Abstract and Keywords

‘Option Pricing in a Single-Period Model’ uses the one-period complete markets model to derive forward prices of European-style options relates them to the forward price of the underlying asset. The authors show that the value of the option depends upon the shape of the pricing kernel, and, in particular, on the shape of the asset-specific pricing kernel, \( \psi(x_j) \). The analysis starts at a general level and then concentrates on an important special case, where the underlying cash flow is lognormal. They establish in this case that a risk-neutral valuation relationship (RNVR) exists between the option price and the price of the underlying asset if the asset-specific pricing kernel, \( \psi(x_j) \), has the property of constant elasticity. This establishes the well known Black-Scholes equation for the value of an option.

Keywords: asset-specific pricing kernel, Black-Scholes, constant elasticity, European-style options, lognormal, pricing kernel, risk-neutral valuation relationship (RNVR)
In this chapter, we use the one-period complete markets model to price European-style options. These options are contingent claims whose payoffs depend upon the terminal cash flow $x_j$ of asset $j$ that occurs at time $t + T$. We show that the value of the option depends upon the shape of the pricing kernel, and in particular on the shape of the asset-specific pricing kernel, $\psi(x_j)$. The analysis starts at a general level and then concentrates on an important special case, where the underlying cash flow is lognormal. We establish in this case that a risk-neutral valuation relationship (RNVR) exists between the option price and the price of the underlying asset if the asset-specific pricing kernel, $\psi(x_j)$, has the property of constant elasticity. This establishes the well known Black–Scholes equation for the value of an option. We also establish sufficient conditions for the asset-specific pricing kernel to exhibit constant elasticity. Throughout the chapter, we derive the forward prices of options and relate them to the forward price of the underlying asset.

3.1 The General Case
Consider an asset $j$ with a payoff $x_{j,t+T}$ at time $t + T$ and an European-style contingent claim on $x_{j,t+T}$ with a payoff function $g(x_{j,t+T})$. From the complete market assumption, the forward price of the contingent claim $g(x_{j,t+T})$ is

\[ F_{t,T}[g(x_{j,t+T})] = \sum q_j g(x_{j,t+T}) \]
\[ = \sum p_{j} g(x_{j,t+T}) \]
\[ = E[g(x_{j,t+T})|x_m] \]

(P.40) Now dropping the time subscripts

\[ F[g(x_j)] = E[g(x_j)|x_m] \]
\[ = E[g(x_j)E[x_m|x_j]] \]
\[ = E[g(x_j)|x_m] \]

Note that there is an important difference between the expectation operator in equation (3.2) and that in (3.1). In (3.1) the expectation is over all the states of the asset, $x_j$ and those of the market cash flow $x_m$, but in (3.2) the expectation is only over $x_j$. We use equation (3.2) in Section 3.3 to evaluate option price assuming lognormality.

As in Chapter 1, equation (3.1) can be expanded using the definition of covariance to give

\[ F[g(x_j)] = E[g(x_j)]E[\varphi(x_m)] + \text{cov}[g(x_j), \varphi(x_m)] \]
Given that $E[\varphi(x_m)] = 1$, we can write

$$\varphi(x) = E[\varphi(x)] + \text{cov}[\varphi(x), \varphi(x_m)]$$  \hspace{1cm} (3.3)

In general, it is difficult to evaluate equation (3.3), given the two functions involved in the covariance term. However, there is a solution in the following example which illustrates the general approach.

3.2 An example: Quadratic Utility and Joint-normal Distribution for $x_j$ and $x_m$

In this example, we make two strong assumptions that allow us to directly evaluate the contingent claim price. The assumptions are the same as those found in Chapter 1 to be sufficient for the CAPM to hold. However, in the case of the contingent claim here, we need both assumptions to hold simultaneously. We assume both quadratic utility, which gives us a linear pricing kernel, and joint-normality of the cash flow and wealth.

Let $\varphi(x_m) = u'(x_m)/E[u'(x_m)] = (A + Bx_m)$ as in the case of quadratic utility, introduced in Chapter 1. Then, from equation (3.3)

$$F[g(x)] = E[g(x)] + \text{cov}[g(x), A + Bx_m].$$  \hspace{1cm} (p.41)

Now assume $x_j$ and $x_m$ are joint-normally distributed. Then, we can invoke Stein’s lemma (see appendix at the end of the book) and obtain

$$F[g(x)] = E[g(x)] + BE[g(x)]\text{cov}(x_j, x_m).$$

Now, assume that the contingent claim is a call option where $g(x_j) = \max(x_j - k, 0)$. In this case $g'(x_j)$ takes the values zero (when the option is out of the money, $x_j < k$) or one (when the option is in the money, $x_j > k$). So $E[g'(x_j)] = \text{prob}(x_j > k)$ and hence

$$F[g(x)] = E[g(x)] + BE[g(x)]\text{cov}(x_j, x_m).$$  \hspace{1cm} (3.4)

The second term on the right-hand side of (3.4) is a covariance weighted by the probability of the option being in the money. If $\text{prob}(x_j > k) = 1$, then the risk premium is identical to the risk premium on the stock. The forward price of aggregate wealth is $F_m = E(x_m) + B\text{var}(x_m)$, which can be solved for the market price of risk, $B$.

Equation (3.4) has an interesting intuitive interpretation. It says the risk premium for the option is a proportion of the risk premium of the underlying asset. The proportion is the
probability that the option will be exercised. Although the pricing relationship is intuitive, the model unfortunately, relies heavily on the use of quadratic utility as well as the normality assumption. Also note that, in order to price the option, we need to know the probability of exercise. In the models below this information is not required.

3.3 Option Valuation When $x_j$ is Lognormal

Returning to the general equation for the value of a contingent claim, (3.2), we see the importance of the asset-specific pricing kernel $\psi(x_j)$ in the valuation of contingent claims. We now assume, more conventionally, that the cash flow $x_j = x_{j,t+T}$ is lognormally distributed. Note that this is the same assumption that is made in the Black and Scholes (1973) model.\textsuperscript{19} We now show that a surprisingly simple valuation relationship between the contingent \textbf{(p.42)} claim price and the price of the underlying asset is obtained if the $\psi(x_j)$ has the property of constant elasticity.\textsuperscript{20} The valuation of the contingent claim is given by what is often termed a risk-neutral valuation relationship (RNVR) between the price of the claim and the price of the underlying asset. If a RNVR holds then the \textit{relationship between the price of the claim and the price of the underlying asset is the same as it would be under risk neutrality}. The most well-known RNVR is the Black–Scholes formula for the price of a call option. The RNVR is of great practical importance, since it allows options to be priced without knowledge of the risk aversion of investors. We now show that the Black–Scholes RNVR exists when $x_j$ is lognormal and $\psi(x_j)$ has constant elasticity, we proceed, using a sequence of steps.

3.3.1 Notation for the Lognormal Case

Suppose a cash flow $x_j$ is lognormal. We now define the mean and variance of $\ln x_j$:

\[
E(\ln x_j) = \mu_x, \\
\text{var}(\ln x_j) = \sigma_x^2.
\]

Using this notation, the probability distribution of $\ln x_j$ is given by (3.5)

\[
f(\ln x_j) = \frac{1}{\sqrt{2\pi} \sigma_x} e^{-\frac{1}{2\sigma_x^2}(\ln x_j - \mu_x)^2}.
\]

Note that $\sigma_x$ here is the non-annualised volatility of $x_j$ over a period of length $T$. It is convenient to work initially with non-annualised variables, since the distance from $t$ to $t + T$ is fixed.

3.3.2 The Asset-Specific Pricing Kernel
We now make an important assumption about the pricing kernel. Here we assume that the asset-specific pricing kernel is a power function of $x_j$:

$$\psi(x_j) = ax_j^\beta,$$

where $\alpha > 0$ and $\beta < 0$ are constants. First note that if this is the case, the asset-specific pricing kernel has constant elasticity. The (p.43) elasticity of the pricing kernel is defined by the relationship:

$$\eta(x_j) = \frac{\partial \psi(x_j)}{\partial x_j / x_j},$$

and hence in this case

$$\eta(x_j) = -\alpha \beta x_j^{\beta-1} x_j / a^\beta = -\beta.$$

One consequence of this constant elasticity assumption, together with the fact that $x_j$ is lognormal, is that the asset-specific pricing kernel is lognormal. We have

$$\psi(x_j) = ax_j^\beta,$$

with $\ln x_j$ normal and hence $\ln \psi(x_j) = \ln \alpha + \beta \ln x_j$, which is also normal. Hence we have

$$\text{var}[\ln \psi(x_j)] = \beta^2 a^\beta,$$

and we denote

$$\text{cov}[\ln x_j, \ln \psi(x_j)] = \sigma_{yx} = \beta a^\beta.$$

Also, for any pricing kernel $\psi(x_j)$, we must have $E[\psi(x_j)] = 1$. It then follows from the lognormal assumption that

$$E[\psi(x_j)] = 1.$$

We will use this result in the derivation below.

3.3.3 The Risk-adjusted PDF

We now derive the forward price of a contingent claim paying $g(x_j)$. From the complete markets assumption, the forward price, at time $t$, for delivery at $t+T$ of the claim is

$$F_{t, t+T}(g(x_j)) = E[g(x_j) \psi(x_j)]$$

(3.6)

To evaluate this expression, we first express the payoff on the contingent claim as a function $h$ of $\ln x_j$. We define the function $h$ by the relation:

$$h(\ln x_j) = g(x_j).$$

As an example, if $g(x_j)$ is the payoff on a call option, with a strike price $k$, we have

$$g(x_j) = \max(x_j - k, 0)$$

and $h(\ln x_j)$ is given by

$$h(\ln x_j) = \max(x_j - k, 0).$$
We can now derive an expression for the value of the claim. The forward price of the contingent claim is given by

\[
F[\phi(x)] = \int_{-\infty}^{\infty} h(lnx) \phi(x) f(lnx) dlnx
\]

where

\[
h(lnx) = \alpha \sqrt{\frac{\sigma^2}{2\pi}} e^{-\frac{1}{2\sigma^2} \left(lnx - \mu\right)^2}
\]

is called the risk-adjusted probability density function (PDF) for the asset. When expectations are taken under this density, we obtain the forward prices for contingent claims on the asset. We will first analyse the properties of this risk-adjusted PDF and then apply the results to the valuation of the contingent claims.

Substituting the assumed form of the asset-specific pricing kernel, we have

\[
\hat{h}(lnx) = \alpha \sqrt{\frac{\sigma^2}{2\pi}} e^{-\frac{1}{2\sigma^2} \left(lnx - \mu\right)^2}
\]

Now substituting the value for \( \alpha \) from equation (3.6) we find

\[
\hat{f}(lnx) = \frac{1}{\sqrt{2\pi} \sigma_x} e^{-\frac{1}{2} \left(lnx - \mu_x\right)^2}
\]

It then follows from completing the square that (see Exercise 3.2)

(3.7)

By comparing (3.5) and (3.7) we note that the risk-adjusted PDF is like the original PDF shifted by the factor \( \beta \sigma \). This is illustrated in Fig.3.1. Since

\[
\psi(x) = \frac{\hat{f}(lnx)}{\hat{f}(lnx)}
\]

the two curves intersect at \( \psi(x) = 1 \). We now proceed to evaluate this shift factor.
3.3.4 The Forward Price of the Underlying Asset under Lognormality

We now analyse the forward price of the underlying asset under the same assumptions. We have, from Chapter 1,

\[ F_j = E[X_j \psi(x_j)] \]

Now, since the product of two lognormal variables is lognormal, \( x_j \psi(x_j) \) is lognormal, and using result 4 in the appendix (in Section 3.8),

\[ F_j = E(x_j)E[\psi(x_j) \exp(\ln x_j \ln \omega)] = E(x_j) \exp(\beta \sigma^2) \]

In terms of the notation introduced above, the expected value of \( x_j \) is given by

\[ E(x_j) = e^{\mu_j + \sigma_j^2/2} \]

and hence, in this case:

\[ F_j = e^{\mu_j + \beta \sigma_j^2} \]

It then follows that

\[ \mu_j + \sigma_j^2/2 = \ln F_j - \beta \sigma_j^2 \]

or

\[ (3.8) \]

\[ \mu_j + \beta \sigma_j^2 = \ln F_j - \sigma_j^2/2. \]

3.3.5 The Lognormal RNVR

Substituting (3.8) into (3.7), we then have the forward price of the option:

\[ (3.9) \]

\[ F \left[ g(x_j) \right] = \int_{-\infty}^{\infty} g(x_j) \frac{1}{\sigma_j \sqrt{2\pi}} e^{-\frac{1}{2\sigma_j^2} \left( \ln x_j - \mu_j \right)^2} \ln x_j^2 \]
Note that the expression for the contingent claim value in (3.9) does not include the pricing kernel parameter $\beta$ or the mean of the asset $\mu_x$. This is an example of what Heston (1993) calls a missing parameters valuation relationship. One parameter of the PDF of the underlying asset ($\mu_x$) and one parameter of the pricing kernel (p.47) ($\beta$) are missing from the valuation formula. In this case the option can be valued without the knowledge of these two parameters.

However, in this case, the relationship between the forward price of the claim and the forward price of the underlying asset is also a RNVR. A RNVR is a relationship which is compatible with risk neutrality. To see this note that under risk neutrality the contingent claim price would be derived by taking the expectation

$$E[(\mu_x)] = E(\mu_x)$$

$$= \int_{-\infty}^{\infty} h(\ln x_j) \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} (\ln x_j - \mu_x^*)^2} \, d\ln x_j,$$

where $\mu_x^*$ is the mean of the asset under risk neutrality. But, under risk neutrality, the expected value of $x_j$ is also the forward price $F_j$ and

$$F_j = E(\mu_x) = \sigma \sqrt{\ln T} \sigma \ln F_j - \frac{1}{2} \sigma^2$$

Substituting this value of $\mu_x^*$ in (3.10) yields exactly the same expression as in (3.9). We see that the claim is priced as if the world was risk neutral. For this reason, the relationship between the option price, $F(g(\mu_x))$, and the forward price of the underlying asset, $F_j$, in (3.9) is referred to as a RNVR.

### 3.4 The Black–Scholes Price of a European Call Option

In this section we apply the general expression for the price of the contingent claim paying $g(x_j)$ to the special case of a call option. A European-style call option, with strike price $k$ has a payoff at time $t+T$:

$$g(x_j) = \max(x_{j+T} - k, 0).$$

We now show that the price of this claim is given by the Black–Scholes formula.

To derive the Black–Scholes formula for the value of a call option, we need to evaluate the RNVR in (3.9). The forward price of the option is given by

$$F(g(\mu_x)) = \int_{-\infty}^{\infty} \max(e^{\mu_x} - k, 0) \hat{h}(\ln x_j) \, d\ln x_j,$$

where the risk-adjusted PDF $\hat{h}(\ln x_j)$ is given by

$$\hat{h}(\ln x_j) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} (\ln x_j - \mu_x^*)^2}.$$
First, note that the integral can be written as

\[ F[g(x)] = \int_{\ln k}^{\infty} (e^{\mu x} - k) \hat{f}(\ln x) d\ln x, \]

since the option pays off nothing if \( \ln x_j \) is less than \( \ln k \), i.e., when \( x_j < k \). It is convenient to split the integral into two parts:

\[ \int_{\ln k}^{\infty} e^{\mu x} \hat{f}(\ln x) d\ln x - k \int_{\ln k}^{\infty} \hat{f}(\ln x) d\ln x. \]

(3.11)

Now, for the normal distribution, closed form expressions exist for the two integrals required in equation (3.11). We require expressions of the general type

\[ \int_{y_0}^{\infty} e^y f(y) dy \]

and

\[ \int_{y_0}^{\infty} f(y) dy. \]

The latter is given by the cumulative density function:

\[ \int_{a}^{\infty} f(y) dy = 1 - \Phi \left( \frac{a - \mu_x}{\sigma_y} \right) = \Phi \left( \frac{\mu_y - a}{\sigma_y} \right). \]

The former expression is less standard but it is shown in the appendix that:

\[ \int_{a}^{\infty} e^y f(y) dy = \Phi \left( \frac{\mu_y - a}{\sigma_y} + \sigma_y \right) \sqrt{2\pi}. \]

Hence we can evaluate the integrals in equation (3.11) in this case using \( y = \ln x_j \) and \( a = \ln k \).

Applying the above results and substituting the mean of \( \hat{f}(\ln x) \), \( \mu_x = \ln F_j - \sigma_x^2/2 \) and \( a = \ln k \), we have

\[ \Phi \left( \frac{\ln F_j - \sigma_x^2/2 - \ln k}{\sigma_x} \right) \]

(p.49) and

\[ \Phi \left( \frac{\ln F_j - \sigma_x^2/2 + \ln k}{\sigma_x} \right) = F_j N \left( \frac{\ln F_j - \sigma_x^2/2 - \ln k}{\sigma_x} \right). \]

Finally, substituting these expressions in the option pricing equation (3.11) yields

\[ F[g(x)] = F_j N \left[ \frac{\ln (F_j/k) + \sigma_x^2/2}{\sigma_x} \right] - k N \left[ \frac{\ln (F_j/k) - \sigma_x^2/2}{\sigma_x} \right]. \]

This is the forward version of the well-known Black–Scholes formula. The forward price of the option is a function of the forward price of the underlying asset, \( F_j \), the (logarithmic) variance, \( \sigma_x \), and the strike price of the option, \( k \). It is a short step now to derive the spot value of the option. We use the notation \( S_t [g(x_j)] \) for the spot price, at time \( t \), of the contingent claim paying \( g(x_j) \) at time \( t+T \). Since the option is
itself a non-dividend paying security, its spot price is given by
the discounted value:

\[
S_d = \frac{K}{e^{rT}} + \frac{\sigma^2 e^{T^2}}{2} - e^{T \sigma^2} \frac{1}{N(d_1)}
\]

or, using conventional notation:

\[
S_d = B_{t,T} [F_t N(d_1) - kN(d_2)]
\]

where

\[
d_1 = \frac{\log \left( \frac{S}{F_t} \right) + \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}}
\]

\[
d_2 = d_1 - \sigma \sqrt{T}
\]

### 3.4.1 Some Applications of the General Black-Scholes Formula

The formula for the price of a call option, in equation (3.13), is
referred to as the ‘general Black–Scholes’ formula because it
applies to call options on many different types of asset. To
illustrate the point, we apply the equation here to a variety of
different assets, including non-dividend paying stocks, and
dividend paying stocks.

**(p.50)**

1. **Non-dividend paying assets**: In this case, spot-
   forward parity for the underlying asset means that the
   spot price of the asset is

   \[
   S_t = F_t B_{t,T},
   \]

   where \( B_{t,T} = e^{-rT} \) and where \( r \) is the continuously
   compounded interest rate and \( T \) is in years. Also if we define
   the annualised volatility of the asset by \( \sigma^2 \) we then
   have:

   \[
   S_d = S_t N(d_1) - ke^{-rT} N(d_2),
   \]

   \[
d_1 = \frac{\log \left( \frac{S_t}{F_t} \right) + rT}{\sigma \sqrt{T}}
   \]

   \[
d_2 = d_1 - \sigma \sqrt{T}
   \]

2. **Assets paying a non-stochastic dividend**: Assume
   that the underlying asset is a stock or bond, paying a
   known dividend \( D_{t+T} \) at time \( t + T \). In this case, spot-
   forward parity implies

   \[
   S_t = (F_t + D_{t+T}) B_{t,T},
   \]

   since the forward contract does not receive the dividend. In
   this case equation (3.13) implies

   \[
   S_d = (S_t - D_{t+T}e^{-rT}) N(d_1) - ke^{-rT} N(d_2),
   \]

   where
3. Assets paying a stochastic proportional dividend: Suppose that the underlying asset pays a dividend proportional to $x_j$ at time $t + T$. While somewhat unrealistic for options on individual stocks, this assumption is often made when considering options on indices of stocks. Let $D_{t+T} = \delta x_j$, then in this case spot-forward parity implies

$$S_t = F_t (1 + \delta) B_{t+T}.$$  

(p.51)

It then follows that

$$S_t [g(x)] = \left( \frac{S_t}{1 + \delta} \right) N(d_1) - ke^{-rT} N(d_2),$$

where

$$d_1 = \frac{\ln[S_t / (1 + \delta)] + \tau r + \sigma^2 T / 2}{\sigma \sqrt{T}},$$

$$d_2 = d_1 - \sigma \sqrt{T}.$$

3.5 The Black-Scholes Model and the Elasticity of the Pricing Kernel

In the above derivation of the Black-Scholes formula for the value of a call option, we made two important assumptions. First, we assumed that the payoff on the underlying asset at time $t + T$ was lognormal. We will see in Chapter 4, that different assumptions regarding the probability distribution of the underlying asset lead to different option pricing models. Second, we assumed that the asset-specific pricing kernel, $\psi(x_j)$, has constant elasticity. Again, in Chapter 4 we will explore the effects of relaxing this assumption. However, we first show that, given the lognormality of $x_j$ the assumption of constant elasticity of the asset-specific pricing kernel is a necessary as well as a sufficient condition for the Black-Scholes formula to hold. In other words if the pricing kernel does not have the property of constant elasticity, then the Black-Scholes formula does not hold.

The argument follows the proof of Brennan (1979), Theorem 1 and Satchel et al. (1997). If

$$F_t [g(x)] = \int g(x) \psi(x) f(\ln x) d\ln x = \int g(x) f(\ln x) d\ln x$$
has to hold for all contingent claims $g(x_j)$ then $\psi(x_j)f(lnx_j)=\tilde{f}(lnx_j)$, where $f(ln x_j)$ is the actual distribution with mean parameter $\mu_x$ and $\tilde{f}(lnx_j)$ is the risk-neutral distribution, with mean parameter $ln\mu_x$. From $\psi(x_j)f(lnx_j)=\tilde{f}(lnx_j)$, we can derive $\psi(x_j)=\tilde{\psi}(x)/f(x)$ which is lognormal (i.e., $\psi = e^{\mu_\psi + \frac{1}{2} \sigma_\psi^2 x_j^2}$ as before), and it follows that $\eta$ is constant.

(p.52) Hence, if $x_j$ is lognormal, the Black-Scholes model (and the RNVR) holds in a single-period economy only if the elasticity of the asset-specific pricing kernel, $\eta$ is a constant (across states), i.e., if $\eta$ is not a constant then the Black-Scholes model (and the RNVR) does not hold. This necessity result is highly significant because it confirms the crucial importance of the assumption of constant elasticity of the asset-specific pricing kernel. Under the assumption of lognormality of $x_j$, the Black-Scholes RNVR holds if and only if $\psi(x_j)$ has constant elasticity. One interesting corollary of this result is that options on a lognormal market cash flow, $x_m$, will be priced by Black-Scholes if and only if the representative investor has power utility. This is the result in Brennan (1979).

3.6 Sufficient Conditions for $\psi(x_j)$ to have Constant Elasticity

In this section, we derive one set of sufficient conditions for the asset-specific pricing kernel, $\psi(x_j)$, to exhibit constant elasticity. We assume that $x_j$ and the pricing kernel, $\varphi(x_m)$ are joint lognormal. We note in passing that this condition will be fulfilled in a representative investor economy, where the investor has power utility and where the aggregate wealth, $x_m$, is joint-lognormal with the cash flow $x_j$. However, there are many other sets of conditions resulting in a lognormal pricing kernel.

As before we denote the terminal payoff on the underlying asset as $x_j$. We again assume that $x_j$ is lognormal with (logarithmic) mean and variance:

$$E[lnx_j]=\mu_x$$

$$var[lnx_j]=\sigma_x^2$$

We further assume that the pricing kernel $\varphi(x_m)$ is also lognormal with (logarithmic) mean and variance:

$$E[ln\varphi(x_m)]=\mu_\varphi$$

$$var[ln\varphi(x_m)]=\sigma_\varphi^2$$

$$cov[lnx_j, ln\varphi(x_m)]=\sigma_{x\varphi}$$

If $x_j$ and $\varphi(x_m)$ are joint-lognormal, then we can write
where $\varepsilon$ is independent of $x_j$. Then the unconditional expectation and variance of $\ln \phi(x_m)$ are given by

$$E[\ln \phi(x_m)] = \mu_{\psi}$$

and

$$\text{Var}(\ln \phi) = \sigma^2_{\psi}$$

respectively. Then, the conditional expectation of $\ln \phi$ is

$$E[\ln \phi(x_m)|x_j] = \mu_{\psi} - \beta \mu_{x} + \beta \ln \left(\frac{x}{x_j}\right)$$

and the conditional variance is

$$\text{Var}[\ln \phi(x_m)|x_j] = \text{Var}(\varepsilon)$$

Hence

$$E[\phi(x_m)|x_j] = e^{E[\ln \phi(x_m)|x_j] + \text{Var}[\ln \phi(x_m)|x_j]}$$

$$= e^{\mu_{\psi} - \beta \mu_{x} + \beta \ln \left(\frac{x}{x_j}\right) + \sigma^2_{\psi} - \frac{\beta^2 \sigma^2_x}{2}}$$

since

$$E[\phi(x_m)] = e^{\mu_{\psi} + \frac{\sigma^2_{\psi}}{2}} = 1.$$  Note that $\psi$ is a lognormal variable, and the exponential term, $e^{[1]}$, is a constant. It follows that $\psi(x_j) = E[\phi(x_m) | x_j]$, the firm-specific pricing kernel, is also lognormal and has constant elasticity.

### 3.7 Conclusion

We have established the Black–Scholes formula for the value of a European-style call option, assuming constant elasticity of the asset-specific pricing kernel. The treatment follows a similar logic (p.54) to that in Huang and Litzenberger (1988). Most other texts prove the Black–Scholes model, using a continuous time model (see e.g. Hull 2003). Cochrane (2001) uses a continuous time process to establish that the pricing kernel has constant elasticity. He then proves the Black–Scholes formula using a similar argument to the one used in this chapter. We will show in the following chapter that the assumption that the stock price follows a lognormal process in continuous time is a sufficient condition for the asset-specific pricing kernel to have constant elasticity. However, the pricing of European-style options is actually a single-period problem, as the treatment here shows.

### 3.8 Appendix: The Normal Distribution
In the text we assume that a cash flow \( x \) is lognormally distributed. If \( y = \ln x \) then \( y \) has a normal distribution. In this appendix we state a number of results that are obtained when a variable is normally distributed. These are stated without proof. The reader is referred to a statistics text, for example, Mood et al. (1974). If \( f(y) \) is normal with \( \mu, \sigma \):

\[
f(y) = \frac{1}{\sigma \sqrt{2\pi}} e^{\frac{-(y-\mu)^2}{2\sigma^2}}.
\]

The expectation that \( y > a \) is

\[
\int_a^\infty f(y) dy = N\left(\frac{\mu - a}{\sigma}\right),
\]

where \( N[\cdot] \) is the standard normal cumulative density distribution function.

We have the following results:

1. The expected value of a lognormal variable:

\[
E(e^y) = e^{\mu + \frac{1}{2} \sigma^2}.
\]

This follows from the moment generating function of the normal distribution. Also it is a special case of result 2 below. It also follows that

\[
E(e^{by}) = e^{b \mu + \frac{1}{2} b^2 \sigma^2}.
\]

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2. The expected value of the truncated lognormal distribution:

\[
\int_a^\infty e^y f(y) dy = N\left(\frac{\mu - a}{\sigma}\right) e^{\mu + \frac{1}{2} \sigma^2}.
\]

Following the proof in Rubinstein (1976), p. 422, we have

\[
E(e^y) = \int_a^\infty e^y f(y) dy = \int_a^\infty \frac{1}{\sqrt{2\pi} \sigma} e^{\frac{-1}{2} (y-\mu)^2} e^y dy = \int_a^\infty \frac{1}{\sqrt{2\pi} \sigma} e^{\frac{y^2}{2} + \frac{1}{2} (\mu - a)^2} dy
\]

and thus the result follows using the definition of the cumulative normal distribution.

3. If \( \ln y \) and \( \ln x_j \) are bivariate normal, we can write the linear regression:

\[
\ln y = a + \beta \ln x_j + \epsilon,
\]

where \( \epsilon \) is independent of \( \ln x_j \).

4. If \( X = e^x \) and \( Y = e^y \) are bivariate lognormal variables, then \( XY = e^{x+y} \) is also lognormal. Hence

\[
E(XY) = e^{E(x+y) + \frac{1}{2} \text{cov}(x,y)} = e^{E(x) + E(y) + \frac{1}{2} \sigma_x \sigma_y} = E(X)E(Y)e^{\sigma xy}.
\]
Also since
\[ E(XY) = E(X)E(Y) + \text{cov}(X, Y), \]
it follows that
\[ \text{cov}(X, Y) = E(X)E(Y)(e^{\sigma Y} - 1). \]

(p.56) Exercises
3.1. Let \( g(x) \) be the payoff function for a call option with strike price \( k \). Show that \( E[g'(x)] = \text{prob}(x > k) \), where \( \text{prob}(x > k) \) is the probability of the call option being exercised.
3.2. Prove that (cf. equations (3.7) and (3.9))
\[ e^{\beta \ln x_j} \frac{1}{\sqrt{2\pi \sigma^2 x_j}} e^{-\frac{1}{2\sigma^2 x_j}(x_j - x_j)^2} = e^{-\frac{1}{2\sigma^2 x_j} \left( x_j - x_j \right)^2}. \]
3.3. Let
\[ \psi(x_j) = a x_j^\beta, \]
where \( x_j \) is lognormal with \( (\mu_x, \sigma_x) \). Derive an expression for \( a \).
3.4. Explain what are the following: (a) \( f(x) \), (b) \( f'(x) \) (c) a RNVR.
3.5. In the following, assume that \( x \) is lognormal with \( \mu_x = 2.5, \sigma_x = 0.3, \beta = 0.8 \).
   (a) Compute \( E(x) \)
   (b) Compute \( E(x^\beta) \)
   (c) Compute \( \text{Pr}(\ln x < a) \) where \( a = 3.5 \)
   (d) Compute \( E(e^{\ln x} | \ln x > 3.5) \)
3.6. Suppose that \( \ln x_j \) has the distribution:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \ln x_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>1.5</td>
</tr>
<tr>
<td>0.5</td>
<td>1.75</td>
</tr>
<tr>
<td>0.25</td>
<td>2.39</td>
</tr>
</tbody>
</table>

What is the payoff on a call option with strike price \( k = 5 \)?

Notes:
(19) In the Black–Scholes model it is assumed that the price of an asset evolves as a geometric Brownian motion over the period \( t \) to \( t + T \). This results in a cash flow to the holder of the underlying asset of \( x_{j,t+T} \) at time \( t + T \) that is lognormal.
(20) Given that \( x_j \) is lognormal, the constant elasticity property of the asset-specific pricing kernel means that \( \psi(x_j) \) will also be lognormal.
(21) In the appendix at the end of this chapter, we state some of the most important properties of lognormal variables.