MULTI-PERIOD ASSET PRICING

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Abstract and Keywords

‘Multi-period Asset Pricing’ expands the analysis of asset prices to a multi-period economy, where an investor has to make consumption decisions in each period which may lead to consumption and wealth being different in the interim periods. The authors consider two distinct approaches to multi-period valuation; the time-state preference approach, where consumptions at different times and in different states were treated as separate assets, and the rational expectations approach that derives a period-by-period equilibrium in which investors form expectations of the price of securities. Here, they value assets relative to the value of bonds. While risk-free interest rate is given exogenously, the prices of these bonds at future points in time can be stochastic.

Keywords: consumption decisions, exogenous interest rate, multi-period economy, period-by-period equilibrium, rational expectations approach, stochastic bond price, time-state preference approach

In this chapter, we extend the analysis of asset prices to a multi-period economy. There are two principal differences from the single-period models presented in Chapter 1. First,
the asset to be valued produces cash flows at a series of dates indexed by \( t + 1, t + 2, \ldots, t + n \). Second, the representative investor has utility for consumption at a series of dates, again indexed by \( t + 1, t + 2, \ldots, t + n \). In a single-period economy, all wealth arising at the end of the period has to be consumed. However, in a multi-period economy, an investor has to make consumption decisions in each period. There may be a difference in this case between consumption and wealth.

We consider two distinct approaches to multi-period valuation. The first approach, termed the *time-state preference approach*, treats consumption at different times and in different states as separate assets and derives an equilibrium in a complete market for time-state claims. The second approach derives a period-by-period equilibrium where investors form expectations of the price of securities. This is known as the *rational expectations approach*.

Throughout this chapter we will assume that the risk-free rate of interest is given exogenously. We are therefore valuing assets relative to the value of bonds. We assume that investors can trade in zero-coupon bonds which pay $1 at times \( t + 1, t + 2, \ldots, t + n \). However, in the multi-period model, the prices of these bonds at future points in time can be stochastic.

### 5.1 Basic Setup

1. We assume again that markets are complete and that there is a finite state space. In the multi-period model, this implies that investors can purchase claims that pay $1 if and only if a given state occurs at a given point in time.
2. As an illustrative example, we assume there are just two periods; period \( t \) to \( t + 1 \) and period \( t + 1 \) to \( t + 2 \). In this case there are three points in time, labeled \( t, t + 1, \) and \( t + 2 \).
3. We wish to value a stock \( j \), which pays dividends \( x_{j,t+1} \) at time \( t + 1 \) and \( x_{j,t+2} \) at time \( t + 2 \).
4. There are \( i = 1, 2, \ldots, I \) states at time \( t + 1 \) and there are \( k = 1, 2, \ldots, K \) states at time \( t + 2 \). The state \( k \) that can occur at \( t + 2 \) depends on the state \( i \) which occurs at \( t + 1 \). For example, we may have a situation where \( k \) takes the value 1 and 2 when \( i = 1 \), whereas \( i = 2 \) leads to \( k = 3 \), etc. The general scheme is illustrated below:
5. Let \( \{p_i\} \) and \( \{p_k\} \) be the sets of unconditional probabilities of state \( i \) and state \( k \).

5.2 A Complete Market: The Multi-period Case

In the multi-period case, we assume that investors can buy claims at time \( t \) on states at time \( t + 1 \) and \( t + 2 \). In addition, investors can buy claims at time \( t + 1 \) on states at time \( t + 2 \). As in Chapter 1, these claims pay out $1, if and only if the state occurs. This assumption is illustrated in Fig. 5.1. In Fig. 5.1(a), we show the forward prices, \( q_i \), of a claim paying $1 in state \( i \) at time \( t + 1 \), and the conditional forward price, \( q_{i,k} \), at time \( t + 1 \) in state \( i \) of a claim paying $1 in state \( k \) at time \( t + 2 \). Note that the conditional probabilities for states \( i \) and \( k \) to occur are \( p_i \) and \( p_{i,k} \) respectively. In addition, we assume that we know, at time \( t + 1 \), the price of a zero-coupon bond paying $1 in every possible state at time \( t + 2 \). These prices are denoted \( B_{t+1,t+2,i} \). These zero-coupon bond prices are stochastic at time \( t \) and are denoted as \( B_{t+1,t+2} \).

In Fig. 5.1(b), we show the forward price, at time \( t \) of purchasing a claim paying $1 in state \( k \) at time \( t + 2 \). Note that (p.79)
this forward price, payable at time $t + 2$, is denoted $q_k$.
Also, $p_k$ is the unconditional probability of state $k$.
At this point, it is useful to introduce a notation for the spot price of a state-contingent claim. Let $q^*$ be the spot price of a state-contingent claim. We then have

\[ q_i^* = q^*B_{t-5}, \]
\[ q_k^* = q^*B_{t-2}, \]
\[ q_i^* = q^*B_{t-2}. \]

These definitions are illustrated in Fig. 5.2.

**Fig. 5.1.** Period-by-period valuation

Now consider the spot price of a state-contingent claim paying $1$ if and only if state $k$ occurs at time $t + 2$. It is immediately apparent that

\[ q_k^* = q^*i, k. \]

This price relationship holds because it is possible in this complete market to buy a claim on state $k$ in two ways. First, the investor can directly buy a claim costing $q_k^*$ at time $t$. Second, the investor could buy $q_i^*i, k$ claims on state $i$ and use them to purchase a claim on $1$ in state $k$. The cost of this strategy is
The equality then follows. Hence, equivalently we can write the following relationship between the forward prices of the claims:

\[(5.1)\]

\[q_k B_{t+2} = q_k B_{t+1} q_k B_{t+2}\]

The importance of this relationship is that it implies that the valuation of a cash flow \(x_{t+2}\) using the unconditional state prices, \(q_k\), must be the same as that which results from a period-by-period valuation using the \(q_i\) and the \(q_{i,k}\). We illustrate this conclusion in the following section.

5.3 Pricing Multi-period Cash Flows

In this section, we value cash flows arising at time \(t + T\); \(x_{j,t+T}\). We price the cash flows, first of all using the unconditional state prices. This approach has traditionally been known as the time-state preference approach, as the unconditional state prices reflect investors preferences for consumption in different states at different times. We then show the alternative pricing from the equivalent period-by-period valuation approach. This approach is generally referred to as the rational expectations approach.

5.3.1 The Time-State Preference Approach

We proceed along the lines followed in Chapter 1. We start with a case where \(T = 1,2\). First, as illustrated above, \(q_i\) is the forward price of \$1 paid if and only if state \(i\) occurs at \(t + 1\). Also \(q_k\) is the forward price of \$1 paid if and only if state \(k\) occurs at \(t + 2\). Note that the \(q_i\) are forward prices for delivery at time \(t + 1\), while the \(q_k\) are forward prices for delivery at time \(t + 2\).

\[(p.81)\] As in Chapter 1, we have \(\Sigma q_i = 1\) and \(\Sigma q_k = 1\) and also the conditions:

\[q_i > 0 \Leftrightarrow p_i > 0 \quad \text{and} \quad q_k > 0 \Leftrightarrow p_k > 0\]

hold. It follows that \(\{q_i\}, \{q_k\}\) are each equivalent probability measures to \(\{p_i\}, \{p_k\}\). From the complete markets assumption and the Law of One Price, the forward prices of the uncertain cash flows \(x_{j,t+1}\), for delivery at \(t + 1\), and of \(x_{j,t+2}\), for delivery at \(t + 2\), are

\[F_{j,t+1} = \sum_{i} x_{j,t+1} q_i\]
\[F_{j,t+2} = \sum_{k} x_{j,t+2} q_k\]
The value at time $t$ of an asset that pays $x_{j,t+1}$ at $t+1$ and $x_{j,t+2}$ at $t+2$, is then given by, using spot-forward parity,

\begin{align}
S_t &= F_{t+1} B_{t+1} + F_{t+2} B_{t+2} \\
&= \sum_j x_{j,t+1} q_{j,t+1} + \sum_k x_{j,t+2} q_{j,t+2}
\end{align}

where $B_{t+1}$ is the price of a $T$-maturity zero-coupon bond at time $t$.

As in Chapter 1, the ‘forward’ pricing kernels in the two-period case may be defined as the probability deflated state forward prices:

$$\phi_{t,t+1} = \frac{q_j}{p_j}, \text{ and } \phi_{t,t+2} = \frac{q_k}{p_k}$$

and in this setup, the forward prices are:

$$F_{t+1} = E\left(\phi_{t,t+1} x_{t+1}\right) \text{ and } F_{t+2} = E\left(\phi_{t,t+2} x_{t+2}\right)$$

and the value of the cash flows is

$$S_t = B_{t+1} E\left(\phi_{t,t+1} x_{t+1}\right) + B_{t+2} E\left(\phi_{t,t+2} x_{t+2}\right)$$

The valuation equation (5.3) has an obvious generalisation to the case where there are $n$ periods' cash flows, as follows. Given the zero-coupon bond prices, $B_{t+T}$, $T = 1, 2, \ldots, n$, we have the value of an asset with cash flows $x_{j,t+1}, x_{j,t+2}, \ldots, x_{j,t+n}$:

\begin{align}
S_t &= \sum_{t=1}^{n} F_{t+T} B_{t+T} \\
&= \sum_{t=1}^{n} E\left(\phi_{t,t+T} x_{t+T}\right)
\end{align}

where

$$F_{t+T} = E\left(\phi_{t,t+T} x_{t+T}\right)$$

or

\begin{align}
S_t &= B_{t+1} E\left(\phi_{t+1,t+2} x_{t+2}\right) + B_{t+2} E\left(\phi_{t+2,t+3} x_{t+3}\right) + \ldots \\
&+ B_{t+n} E\left(\phi_{t+n,t+n} x_{t+n}\right)
\end{align}

However, as in the single-period case, until we specify further the properties of $\phi_{t,t+T}$, the asset valuation equation is little more than a necessary condition for prices to satisfy in a complete market model.

5.3.2 The Rational Expectations Model

We now derive prices using a period-by-period approach, where agent's form expectations of $t+t$ prices, by 'solving the model’ for time $t+t$ prices. This is known as the rational expectations approach. As an example of the general case, we first price the cash flow of firm $j$ which arises at time $t+2$, denoted $x_{j,t+2}$.
We start by pricing the cash flow, \( x_{j,t+2} \), at time \( t + 1 \), in state \( i \). From the complete markets assumption, this is given by

\[
S_{j,t+1} = B_{t+1} \sum_k p_{i,k} x_{j,t+2k}
\]

where the \( q_{i,k} \) are the conditional state forward prices. We now define the period-by-period, or conditional, pricing kernel by the relationship

\[
\varphi_{t+2k} = \frac{q_{i,k}}{p_{i,k}}.
\]

\textbf{(p.83)} As in the single-period case these are the probability deflated state forward prices. Using this definition we can rewrite equation (5.6):

\[
S_{j,t+1} = B_{t+1} \sum_k p_{i,k} \varphi_{t+2k} x_{j,t+2k} = B_{t+1} E_{t+1} \left\{ \varphi_{t+2k} x_{j,t+2k} \right\}
\]

where the notation \( E_{t+1} \) means the expected value at time \( t + 1 \), in state \( i \).

We now value the cash flow \( x_{j,t+2} \) at time \( t \), assuming that investors expectations of the price of the cash flow are given by equation (5.7). We have, again using the complete markets assumption:

\[
S_{j,t} = \sum_k B_{j,t} q_{j,t} S_{j,t+1} = B_{j,t} \sum_k p_{i,k} \varphi_{t+2k} \left\{ \varphi_{t+2k} x_{j,t+2k} \right\} = B_{j,t} E_{t} \left\{ \varphi_{t+2k} \varphi_{t+2k} x_{j,t+2k} \right\}
\]

Equation (5.8) says that the value of a cash flow is the discounted expectation at time \( t \) of a further discounted expectation taken at time \( t + 1 \). Here again we emphasise the fact that the zero-coupon bond price at time \( t + 1 \) is a stochastic variable.

The analysis now extends in a fairly straightforward manner to the valuation of a cash flow arising at time \( t + T \) and then to multiple cash flows. The spot value of \( x_{j,t+T} \) is given by:

\[
S_{j,t} = B_{j,t} E_{t} \left\{ \varphi_{t+2k} \varphi_{t+2k} x_{j,t+2k} \right\} \cdots \hat{B}_{j,t+T} E_{t+T} \left\{ \varphi_{t+2k} \varphi_{t+2k} x_{j,t+2k} \right\}
\]

where \( \varphi_{t+1} \) is the period-by-period pricing kernel.

Now we find the spot value of a firm \( j \) that produces a stream of cash flows

\[
\{ x_{j,t+1}, x_{j,t+2}, \ldots, x_{j,T} \}.
\]
We denote the spot value of all the cash flows subsequent to time \( t + \tau \) as \( S_{j,t+\tau} \). Again for simplicity, we first take the case where \( n = 2 \). First, from equation (5.7), the value of \( x_{j,t+2} \) at time \( t + 1 \) is given by

\[
S_{j,t+1} = B_{t+1,t+2}\phi_{t,n+1}^1 x_{j,t+2}^1.
\]

Now, since investors receive the cash flow, \( x_{j,t+2} \), as well as the value of \( x_{j,t+2} \), their total payoff at \( t + 1 \) in state \( i \) is \( S_{j,t+1,i} + x_{j,t+1,i} \). Therefore, valuing this payoff at time \( t \) using the pricing kernel \( \phi_{t,t+1} \) yields

\[
S_{j,t} = B_{t+1,t}E_i\left[\phi_{t+1,1}^1 (x_{j,t+1} + S_{j,t+1})\right]
\]

Substituting the value of \( S_{j,t+1} \) from (5.7), we then have

\[
S_{j,t} = B_{t+1,t}E_i\left[\phi_{t+1,1}^1 (x_{j,t+1}) + B_{t+1,t}E_i\left[\phi_{t+1,2}^1 (x_{j,t+2})\right]\right]
\]

In general, the value of \( T = n \) cash flows is given by

\[
(5.9)
\]

5.3.3 The Relationship between the Time-State Preference and the Rational Expectations Equilibria Prices

The prices of the cash flows in equation (5.5) from the time-state preference approach and in equation (5.9) from the rational expectations approach are actually the same. The two approaches are just different ways of valuing the cash flows. In equation (5.5), the cash flows are valued using forward prices and long-bond prices. In equation (5.9) the cash flows are valued using the conditional pricing kernels and the stochastic one-period zero-coupon bond prices.

In order to illustrate this, we return to the relationship between the state forward prices in equation (5.1). We have:

\[
B_{t+2,t}^1 = B_{t+1,t}^1 q_{t+1,t+2}^1 q_{j,k}^1
\]

This relationship can be re-written in terms of the pricing kernels as follows:

\[
B_{t+2,t}^1 \phi_{t+2}^1 = B_{t+1,t}^1 \phi_{t+1}^1 B_{t+1,t+2}^1 \phi_{t+2}^1 q_{t+1,t+2}^1 q_{j,k}^1
\]

It then follows that for the single cash flow, \( x_{j,t+2} \),
and the left-hand side of the equation can be written as

\[ \sum B_{t+2,t} p_i x_{i,t+2} = B_{t+2,t} E_{t+1} \left( \varphi_{t+1,t+2} x_{t+2} \right) \]

Hence, the values of the cash flow \( x_{j,t+2} \) are the same under the two approaches. We will see in the following chapter that the time-state preference approach to valuation is particularly useful when computing forward prices. In contrast, the rational expectations approach is more convenient when computing futures prices.

### 5.3.4 The Relationship between the Pricing Kernels when Interest Rates are Non-stochastic

We now impose the further restriction that future interest rates are known at time \( t \). The relationship between the pricing kernels in the time-state preference and in the rational expectations approach is particularly simple if interest rates, and hence zero-coupon bond prices, are non-stochastic. As an example, consider the two-period case. As above, the cost of purchasing a claim on a state \( k \) is \( B_{t,t+2} q_k \). Alternatively, the claim could be secured by first purchasing a claim on a state \( i \) at time 1 and then using this to buy a claim on state \( k \). The cost of this strategy is \( B_{t+1,t+2} q_i B_{t+1,t} q_{i,k} \). We must then have:

\[ \text{since both strategies result in the same payoff, i.e., } \$1 \text{ in state } k \text{ at time 2. Now, dividing both sides by the probability of state } k \text{ occurring} \]

\[ B_{t,t+2} q_k = B_{t+1,t} q_i B_{t+1,t+2} q_{i,k} \]

\[ \text{(p.86)} \] and hence

\[ B_{t,t+2} \varphi_{t+1,t+2} = B_{t+1,t} \varphi_{t+1,t+2} \varphi_{t+1,t+2} \]

But since in this case, \( B_{t,t+2} = B_{t+1,t+1} B_{t+1,t+2} \), we have

\[ (5.10) \]

\[ \varphi_{t+1,t+2} = \varphi_{t+1,t+2} \varphi_{t+1,t+2} \]

The forward price of obtaining \$1 in state \( k \) is the same as the price paid if two successive forward contracts are made. Equation (5.10) will be important when considering values in the two approaches. Generalising (5.10), we have

\[ \varphi_{t,t+T} = \varphi_{t+1,t+T} \varphi_{t+1,t+T} \]
This equation gives some intuition for the pricing under the two approaches. If interest rates are non-stochastic, the pricing kernel over any long time period, from \( t \) to \( T \), is the product of the pricing kernels over the successive short periods from \( t \) to \( t + 1 \), \( t + 1 \) to \( t + 2 \) and so on. However, when interest rates are stochastic, this simple product relationship will not hold.

5.4 Multi-Period Valuation Equilibrium: Joint-Normal Cash Flows

In this section, we follow the methodology used in Chapter 1 and evaluate the covariance terms in the valuation equation of the rational expectations approach. We pursue the rational expectations rather than the alternative time-state preference approach, because it is possible under this approach to price a series of cash flows, using an extension to the multi-period economy of the single-period CAPM.

To illustrate the valuation, we again consider a two-period example, where there are three dates, \( t \), \( t + 1 \), and \( t + 2 \). There is only one cash flow from firm \( j \) that arises at time \( t + 2 \), and is denoted as \( x_{j,t+2} \). When we refer to the aggregate cash flow at time \( t + 2 \) of all the firms we use \( x_{m,t+2} \). We assume that the aggregate cash flow of all the firms at time \( t + 1 \) is \( x_{m,t+1} \). We let \( S_{j,t+1} \) and \( S_{m,t+1} \) be the values at \( t + 1 \) of the firm \( j \) and of the market, respectively. Note that the value \( S_{m,t+1} \) includes the cash flow \( x_{m,t+1} \) as well as the time \( t + 1 \) value of \( x_{m,t+2} \).

Under the rational expectations approach, the value of the cash flow at time \( t \) is

\[
S_{jt} = B_{jt+1} E_t \left[ \phi_{jt+1} B_{jt+2} E_{t+1} \left[ \phi_{jt+2} x_{jt+2} \right] \right]
\]

Alternatively, this pricing equation can be written as

\[
S_{jt} = B_{jt+1} E_t \left[ \phi_{jt+1} S_{jt+1} \right]
\]

where

\[
S_{jt+1} = B_{jt+1} E_{t+1} \left[ \phi_{jt+1} x_{jt+2} \right]
\]

Note that all the variables in this latter equation are conditional on state \( i \) at time \( t + 1 \). Now, using the definition of covariance and the property that \( E_{t+1}(\phi_{t+1,t+2}) = 1 \), for all \( i \), we have

\[
S_{jt+1} = B_{jt+1} E_{t+1} \left[ \phi_{jt+1} x_{jt+2} \right] + \text{cov}_{ij} x_{jt+2}
\]

Following the same logic as in Chapter 1, we now write the period 2 conditional pricing kernel:
Now, again following the argument in Chapter 1, if the cash flow $x_{j,t+2}$ and the aggregate market cash flow are joint-normal, then Stein’s lemma (see appendix at the end of the book) allows us to write

$$\varphi_{t+1,t+2} = \varphi_{t+1,t+2}(x_{m,t+2})$$

and defining the market price of risk, $\lambda_{t+1,i} = -E_{t+1,i}(\varphi'_t + 1, t+2, i)$ we have the CAPM relationship:

$$(5.11)$$

$$S_{j,t+1} = B_{t+1,t+2}[E_{t+1}(x_{j,t+2}) - \lambda_{t+1,i} \text{cov}_{t+1}(x_{m,t+2}, x_{j,t+2})]$$

It is not surprising that the single-period CAPM holds, in equation (5.11) over the second period, since we have in effect assumed a one-period world, with normally distributed cash flows from time $t + 1$ to time $t + 2$. Stein’s lemma and the assumption that the pricing kernel is a function of $x_{m,t+2}$ is then sufficient to establish the CAPM.

We now consider the pricing of $x_{j,t+2}$ over the first period. First, it is important to note that the value of the cash flow in state $i$, $S_{j,t+1,i}$, depends on four variables, each of which could be state dependent. However, given joint-normality of $x_{j,t+2}$ and $x_{m,t+2}$, (p.88) the covariance term $\text{cov}_{t+1,i}(x_{m,t+2}, x_{j,t+2})$ is non-stochastic. If we assume that $\lambda_{t+1,i}$ the market price of risk, is also non-stochastic, then the risk adjustment term in (5.11) is non-stochastic and since normality of $x_{j,t+2}$ guarantees that $E_{t+1}(x_{t+2})$ is also normal, then the forward price of the cash flow at time $t + 1$ is also normal. However, in order to price the cash flow at time $t$, using the methods of chapter 1, we require the value $S_{j,t+1}$ to be normally distributed. This will be the case if, in addition, the bond price $B_{t+1,t+2}$ is non-stochastic. In the following derivation we make this additional assumption.

Under the rational expectations approach, the value of the cash flow $x_{j,t+2}$ at time $t$, is given by

$$S_{j,t} = B_{t+1,t}[E_t(S_{j,t+1}) + \text{cov}_t(\varphi_{t+1,t+1}, S_{j,t+1})]$$

Given the additional assumptions made above, $S_{j,t+1}$ is normally distributed. Now consider the purchase of claims at $t$ on the value of the cash flow at $t + 1$, $S_{j,t+1}$. We now assume that the pricing kernel $\varphi_{t+1,t+1} = \varphi_{t+1}(S_{m,t+1})$, where $S_{m,t+1}$ is the value of the market portfolio at time $t + 1$, including cash flows arising at time $t + 1$. The assumptions above are
sufficient for the prices $S_{j,t+1}$ and $S_{m,t+1}$ to be joint-normally distributed. It then follows, again from Stein's lemma that

$$\text{cov}(\varphi_{t+1}, S_{j,t}) = -\lambda_t \text{cov}(S_{m,t+2}, S_{j,t+1}).$$

Hence, a CAPM relationship holds over the first period and the value of the cash flow is given by:

$$S_{j,t} = B_{j,t+1}[E(S_{j,t+1}) - \lambda_t \text{cov}(S_{m,t+2}, S_{j,t+1})].$$

where

$$S_{j,t+1} = B_{j,t+2}[E_r(S_{j,t+2}) - \lambda_{t+1} \text{cov}_{t+1}(X_{m,t+2}X_{j,t+2})].$$

It is interesting to note that the conditions required for the CAPM to hold on a period-by-period basis are much stronger than in the case of the previous time-state preference equilibrium. The assumption that $\lambda_{t+1}$ is non-stochastic is a very strong assumption. In Sections 5.5 and 5.6 we derive an equilibrium where this condition holds, but the result requires a utility function with CARA, (p.89) as well as normality of the cash flows. Also, the assumption that interest rates are non-stochastic obviously flies in the face of reality. Various extensions to the CAPM have been suggested to cope with the more general case where the market price of risk or interest rates are stochastic.\(^{32}\)

Finally, in this section, for completeness we state the general case, where the asset to be priced produces a cash flow, $x_{j,t+n}$, at time $t+n$. Again we assume that bond prices and market prices of risk are non-stochastic. We then have

\[(5.12)\]

$$S_{j,t} = B_{j,t+1}[E(S_{j,t+1}) - \lambda_t \text{cov}(S_{j,t+2}, S_{m,t+1})].$$

where

$$S_{j,t+1} = B_{j,t+2}[E_r(S_{j,t+2}) - \lambda_{t+1} \text{cov}_{t+1}(S_{j,t+2}, S_{m,t+2})],$$

$$S_{j,t+2} = B_{j,t+3}[E_r(S_{j,t+3}) - \lambda_{t+2} \text{cov}_{t+2}(S_{j,t+3}, S_{m,t+3})],$$

... 

$$S_{j,t+n} = B_{j,t+n}[E_r(S_{j,t+n}) - \lambda_{t+n} \text{cov}_{t+n}(S_{j,t+n}, S_{m,t+n})].$$

5.5 Time-State Preference: Pricing Kernels in a Multi-period Equilibrium

In this section, we derive an equilibrium in which the representative investor's marginal utility function provides an example of the pricing kernels required for the complete markets multi-period model. The model assumes that the investor has a time-separable utility function for consumption in each period.\(^{33}\) We present a two-period example in which
we derive the pricing kernels $\phi_{t,t+1}$ and $\phi_{t,t+2}$ in a time-state preference equilibrium.

Let $c_{t+T}$ be the consumption of the representative agent at time $t + T$. Now we assume time additive utility, where $u(c_{t+1}, c_{t+2}) = u_1(c_{t+1}) + u_2(c_{t+2})$, such that $u_2$ is not dependent on $c_{t+1}$.

(p.90) The agent’s problem is to maximise:

$$\max_{\{w_t : \sum_{i} q_{it} = 1\}} \mathbb{E}[u(c_{t+1}, c_{t+2})] = \mathbb{E}[u(c_{t+1})] + \mathbb{E}[u(c_{t+2})],$$

where $w_t$ is the agent’s wealth at time $t$. We solve the optimisation problem using the Lagrange multiplier method. Form the Lagrangian:

$$L = \mathbb{E}[u(c_{t+1})] + \mathbb{E}[u(c_{t+2})] + \lambda \left( w_t - \sum_{i} q_{it} B_{i,t+1} - \sum_{k} p_k u(c_{t+2}) \right) = \sum_{i} \mu_i(c_{t+1}) + \sum_{k} p_k u(c_{t+2}) + \lambda \left( w_t - \sum_{i} q_{it} B_{i,t+1} - \sum_{k} p_k u(c_{t+2}) \right),$$

where $\lambda$ is a continuous variable for each period.

The first-order conditions are:

$$\frac{\partial L}{\partial \mu_i} = p_i u(c_{t+1}) - \lambda B_{i,t+1} = 0, \quad i = 1, 2, \ldots, I,$n
$$\frac{\partial L}{\partial \mu_k} = p_k u(c_{t+2}) - \lambda B_{i,t+2} = 0, \quad k = 1, 2, \ldots, K.$$

Taking the expectation and adding up over the states, we have

$$\mathbb{E}[u(c_{t+1})] = \lambda \sum_{i} q_{it} B_{i,t+1} = \lambda B_{t+1},$$

since $\sum q_{it} = 1$. Similarly,

$$\mathbb{E}[u(c_{t+2})] = \lambda B_{t+2},$$

and

$$\lambda = \frac{\mathbb{E}[u(c_{t+1})]}{\mathbb{E}[u(c_{t+2})]} = \frac{\mathbb{E}[u(c_{t+1})]}{\mathbb{E}[u(c_{t+2})]}.$$

(p.91) Now, substitute the value for $\lambda$ into (5.13) and (5.14), and we get

$$q_i = p_i \frac{u(c_{t+1})}{\mathbb{E}[u(c_{t+1})]} = p_i \phi_{t+1},$$

and

$$q_k = p_k \frac{u(c_{t+2})}{\mathbb{E}[u(c_{t+2})]} = p_k \phi_{t+2}.$$
\( \phi_{t,t+1} = \phi_{t,t+1} \) given that \( E_t(\phi_{t,t+1}) = E_t(\phi_{t,t+2}) = 1 \). Now, if we substitute the marginal utilities in \( \phi_{t,t+1} \), then assume that \((x_{j,t+1}, c_{t+1})\) are joint-normal applying Stein’s lemma (see appendix at the end of the book) we find

\[
(5.17) \quad \text{cov}(x_{j,t+1}, \phi_{t,t+1}) = \text{cov}(x_{j,t+2}, c_{t+1})
\]

where \( \lambda_{t,t+1} \) is some constant. Similarly, if \((x_{j,t+2}, c_{t+2})\) are joint-normal, again applying Stein’s lemma yields

\[
(5.18) \quad \text{cov}(x_{j,t+2}, \phi_{t,t+2}) = -\lambda_{t,t+2}\text{cov}(x_{j,t+2}, c_{t+2})
\]

Now, if we substitute (5.17) and (5.18) into the forward price equations, (5.15) and (5.16), and then substitute into the spot price equation (5.2). We have

\[
(5.19) \quad S_{t} = B_{2,t-1}[E(x_{j,t+1}) + \lambda_{t,t+1}\text{cov}(x_{j,t+1}, c_{t+1})] + B_{1,t-1}[E(x_{j,t+2}) + \lambda_{t,t+2}\text{cov}(x_{j,t+2}, c_{t+2})]
\]

where \( c_{t+1} \) and \( c_{t+2} \) are the consumption levels of the representative agent. Equation (5.19) is a form of CAPM, often referred to as the ‘consumption CAPM’. Note that there is no interaction between the first and the second periods in this model. As one can see from the derivation, it is very much like working out the single-period model twice with two separate time periods. Also, the analysis had not as yet solved for the optimal consumption level. Sometimes, the model is closed, somewhat artificially, by assuming that aggregate consumption has to be equal to some exogenous cash flow supply, \( x_{m,t+T} \). We now move on to solve a model where consumption levels are derived within the model.

5.6 Marginal Utility of Consumption and Wealth in a Normal Distribution and Exponential Utility Model

In the previous section we priced cash flows within the framework of a multi-period CAPM. However, the price of risk depends there on the marginal utility of consumption in each period. In an important sense, therefore, the model is incomplete, since consumption levels still need to be determined in the multi-period world. In this section we solve for consumption and show that the market price of risk can be re-expressed in terms of the marginal utility of wealth.
Given exponential utility and joint-normality of the cash flows, it turns out that the market price of risk is non-stochastic, as it was assumed in Section 5.4. We also assume non-stochastic interest rates, which once again, as in Section 5.4, leads to the one-period CAPM holding on a period-by-period basis.

We assume a particular form for the utility functions \( u_1 \) and \( u_2 \), and solve explicitly for consumption, \( c_{t+1} \), as a function of wealth, \( w_{t+1} \). We assume an additive exponential utility function for the representative investor, as shown below:

\[
\ln c_{t+1} = -e^{-\alpha t} - \rho e^{-\alpha w_{t+1}};
\]

(p.93) Here the constant \( \rho \) can be interpreted as a discount or impatience factor for delayed consumption. We assume also that the investor can reinvest at a gross risk-free rate \( R \) at time \( t + 1 \), where \( R = 1/B_{t+1} \). We assume also that \( R \) is non-stochastic. Then the optimisation problem is one that solves the consume versus save decision at time \( t + 1 \) such that utility at that point in time is maximized, i.e.,

\[
\ln w_{t+1} = c_{t+1} \ln [1 - e^{-\alpha w_{t+1}} - \rho e^{-\alpha w_{t+1}}];
\]

In equation (5.20), the utility of wealth, \( u(w_{t+1}) \), is the (maximum) derived utility of consumption, given the wealth level.

In this optimisation, the investor chooses to consume \( c_{t+1} \) and invest \( (w_{t+1} - c_{t+1}) \). The return on investment is risk free and the investor gets to consume \( (w_{t+1} - c_{t+1}) R \) at time \( t + 2 \). The analysis extends fairly easily to the case where reinvestment is in a single, risky asset.\(^{34}\)

Differentiating (5.20), yields the first-order condition:

\[
\frac{\partial u(w_{t+1})}{\partial c_{t+1}} = -e^{-\alpha t} - \rho e^{-\alpha w_{t+1}} = 0.
\]

Cancelling out \( \alpha \) and taking the logarithm of both sides gives

\[
\ln c_{t+1} = -\ln(1 - e^{-\alpha w_{t+1}} - \rho e^{-\alpha w_{t+1}}) = \ln R + \ln (1 + \frac{1}{2})
\]

where \( k \) is a constant. Note that \( R \) is 1 plus the risk-free rate, so the amount \( R/1+R \) is quite close to \( 1/2 \). So roughly, the investor consumes half of the wealth in year 1. Previously, from the first-order conditions in the time-state preference equilibrium, we found that
\[ \varphi_{t+1} = \frac{u(c_{t+1})}{E[u(c_{t+1})]} \]

**Exercise:** Assume a state space as follows:

\[ \text{cov}[\varphi_{t+1} (S_{j,t+1} + x_{j,t+1})] = \frac{\mu_u(c_{t+1})}{\mu_u(c_{t-1})} \{ \lambda \} \text{cov}[w_{t+1} (S_{j,t+1} + x_{j,t+1})] = -\lambda \text{cov}[w_{t+1} (S_{j,t+1} + x_{j,t-1})], \]

where \( \lambda \) is a constant. Note that this result is consistent with the general model in Section 5.4. The difference here is that in this model we have a more detailed representation of the market price of risk.

5.7 Conclusions

Multi-period models of asset pricing are inevitably complex. The multiple time periods imply that covariances across cash flows arising at different times are relevant. Also, complications arise due to possibly stochastic market prices of risk and interest rates. In this chapter, we have valued assets using both the time-state preference approach and the rational expectations approach.

The time-state preference approach which leads to the ‘consumption CAPM’ is dealt with extensively in Huang and Litzenberger (1988) and Cochrane (2001). Pliska (1997) relies exclusively on the rational expectations approach. While these approaches are usually treated as separate models, it is important to recognise that they are just different approaches to the same problem. Given similar assumptions, the two approaches yield the same prices.

**Exercises**

5.1. Assume a state space as follows:
Let the state prices, \( q^*_i = q_i \) 
\( B_{t,t+1} \) and \( q^*_k = q_k B_{t,t+2} \)

\[
q^*_i = \begin{bmatrix} q_1^* \\ q_2^* \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}
\]

and

\[
q^*_k = \begin{bmatrix} q_1^* \\ q_2^* \\ q_3^* \end{bmatrix} = \begin{bmatrix} 0.16 \\ 0.20 \\ 0.20 \end{bmatrix}
\]

(a) Compute the bond prices \( B_{t,t+1} \) and \( B_{t,t+2} \).

(b) Compute the conditional state prices \( q_{i,k} \) and the conditional bond prices \( B_{1,2,i} \).

(c) If the payoff on an asset at \( t = 2 \) is given by the vector

\[
x_3 = \begin{bmatrix} 10 \\ 9 \\ 8 \\ 7 \end{bmatrix}
\]

what is the value of the asset, \( S_{t,t+2} \), and what is its (time 2) forward price?

(d) Compute the forward state prices, \( q_k \) and re-compute the forward price of the asset, \( F_{t,t+2} \).

5.2. If the value of \( x_1 \) at time \( t \) is given by

\[
S_{t=1} = B_{t=1}E\left[\phi_{t=1} x_1\right]
\]

and the value of \( x_2 \), at time \( t \), is

\[
S_{t=2} = B_{t=1}E\left[\phi_{t=1} S_{t=1} + \phi_{t=2} x_2\right]
\]

show that the value of \( x_1 \) and \( x_2 \) at time \( t \):

\[
S_t = S_{t=1} + S_{t=2} = B_{t=1}E\left[\phi_{t=1} \left( x_1 + S_{t=1,2} \right)\right]
\]

5.3. Assume that

\[
\alpha = e^{\omega_1} - e^{\omega_2}
\]

and that the representative investor can invest in a single risky asset from \( t = 1 \) to \( t = 2 \). Show that
\[ c_t = \frac{w'tr}{1+r} + k \]
for some constant, \( k \), and \( r \) is the return on the risky asset.

5.4. Explain the difference between the period-by-period pricing kernels \( \phi_{t,t+1}, \phi_{t+1,t+2} \) and the pricing kernel \( \phi_{t,t+2} \). When and why is one or the other type of pricing kernel used in the valuation of cash flows?

Notes:
(29) Hence, the forward price \( F_{t,t+T} \) is sometimes written as \( E_{Q_T}(x_{t+T}) \), where \( E_{Q_T} \) symbolises an expectation under the \( T \)-period EMM.

(30) The material in this and the following two sections is based on the model developed in Stapleton and Subrahmanyam (1978).

(31) Non-stochastic variances and covariances are a property of joint-normal variables. See for example Mood et al. (1974).

(32) See for example the inter-temporal CAPM of Merton (1973).

(33) This is quite a critical assumption for ‘convenience’. It assumes that the utility for different period consumptions are not related.

(34) See Exercise 5.3.

(35) In fact, we only really need \( w_{t+1} \) and the sum of \( S_{j,t+1} \) and \( x_{j,t+1} \) to be joint-normal. However, it is sufficient to assume that \( w_{t+1}, S_{j,t+1} \) and \( x_{j,t+1} \) are joint-normal. Also, note that we proved above that \( S_{j,t+1} \) is normally distributed.