two-period forward, we show in (7.14) that the drift is

$$E_{t}^{\mathbb{Q}_{2|3}} \left[ f_{t+1|2} \right] = -\text{cov} \left( f_{t+1|2} \frac{1}{1+y_{t+1}}, \frac{1}{1+f_{t+1|2}} \right) \times \left( 1+f_{t+1} \right).$$

Show that a similar relationship holds for the three-period forward

$$E_{t}^{\mathbb{Q}_{3|4}} \left[ f_{t+1|2} \right] = -\text{cov} \left( f_{t+1|2} \frac{1}{1+y_{t+1}}, \frac{1}{1+f_{t+1|2}} \right) \times \left( 1+f_{t+1} \right) \left( 1+f_{t+2|2} \right).$$

7.4. The drift of the two-period forward rate depends upon the term
Write down all the steps showing that this equals 7.5.

In the LIBOR market model, the drift of the forward rate is the sum of a set of discounted covariances of the forward rates [see equation (7.19)]. Discuss the assumptions that have been made in deriving this result and their significance in the argument.

(p.131) Appendix: Stein's lemma

Since Stein's lemma (1973) is crucial to pricing relationships derived in several chapters, a summary of Stein's lemma is shown in this appendix.

First define

\[(A.1)\]

\[h(y) = \frac{y - \mu_y}{\sigma^2} f(y),\]

where \(f(y)\) is the normal density function. Note that (A.1) implies

\[(A.2)\]

\[h(y) = -f(y),\]

since if

\[h(y) = -\frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}(\frac{y-\mu_y}{\sigma})^2}\]

\[h(y) = \frac{y - \mu_y}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}(\frac{y-\mu_y}{\sigma})^2}\]

\[= \frac{y - \mu_y}{\sigma^2} f(y).\]

Lemma 9 (Integration by parts)

\[
\int fg = \int gf
\]

Proof

\[(A.3)\]

\[h \cdot g = \int (h \cdot g) = \int [h \cdot g + gh] = \int h \cdot g + \int g
\]

\[= \int h \cdot g - \int g \cdot h.
\]

Now since

\[\int_a^b h \cdot g = h(f(b)) - h(f(a)),\]

(p.132) from (A.3)

\[\int h \cdot g = h(\infty) - h(-\infty) - \int g \cdot h
\]

\[= h(\infty)g(\infty) - h(-\infty)g(-\infty) - \int g \cdot h.\]
Given that \( f(\infty) = f(-\infty) = 0 \) then from (A.2) \( h(\infty) = h(-\infty) = 0 \).

Then, if \( g(y) \) is bounded, we get, using (A.2),
\[
\int hg = -\int gh = \int gf.
\]

Stein's lemma specifies the covariance of \( x \) and a function of \( y \) when \( x \) and \( y \) are a pair of bivariate normal variables as follows.

**Lemma 10 (Stein's lemma)**

\[
\text{cov}[x, g(y)] = E[g(y)]\text{cov}(x, y)
\]

**Proof**

Consider

(A.4)
\[
E[xg(y)] = E(x)E[g(y)] + \text{cov}[x, g(y)]
\]

Rewrite the LHS of (A.4) in terms of conditional expectations:

(A.5)
\[
E[xg(y)] = E[g(y)E(x|y)]
\]

Since \( x \) and \( y \) are joint-normal

(A.6)
\[
x = a + by + \epsilon
\]
\[
E(x) = a + bE(y)
\]
\[
E(x|y) = a + by
\]
\[
E(x) - bE(y) + by.
\]

Hence (A.5) becomes
\[
E[xg(y)] = E[g(y)[\mu_x + b(y - \mu_y)]]
\]
\[
= E[g(y)\mu_x] + bE[g(y)(y - \mu_y)]
\]

(p.133)

Now substitute into LHS of (A.4):

(A.7)
\[
E[g(y)\mu_x] + bE[g(y)(y - \mu_y)] = \mu_xE[g(y)] + \text{cov}(x, g(y))
\]
\[
\text{cov}[x, g(y)] = bE[g(y)(y - \mu_y)]
\]

Since from bivariate normal regression in (A.6):

\[
b = \frac{\text{cov}(x, y)}{\sigma_y^2},
\]

so

(A.8)
\[
\text{cov}[x, g(y)] = \text{cov}(x, y)E\left[\frac{g(y)(y - \mu_y)}{\sigma_y^2}\right]
\]
\[
= \text{cov}(x, y)\int g(y)(y - \mu_y)\sigma_y^2dy
\]
\[
= \text{cov}(x, y)\int g(y)h(y)dy.
\]

Using lemma (9)
\[
\text{cov}[x, g(y)] = \text{cov}(x, y)\int g(y)h(y)dy
\]
\[
= \text{cov}(x, y)E[g(y)]
\]
(p.134)

Notes:

(49) It is important to note that this is only true for the one-period-ahead prices.

(50) This follows from the pricing of a $T$-maturity FRA.

(51) Note that this expression uses the fact that

$$\text{cov}(\ln \delta f_{i,t+1}, \ln \delta f_{i,t+1}) = \text{cov}(\ln f_{i,t+1}, \ln f_{i,t+1})$$

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