Risk Aversion, Background Risk, and the Pricing Kernel

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Abstract and Keywords
‘Risk Aversion, background Risk, and the Pricing Kernel’ looks in more detail at utility functions and their effect on the shape of the pricing kernel. The authors discuss the meaning of risk aversion and, in particular, ‘relative risk aversion’ and show that if relative risk aversion is constant at different levels of wealth, then the pricing kernel exhibits constant elasticity. They then show that the introduction of ‘background risk’, that is, non-hedgeable risks, causes the pricing kernel to exhibit declining elasticity. This effect on the pricing kernel is particularly significant for the pricing of options.

Keywords: background risk, constant elasticity, declining elasticity, relative risk aversion, utility
We have seen in Chapter 1 that asset prices depend on the characteristics of the pricing kernel, \( \varphi(x_m) \). In the simple case, where all investors are identical, we can model the pricing kernel using the utility function of the ‘representative investor’. In this chapter we look in more detail at utility functions and their effect on the shape of the pricing kernel. We discuss the meaning of risk aversion, in particular ‘relative risk aversion’ and show that if relative risk aversion is constant at different levels of wealth, then the pricing kernel exhibits constant elasticity. We then show that the introduction of ‘background risk’, i.e., non-hedgeable risks, causes the pricing kernel to exhibit declining elasticity. This effect on the pricing kernel is particularly significant for the pricing of options.

2.1 Risk Aversion and Declining Marginal Utility of Wealth

Consider the well-known St. Petersburg Paradox (Bernoulli, 1738) that describes the probability game where one tosses a coin till a head is obtained. The game offers the investor $1 if a head is obtained on the first toss, $2 if the head is first obtained on the second, $4 if head is first obtained on the third, $8 on the fourth, and so on. If the game consists of \( n \) possible tosses, the expected value of the game is

\[
(0.5)(1) + (0.5)(2) + (0.5)(4) + \cdots + (0.5)(2^{n-1}) = \frac{S_n}{2}.
\]

Now assume that \( n \to \infty \), i.e., there is no limit on the number of tosses of the coin. The expected payoff from the game is infinite. However, although the expected value of the game is infinite, reasonable individuals would be willing to pay at most a few dollars to play a game with an expected payoff of infinity. One possible explanation is that most investors are risk averse. The amount that a risk averse individual will pay is less than the expected return of the game, since risk represents a source of disutility. The utility of wealth describes the level of satisfaction gained from a given amount of wealth. A utility function defines the relationship between the amount of wealth and the utility the investor derives from it. There are some commonly accepted characteristics of utility functions. **Insatiability** means an individual will prefer more wealth to less, i.e., the first derivative of the utility function is positive. **Diminishing marginal utility** means the additional utility derived from an additional unit of wealth decreases as wealth increases, i.e., the second derivative of utility function is negative.
We will assume that an individual's utility function for wealth \( u(w) \) is three times differentiable, i.e., the derivatives \( u'(w), u''(w), \) and \( u'''(w) \) exist. A rational investor is averse to risk if \( u'(w) > 0 \) and \( u''(w) < 0 \). This is the case where the marginal utility of wealth is positive and declining. Such an investor will decline to participate in a fair game. The general shape of the utility function is illustrated in Fig. 2.1.

As an example, assume that an investor has a utility function:

\[
 u(w) = w^\gamma,
\]

with \( 0 < \gamma < 1 \).

This utility function has the properties:

\[
 u(w) = \gamma w^{\gamma - 1} > 0, \quad u'(w) = \gamma (\gamma - 1) w^{\gamma - 2} < 0.
\]

(p.21) Assume that this investor faces a fair gamble of \( w_0 \) in return for \( (w_0 + \varepsilon) \) with probability \( p = 0.5 \), and \( (w_0 - \varepsilon) \) with probability \( 1 - p = 0.5 \). The expected value of the incremental utility from taking the gamble is

\[
 E[u(\varepsilon)] = 0.5(w_0 + \varepsilon)^\gamma + 0.5(w_0 - \varepsilon)^\gamma - w_0^\gamma.
\]

Differentiating with respect to \( \varepsilon \), we find

\[
 \frac{\partial E[u(\varepsilon)]}{\partial \varepsilon} = 0.5(w_0 + \varepsilon)^{\gamma - 1} - 0.5(w_0 - \varepsilon)^{\gamma - 1} < 0.
\]

The investor faces an expected reduction in utility if he or she accepts a gamble with \( \varepsilon \neq 0 \). This is because of the non-linearity (i.e., convexity) of the utility function. In this example if we assume, as in Chapter 1, that the investor maximises the expected utility of wealth, then the fair gamble will be rejected.

2.2 Absolute Risk Aversion

One of the frequently encountered assumptions in finance is that investors have exponential utility functions. Here we assume that the representative investor has an exponential utility function of the form:

\[
 u(w) = A - e^{-yw}.
\]
It follows that

\[ u(w) = ae^{-\alpha w} > 0, \]
\[ u(w) = -ae^{-\alpha w} < 0. \]

We now explore the implications of this function for the risk premium. First, for the representative investor, wealth \( w = x_m \), the aggregate cash flow of the firms in the economy. In the previous chapter we found that, under the assumption of normality, the risk premium for a cash flow was proportionate to \( E[\varphi'(x_m)] \), where

\[ \varphi(x_m) = \frac{u(x_m)}{E[u(x_m)]}. \]

Evaluating this expression in the case of exponential utility we have:

\[ E[\varphi(x_m)] = \frac{E[u(x_m)]}{E[u(x_m)]} - \frac{E[ae^{-\alpha w}]}{E[ae^{-\alpha w}]} = -\alpha, \]

\( (p.22) \) a constant. Hence, in the context of the CAPM, exponential utility is of central importance. If the representative investor has exponential utility, the market price of risk, \( E[\varphi'(x_m)] \), is non-stochastic. We will use this property in Chapter 5, when deriving a tractable multi-period asset pricing model.

The above property of exponential utility is closely related to the fact that for this function

\[ -\frac{u'(w)}{u(w)} = \frac{\alpha e^{-\alpha w}}{\alpha e^{-\alpha w}} = \alpha. \]

In general, for any utility function, the ratio \( a(w) = -u'(w)/u' \) (w) is known as the coefficient of absolute risk aversion. The degree of risk aversion is measured by \( a(w) \). In the case of exponential utility, \( a(w) = \alpha \), a constant. However, for utility functions other than the exponential, \( a(w) \) is stochastic and dependent on wealth. For this reason exponential utility is often referred to as Constant Absolute Risk Aversion utility, or CARA.

The degree of absolute risk aversion also determines the changes in the absolute amount of risky investment an investor will make as wealth increases. Absolute risk aversion could be decreasing, constant or increasing. If the investor increases the absolute amount invested in risky assets as his or her wealth increases, then the investor is said to exhibit decreasing absolute risk aversion.

In practice, one would expect, for most investors, that as wealth increases the dollar amount invested in risky assets
will increase. This explains why declining absolute risk aversion is a commonly made assumption. This assumption restricts the possible utility function that could describe the investor's preferences to a reasonable set.\(^{10}\)

### 2.2.1 Relative Risk Aversion

Another frequently encountered utility function is the power function. The most convenient form of this function is as follows:

\[
    u(w) = \frac{1 - \gamma}{\gamma} \left( \frac{w}{1 - \gamma} \right)^\gamma, \tag{p.23}
\]

where \( \gamma \) is a constant such that \( \gamma < 1 \) and \( \gamma \neq 0 \).

Taking the first derivative:

\[
    u'(w) = \left( \frac{w}{1 - \gamma} \right)^{\gamma - 1}.
\]

Further,

\[
    u''(w) = \frac{-\gamma}{1 - \gamma} \left( \frac{w}{1 - \gamma} \right)^{\gamma - 2},
\]

and the coefficient of absolute risk aversion is

\[
    \alpha(w) = -\frac{u''(w)}{u'(w)} = \frac{1 - \gamma}{w}.
\]

This is an example of declining absolute risk aversion. Further, since \( \alpha(w) \) declines in proportion to wealth, we say that the power function exhibits constant relative risk aversion (CRRA).

We define the coefficient of relative risk aversion as\(^{11}\)

\[
    r(w) = -\frac{u''(w)}{u'(w)} w.
\]

In the case of the power function above, the relative risk aversion is:

\[
    r(w) = 1 - \gamma.
\]

If the investor increases the proportionate amount invested in risky assets as wealth increases, then the investor is said to exhibit decreasing relative risk aversion. Theoretically, investors may also exhibit CRRA or increasing relative risk aversion. While there is a general agreement that most investors exhibit decreasing absolute risk aversion, there is much less agreement concerning relative or proportional risk aversion.

The importance of the power function in the context of asset pricing is illustrated below. Assuming again a representative
investor with utility $u(x_m)$, we now assume that the investor has power utility:

$$u(x_m) = \frac{1-\gamma}{\gamma} \left( \frac{x_m}{1-\gamma} \right)^\gamma.$$  

It then follows that the pricing kernel in this case is

$$\phi(x_m) = \frac{u(x_m)}{u(x_m) + \frac{\gamma}{1-\gamma}}.$$  

Assume now that a cash flow, $x_j$ and the aggregate market cash flow, $x_m$ are joint-lognormally distributed. The logarithmic mean of $x_j$ is $E(\ln x)$ and the logarithmic variance is $\text{var}(\ln x)$. Also, the logarithmic mean of the pricing kernel is $E(\ln \phi)$ and the logarithmic variance is $\text{var}(\ln \phi)$. If $y$ is a lognormal variable,

$$E(y) = \rho E(\ln y) \pm \sqrt{\text{var}(\ln y)}.$$  

We then have the forward price

$$F_j = E[x_j \phi(x_m)] = e^{E[\ln(x_j) + \ln \phi(x_m)]},$$

where

$$E[\ln(x_j \phi(x_m))] = E[\ln x_j + \ln \phi(x_m)].$$

Now, using the relationship

$$\text{var}[\ln(x_j \phi(x_m))] = \text{var}[\ln x_j + \ln \phi(x_m)] = \text{var}[\ln x_j] + \text{var}[\ln \phi(x_m)] + 2 \text{cov}[\ln x_j, \ln \phi(x_m)]$$

and the property $E(\phi) = 1$, we find

$$F_j = E(x_j) \text{cov}[\ln x_j, \ln \phi(x_m)].$$

and hence

$$\frac{F_j}{E(x_j)} = e^{\text{cov}[\ln x_j, \ln \phi(x_m)]}.$$  

Hence, using the pricing kernel:

$$\phi(x_m) = \left( \frac{x_m}{(1-\gamma)^{1-\gamma}} \right)^{\gamma},$$

it follows that

$$\text{cov}[\ln x_j, \ln \phi(x_m)] = (\gamma - 1) \text{cov}(\ln x_j, \ln x_m)$$

and

$$\frac{F_j}{E(x_j)} = e^{\gamma - 1} \text{cov}(\ln x_j, \ln x_m).$$

In this model, where the cash flows $x_j$ and $x_m$ are joint lognormal, and where the representative investor has power utility, the risk premium, expressed as the ratio of the forward price to the expected value of the cash flow, is a constant.
equation (2.1) \( \gamma \) and the covariance term are constants. This stems from the CRRA property, which implies that the risk premium is the same at all levels of the cash flow.\(^{12}\)

2.2.2 The Elasticity of the Pricing Kernel

A precise measure of the degree to which the pricing kernel reflects constant or declining relative risk aversion is its elasticity. We may define the elasticity of the pricing kernel by the following relationship:

\[
\nu(x_m) = -\frac{\partial \phi(x_m)}{\partial x_m / x_m}.
\]

If the representative investor has power utility, we saw above that the relative risk aversion is constant. It follows that in this case, the elasticity of the pricing kernel is constant. We have

\[
\nu(x_m) = -\frac{\partial \phi(x_m)}{\partial x_m / x_m} = \left[ -\frac{u(x_m)}{u'(x_m)} \right] x_m.
\]

Hence, in the representative investor economy, the elasticity of the pricing kernel is simply the degree of relative risk aversion of the investor. Hence, in the case of the pricing kernel \( \varphi(x_m) \) we (p.26) use the terms elasticity of the pricing kernel and the relative risk aversion of the representative investor interchangeably. However we will also discuss in Chapter 3 about the elasticity of the firm-specific pricing kernel, defined in a similar manner, using \( \psi(x_j) \) rather than \( \varphi(x_m) \).

The elasticity of two pricing kernels is illustrated in Fig. 2.2, for the cases of CARA and CRRA. The case of CARA assumes that \( u(x) = -e^{-ax} \) with \( a = 0.28 \). The case of CRRA assumes that \( u(x) = [(1 - \gamma)/\gamma] \frac{x}{(1 - \gamma)} \) with \( \gamma = -3.98 \). From the graph it can be observed that the pricing kernel in the case of CRRA has the property of constant elasticity. In the case of the CARA, the elasticity is non-constant, and is in fact increasing with \( x \).
2.2.3 Prudence

An important characteristic of the pricing kernel, which determines the pricing of options, is the rate of decline of its elasticity.

Mathematically, this decline is determined by the third derivative of the utility function of the representative investor.\(^{13}\) It is \(p.27\) given by

\[
\nu(w) = -\left[\frac{u'(w)u''(w) - u''(w)u'(w)}{u(w)u'(w)}\right] - \frac{u'(w)}{u(w)}
\]

Rearranging this derivative:

\[

\nu(w) = \dot{d}(w) \left[1 - \left[\frac{-u'(w)}{u(w)} - d(w)\right]\right]
\]

Defining

\[
p = \frac{-u'(w)}{u(w)}
\]

as the absolute prudence of the utility function and \(wp(w)\) as the relative prudence, we obtain

\[
\nu(w) = \dot{d}(w) \left[1 - \left[wp(w) - wd(w)\right]\right]
\]

It follows that \(\nu' < 0\) if the difference between relative prudence and relative risk aversion exceeds unity.

Note that absolute prudence is analogous to absolute risk aversion, but involves the second and third derivatives of the utility function rather than the first and second derivatives. Also the more prudent is the utility function, given the absolute risk aversion, the more the pricing kernel elasticity declines. To illustrate the calculation of absolute prudence, we take power utility as an example,
Note that, in this case, absolute prudence exceeds absolute risk aversion and

\[ wp(w) - wd(w) = 1, \]

which is consistent with constant elasticity.

The coefficient of absolute risk aversion, \( a(w) > 0 \) indicates positive risk aversion and the coefficient of absolute prudence, \( p(w) > 0 \) indicates positive prudence. When \( a'(w) < 0 \), investors become less risk averse as they get wealthier. \( p'(w) < 0 \) indicates declining prudence; investors become less prudent as they get wealthier. An investor is said to be standard risk averse if \( a(w) > 0, a'(w) < 0, p(w) > 0 \) and \( p'(w) < 0 \). The HARA class of functions, excluding the exponential utility function, are standard risk averse. The significance of standard risk aversion is as follows: a standard risk averse investor will act in a more risk averse manner towards risky assets when faced with a zero-mean, additive background risk.\(^{14}\)

To illustrate non-constant elasticity of the pricing kernel, we now introduce a wider class of utility functions. Most utility functions used in Finance, are members of the hyperbolic absolute risk averse (HARA) class.

First, we define the HARA class. A HARA utility function is of the form:

\[ u(w) = \frac{1}{\gamma} \left( \frac{A+w}{1-\gamma} \right)^\gamma, \]

if \( \gamma \neq 0 \), where \( A \) and \( \gamma \) are constants such that (i) \( \gamma < 1 \) and \( A + w > 0 \), or (ii) \( \gamma = 2 \).

If \( \gamma = 0 \), the HARA utility function is defined by the marginal utility function

\[ u(w) = \left( \frac{A+w}{1} \right)^{\gamma-1} = \frac{1}{A+w}. \]

\(^{15}\) Note that the power function (CRRA) is a special case where \( A = 0 \). Also \( u(w) = \ln(w) \) when \( A = 0 \) and \( \gamma = 0 \). Also, the exponential utility function is a special case where \( \gamma \rightarrow -\infty \).

The relative risk aversion of the utility function is directly affected by the constant \( A \). If \( A < 0 \) the utility function exhibits
declining relative risk aversion. To see this, differentiate the marginal utility function and obtain

$$u(w) = -\left(\frac{A+w}{1-\gamma}\right)^{-2}.$$ 

The relative risk aversion is hence

$$r(w) = \frac{1-\gamma}{A+w},$$

which declines in $w$ when $A < 0$. Differentiating again,

$$u'(w) = -\left(\frac{y-2(A+w)}{(1-\gamma)^2}\right)^{-3}.$$ 

The absolute prudence is then given by

$$p(w) = \frac{2-\gamma}{A+w}.$$ 

Also, the difference between the relative prudence and relative risk aversion is

$$\frac{w}{A+w}.$$ 

Since $r(w)$ is a constant, log utility is an example of the class of Constant Relative Risk Averse utility functions. $a(w)$ is positive but $a'(w) = -1/w^2 < 0$, which means that the investor is decreasing absolute risk averse. A log utility investor is one who is myopic.

(p.30) which is greater than 1 when $A < 0$, confirming that the elasticity of the pricing kernel declines when $A < 0$.

In Fig. 2.3, we illustrate the effect of the constant $A$ factor on the elasticity of the pricing kernel. The solid line plots the pricing kernel under DRRA, with $A = -5$ and $\gamma = -2$. The dotted line plots the pricing kernel under CRRA, with $A = 0$ and $\gamma = -3.98$. The example assumes a uniform distribution for $x_m = 10, 11, \ldots, 19$ and is calibrated so that the two pricing kernels have expected values equal to unity and produce the same forward price. In both cases, $F = E[x_m \phi(x_m)] = 11.92$. Note that the two pricing kernels intersect twice.
The example illustrates the fact that two pricing kernels, calibrated to give the same forward price of an asset, one with constant relative risk aversion and one with declining relative risk aversion, must intersect twice. This is important in the valuation of options, since, as we will see in Chapter 4, the convexity of the pricing kernel is a crucial determinant of the relative pricing of options.

2.3 Background Risk and the Pricing Kernel

In this section, we analyse the effect on investors' attitudes towards marketable risky assets of a second non-hedgeable risk, such as labour income uncertainty. These secondary risks are called background risks. The CAPM, derived in Chapter 1, assumes that investors are risk averse, i.e., $u'(w) > 0$, $u''(w) < 0$ and derives an equilibrium in which the beta of a company's stock determines its cost of capital. Risk aversion implies that the pricing kernel is a declining function of aggregate wealth, $\varphi'(w) = \varphi'(x_m) < 0$, as shown in the analysis of the representative investor's utility. However, an individual's attitude to market risk can be affected by background risk. Generally, the effect of background risk is to increase the risk aversion of the investor towards marketable risks. This increases the slope of the pricing kernel. However, as we will show, it also changes the shape of the pricing kernel.

Investors may be subject to many different background risks, not just labour income uncertainty, which can affect their demand for risky assets and hence their prices. Examples of other background risks that can affect investors are uncertain
bequests, uncertain medical bills, and the returns on non-marketable stocks.

Here, we look at the effect of zero-mean (or pure) background risks by analysing a derived utility function for market portfolio wealth. The derived utility is the utility function of an investor who faces a marketed risk in the presence of a second non-hedgeable risk. It has been shown in the literature that the risk aversion of the derived utility function exceeds the risk aversion, in the case where there is no background risk.

So far we have assumed complete markets for cash flows of firms. Now in the case of non-hedgeable background risks, we introduce an element of market incompleteness. However, we continue to assume that the marketable risks of firms are traded in a complete market.17

2.3.1 Consumption Optimisation Under Background Risk

We now analyse the effect of background risk on the pricing kernel, by looking at the portfolio demand of a representative investor. Background risk refers to a second, non-hedgable, zero-mean, independent risk to which the investor is subject. We show here that an investor with a constant relative risk aversive (power) utility function, faced with background risk, acts towards the market risk like an investor without background risk, but with declining relative risk aversive utility.

Following Franke et al. (1998) consider a representative investor whose wealth at time $t + T$ is given by $w = x_m + e$, where $x_m$ is the aggregate market cash flow, and $e$ is a background risk. Utility is given by

$$u(w) = u(x_m + e),$$

where a complete market exists for $x_m$, and $e$ is a non-hedgeable background risk. In this case, the amount the investor can consume depends not only on the risky payoff, but also on the background risk. The background risk $e$ is independent from $x_m$. We also assume that $E(e) = 0$ so that the non-hedgeable income is a pure risk.18

The maximisation problem is:

$$\max_{\{x_m\}} u(x_m + e)$$

subject to the same budget constraint used before in Chapter 1, i.e.,

$$\sum_i x_m q_i = w_i B_{i+T}.$$

First, we write
Then, by analogy with the no background risk case in Chapter 1, the first-order condition is

\[ p E_x[u(x_m + e)] - \lambda q_i = 0 \text{ for all } i \]

and, summing over \( i \)

\[ E_x[E_x[u(x_m + e)]] = \lambda \]

Finally, substituting for \( \lambda \) we have

\[ q_i = \frac{E_x[u(x_m + e)]}{E_x[E_x[u(x_m + e)]]} = \phi(x_m) \text{ for all } i. \]

Background risk changes the pricing kernel in the following way. As equation (2.2) shows, the numerator is now an expectation of marginal utility, over the background income states. As we will see below, the presence of background risk can have significant effects on the \( \phi(x_m) \) function.

### 2.3.2 The Precautionary Premium and the Shape of the Pricing Kernel

In order to analyse the impact of background risk on the pricing kernel, it is useful to introduce the concept of the precautionary premium. Kimball (1990) defines the precautionary premium, \( \theta(x_m) \), by the relation:

\[ E_x[u(x_m + e)] = u(x_m - \theta(x_m)] \]

Hence, \( \theta(x_m) \) is the amount of the deduction from \( x_m \) that makes the marginal utility equal to the conditional expected marginal utility in the presence of the background risk, \( e \). The precautionary premium is analogous to the risk premium, but applies to marginal utility instead of the utility itself.

In the appendix, we prove two results concerning the precautionary premium for the general class of HARA utility functions. We show there that \( \theta(x_m) > 0 \) when a background risk exists. Also, except in the case of exponential utility, \( \theta'(x_m) < 0 \). The non-negativity of \( \theta(x_m) \) reflects the fact that the risk premium is also non-negative. Intuitively, \( \theta(x_m) < 0 \) follows from the fact that (p.34) a given background risk has less effect at high income level than at low income level. Similarly, we would expect rich individuals to have a smaller precautionary premium than poor individuals. The shape of the precautionary premium for two levels of background risk is illustrated in Fig. 2.4.
Figure 2.4 illustrates the properties of the precautionary premium for two possible levels of background risk. The higher curve reflects a higher level of background risk. For each level, the value of $\theta(x_m)$ is positive and declining in $x_m$.

We can now analyse the effects of the precautionary premium on the pricing kernel. First, rewriting equation (2.2) using $\theta(x_m)$, we have

$$
(2.3) \quad \frac{u[x_m - \theta(x_m)]}{E_n[u(x_m - \theta(x_m))]} = \frac{q}{p_i} = \theta(x_m) \text{ for all i.}
$$

Since $\theta$ is positive, the effect of the precautionary premium is to reduce the impact of the constant $A$ on the pricing kernel. Since a positive value of $A$ indicates increasing relative risk aversion, the effect of $\theta(x_m)$, in this case, is to reduce the rate of increase of the derived relative risk aversion. If $A = 0$, the case of CRRA, (p.35) the effect of $\theta(x_m)$ is to induce declining relative risk aversion in the derived function. If $A < 0$, and $u$ exhibits declining relative risk aversion, the effect of $\theta(x_m)$ is to increase the rate of decline in the derived function.

However, the pricing kernel also reflects the fact that $\theta(x_m)$ declines as $x_m$ increases. This has the effect of increasing the values of $\varphi(x_m)$ in the low $x_m$ states relative to the high $x_m$ states.

Background risk has two separate effects on the derived utility of the representative investor and the pricing kernel. First, since the marginal utility of the derived utility function exceeds that of the original function, the aversion to market
risks increases. This is reflected in a steeper slope of the pricing kernel. The consequence is that the risk premium increases, compared to the no-background-risk case. This is the effect analysed by Weil (1992). Second, the pricing kernel is more likely to exhibit declining elasticity. This effect will have an impact on the relative value of contingent claims, as we show in Chapter 4.

2.4 Conclusion
In a representative agent economy, the pricing kernel is determined by the relative marginal utility of the agent. In this chapter, we have investigated the properties of utility functions and defined various measures of risk aversion, which are important for asset pricing. We have also defined the prudence of a utility function, which is important for option pricing. Further analysis of the utility foundations of asset pricing are found in Gollier (2000). For a detailed justification of the representative agent economy assumption, see Huang and Litzenberger (1988), chapter 5.

We have emphasised here the impact of non-hedgeable background risks on the prudence of the representative investor and hence on the shape of the pricing kernel. We will see in Chapters 3 and 4, the shape and in particular the decline in the elasticity of the pricing kernel is critical for the pricing of contingent claims.

2.5 Appendix: Properties of the Precautionary Premium
In Section 2.3.2 we showed the effects of a positive, declining precautionary premium, θ, on the pricing kernel. The proofs that θ has these properties in the case of HARA utility functions are shown (p.36) in Franke et al. (1998). Here we summarise the proofs of these properties.

**Lemma 1** With background risk, if \( u(x_m) \) is HARA with \( -\infty < \gamma < 1 \) and \( \theta \) is two times differentiable, then

\[
\theta > 0, \quad \frac{d\theta}{dx_m} < 0.
\]

**Proof** For the HARA utility function, the marginal utility function \( u' \) is a strictly convex function, since \( u'' > 0 \). It then follows from Jensen's inequality

\[
\begin{align*}
    u(x_m - \theta) &= E[u(x_m + e)] \\
    > u[E(x_m + e)] \\
    &= u(x_m).
\end{align*}
\]
Hence, $\theta > 0$, since $e$ has zero mean and $u'$ is strictly decreasing in $x_m$.

To establish the second property, note that for the HARA utility, absolute prudence and absolute risk aversion have the same sign, since

$$p(x_m) = \frac{2 - \gamma}{(1 - \gamma)}$$

Also, the derivatives of $p(x_m)$ and $a(x_m)$ with respect to $x_m$ have the same sign. Pratt (1964) established that the risk premium declines with $x_m$, except in the case of exponential utility. By analogy, the precautionary premium also declines in $x_m$, except in the case of exponential utility, where it is a constant.

(p.37) Exercises
2.1. Assume that the utility function of the representative investor is given by

$$u = A - e^{\alpha x}.$$ 

Assume that $A = 10$, $\alpha = 0.1$, $w = 1, 2, 3, 4$ each with probability 0.25. Compute the distribution of the pricing kernel.

2.2. Assume that utility of consumption is given by:

$$u(w) = \frac{w}{1 - \gamma w^{1-\gamma}}.$$ 

(a) Compute $u'$, $u''$, $a(w)$, and $\gamma(w)$.

(b) What do you conclude from these calculations?

2.3. Assume a probability distribution for $x$ as follows:

- $p_1 = \frac{1}{3}, x_1 = 10$
- $p_2 = \frac{1}{3}, x_2 = 12$
- $p_3 = \frac{1}{3}, x_3 = 14$

and an independent distribution for a background risk, $e$:

- $p_1 = \frac{1}{2}, e_1 = +1$
- $p_2 = \frac{1}{2}, e_2 = -1$.

Assume $u(w) = \ln(w)$, where $w = x + e$

(a) Compute $E_e[u(x + e)]$ for $x_1$, $x_2$, and $x_3$;

(b) Compute $\lambda$ in the first-order condition;

(c) evaluate the precautionary premium, $\theta$ for $x_1 = 10$.

2.4. Let $u(w) = w^{1/2}$. Show that an investor with such a utility function will not pay $100 to play a game that pays $150 or $50 with equal probability.

2.5. Assume that utility is HARA, with
Show that relative risk aversion declines in $w$ when $A < 0$.

2.6. Reproduce the example in Fig. 2.3. Assume that $x$ has a uniform distribution and $u(x)$ is HARA.

(p.38)

Notes:

(10) Pratt (1964) shows that when two investors, indexed as 1 and 2, are facing the choice of investment in one risky and one risk-free asset and if $a_1(w) > a_2(w)$, investor 1 will invest less money in the risky asset than investor 2, regardless of his/her level of wealth.

(11) Merton (1969) shows that two investors with different levels of wealth, $w_1$ and $w_2$, will invest the same proportion of their wealth in the risky asset and risk-free asset if $r_1(w) = r_2(w)$.

(12) The same property implies that in inter-temporal models the proportionate risk premium is non-stochastic. See, for example, the model of Rubinstein (1976).

(13) A positive third derivative of the utility function indicates a precautionary saving motive. It reflects how uncertainty about future income will reduce current consumption and investment in risky assets and increase current saving. See Kimball (1990).

(14) For a definition of background risk, see Section 2.3. An example of a non-standard risk averse investor is one with exponential utility. Such an investor has $a'(w) = 0$ and $p'(w) = 0$. So the exponential utility investor will not react to background risk by becoming more risk averse to marketable risks.

(15) In the case of the logarithmic utility function, we have:

$$u(w) = \ln w,$$

$$u'(w) = \frac{1}{w},$$

$$u''(w) = -\frac{1}{w^2},$$

$$a(w) = -\left(\frac{1}{w} + \frac{1}{2}\right) = \frac{1}{w},$$

$$r(w) = w \times a(w) = 1.$$
Since \( r(w) \) is a constant, log utility is an example of the class of Constant Relative Risk Averse utility functions. \( a(w) \) is positive but \( a'(w) = -1/w^2 < 0 \), which means that the investor is decreasing absolute risk averse. A log utility investor is one who is myopic.

(16) Examples of background risk in finance go beyond risks that affect individual investors. Franke et al. (1998) give several examples. Consider, for example, the case of a multinational company with foreign exchange and interest rate risks which are hedgeable ‘market’ risks and operational risk which is not hedgeable. Here the operational risks are background risks which affect the demand for foreign exchange and interest rate hedging. Also, consider the case of a fund manager who is judged on his fund’s absolute and relative performance. The portfolio risk is hedgeable, but his performance relative to his peers is not. The latter can be treated as a background risk.

(17) The existence of incomplete markets for background risks has the potential for solving at least three major puzzles in finance: the equity premium puzzle (Weil, 1992); the demand for options, which is supposed to be a redundant asset in equilibrium models (Franke et al., 1998); and the herd-like behaviour and seeming underperformance of portfolio managers.

(18) Gollier and Pratt (1990) provide a good summary of the literature on background risk and its effect on risk taking. The focus of their paper is on adverse (unfair) risks, where \( E[e] \leq 0 \). FSS (1998) follow Kimball (1993) and set \( E[e] = 0 \).