ASSET PRICES IN A SINGLE-PERIOD MODEL

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Abstract and Keywords

‘Asset Prices in a Single-period Model’ derives asset prices in a one-period model. The authors derive a version of the Capital Asset Pricing Model (CAPM) using a complete market, state-contingent claims approach. They define the forward pricing kernel and then use the assumption of joint normality of the cash flows and Stein's lemma to establish the CAPM. They then derive the pricing kernel in an equilibrium representative investor model.

Keywords: Capital Asset Pricing Model (CAPM), complete market, state-contingent claims, forward pricing kernel, Stein's lemma

This chapter derives asset prices in a one-period model. We derive a version of the capital asset pricing model (CAPM) using a complete market, state-contingent claims approach. We define the forward pricing kernel and then use the assumption of joint normality of the cash flows and Stein's
lemma to establish the CAPM. We then derive the pricing kernel in an equilibrium representative investor model.

1.1 Initial Setup and Key Assumptions
In this section we establish the value of a firm $j$ which generates a cash flow $x_j$ at a single point in time. There are $j = 1, 2, ..., J$ firms in the economy and the sum total of the cash flows $x_m = x_1 + x_2 + ... + x_J$ is the aggregate, or market cash flow. In the model, states of the world are represented by outcomes of the cash flows of the firms. We make the following assumptions:

1. We assume a single period extending from time $t$ to time $t + T$. Each firm pays a dividend equal to its cash flow at $t + T$.
2. Next, we assume that forward parity holds. Since no dividends are paid between $t$ and $t + T$, this means that the spot price $S_{j,t}$ of asset $j$ is given by

\[ S_{j,t} = F_{j,t,t+T} B_{t,t+T} \]  

where $B_{t,t+T}$ is the price at $t$ of a zero-coupon bond paying $1$ at time $t + T$ and $F_{j,t,t+T}$ is the forward price at time $t$ for the delivery of asset $j$ at time $t + T$. Note that, if this equality does not hold (i.e. forward parity is violated), arbitrage profits can be obtained by trading $S$, $F$, and $B$.

3. We assume that there are a finite number of states of the world at time $t + T$, indexed by $i = 1, 2, ..., I$, each with a positive probability of occurring. Let $p_i$ be the probability of state $i$ occurring. A state-contingent claim on state $i$ is defined as a security which pays $1$ if and only if state $i$ occurs.
4. We now assume that the markets are complete. Specifically, we assume that it is possible to buy a state-contingent claim with a forward price $q_i$ for state $i$.\(^1\) In complete markets, the $q_i$ prices exist, for all states $i$.\(^2\)
5. Assume that the investors have homogeneous expectations. This means that they agree on the probability of a state occurring and on the cash flow of each firm in each state.
6. Assume that the price of a portfolio (or a package) of contingent claims is equal to the sum of the prices of the individual state-contingent claims.\(^3\) It follows that
an asset $j$, which has a time $t + T$ payoff $x_{j,t+T,i}$ in the state $i$, has a forward price

\begin{equation}
F_{j,t+T} = \sum_i q_i x_{j,t+T,i}
\end{equation}

For simplicity, when there is no ambiguity, we drop the time subscripts and write $F_j = \sum_i (q_i x_{j,i})$.

We will show that the above set of assumptions is sufficient to establish the pricing of assets. However, other sets of assumptions are possible. For example Pliska (1997) assumes, at a more fundamental level, just the absence of arbitrage in financial markets. LeRoy and Werner (2001), equivalently, assume that a set of assets exists which span the state space.

1.2 Properties of the State Price, $q_i$

We first establish some important properties of the state forward prices. Note first that $q_i$ is a pricing function. We can write $q_i = q(i)$, where convenient, to emphasise this fact. We have:

1. The state price, $q_i$, is always greater than zero. Since $q_i$ represents the price of a claim which pays $1 if a state with positive probability occurs, it is a claim with positive utility and thus must have a positive price, i.e. $q_i > 0$.

2. The state prices sum to 1, i.e. $\sum_i q_i = 1$.

To prove that $\sum_i q_i = 1$, we use the relation in equation 1.2. If $x_j$ is a certain cash flow, for example the payoff on a zero-coupon bond, $x_{j,i} = 1$ for all $i$. In this case, the forward price must be equal to $1$, which means from (1.2) that $F_j = \sum_i 1 \cdot q_i = 1$ or $\sum_i q_i = 1$.

A set \{ $q_i$ \} which is positive and sums to unity is a ‘probability’ measure. Note that it is similar in many respects to the set of probabilities \{ $p_i$ \} which is also positive and sums to unity. In the literature, $q_i$ is often referred to as the risk-neutral measure.\(^4\)

1.3 A Simplification of the State Space

So far, we have defined the state space as the product of the states of all the individual firms in the economy. We now simplify the state space, defining the states of the world by different outcomes of $x_m$, the aggregate market cash flow.
We first illustrate the state space assumed so far, using an example with just two firms and three states for each cash flow. In Fig. 1.1, there are three states for each firm and nine in all. These result in nine different states for the market portfolio. Note that firm 2 can be regarded as the combination of all the other firms in the economy. (p.4)

1.4 The Pricing Kernel, \( \phi_i \)

In this section we define a variable, often known as the pricing kernel, \( \phi \). We then establish the essential properties of \( \phi \). It is defined by

\[
\phi_i = \frac{q_i}{p_i},
\]

i.e. it is the forward price of a state-contingent claim relative to the probability of the state occurring. It is sometimes, therefore, referred to as the ‘probability deflated’ state price. Note that the pricing kernel here is more precisely described as the ‘forward pricing kernel’, since \( q_i \) is the forward state price. Often, we will write \( \varphi = \varphi(i) \) in functional form. The properties of \( \phi_i \) are as follows:

1. Since \( p_i > 0 \) and \( q_i > 0 \), this means the pricing kernel \( \varphi_i \) is a positive function. (p.5)

*Fig. 1.1. The state space*

Notes:

1. Firm 1 has a cash flow \( x_{1,g} \) in its good state, \( x_{1,b} \) in its bad state, and \( x_{1,o} \) in its OK state.
2. Firm 2 has a cash flow \( x_{2,g} \) in its good state, \( x_{2,b} \) in its bad state, and \( x_{2,o} \) in its OK state.
3. There are 9 states in all, indicated by \( (g,g), (g,o), \ldots, (b,b) \).
4. The market cash flow in state 1 is \( x_{m,1} = x_{1,g} + x_{2,g} \), and is \( x_{m,2} \) in state 2 and so on.
2. \( E(\varphi) = 1 \). This follows immediately from the fact that the sum of the state prices is 1. We have
\[
E(\varphi) = \sum p_i \varphi_i \\
= \sum p_i \frac{q_i}{p_i} \\
= \sum q_i = 1.
\]

In Fig. 1.2, we illustrate the state prices, probabilities, and the pricing kernel using the same example introduced in Fig. 1.1. Note that there is a state price, \( q_i \) and a joint probability, \( p_i \), for each

![Fig. 1.2. State space, state prices, and pricing kernel](image)

Notes:

(1.) Firm 1 has a cash flow \( x_{1,g} \) in its good state, \( x_{1,b} \) in its bad state, and \( x_{1,o} \) in its OK state.

(2.) Firm 2 has a cash flow \( x_{2,g} \) in its good state, \( x_{2,b} \) in its bad state, and \( x_{2,o} \) in its OK state. \( x_m \) is the sum of \( x_1 \) and \( x_2 \).

(3.) There are 9 states in all of the market cash flow.

(4.) \( q_i \) is the state price, \( p_i \) is the probability of the state, and \( \varphi_i \) is the probability deflated state price or pricing kernel.

(5.) In this example, the forward price of the cash flow \( x_1 \) is given by
\[
F_1 = x_{1,1}(q_1 + q_2 + q_3) + x_{1,2}(q_4 + q_5 + q_6) + x_{1,3}(q_7 + q_8 + q_9)
\]
joint outcome of the firm cash flow and the market portfolio. This illustrates one potential problem. There is nothing to prevent two of the outcomes leading to the same value of $x_m$. In this case we will assume that the pricing kernel has the same value in both states. Note that although the state prices will not usually be the same, it is reasonable to assume that the probability deflated state prices are the same. In this case we can write the pricing kernel as a function of the aggregate cash flow, i.e., $\phi = \phi(x_m)$.

Given our definition of the pricing kernel, we find, rewriting equation (1.2), that the forward price of the asset $j$ is

\begin{equation}
F_j = \sum_i p_i x_{ij} = \sum_i p_i \phi(x_m) x_{ij} = \mathbb{E}\left[\phi(x_m) x_{ij}\right]
\end{equation}

It follows that the case where $\phi_i = 1$, for all $i$, is of particular significance. In this case we would have:

\begin{align*}
F_j &= \sum_i p_i x_{ij} \\
&= \sum_i p_i \phi x_{ij} \\
&= \mathbb{E}(x_j)
\end{align*}

Here, the forward price equals the expected value of the cash flow. This occurs if the cash flow can be priced under the assumption of risk neutrality. Hence the case where $\phi_i = 1$, for all $i$, equates to the case of risk neutrality.

In order to appreciate the importance of the pricing kernel, consider the following expansion of equation (1.3). Using the definition of covariance, the forward price is

\begin{align*}
F_j &= \mathbb{E}\left[\phi(x_m) x_{ij}\right] \\
&= \mathbb{E}[\phi(x_m)] \mathbb{E}(x_j) + \text{cov}[\phi(x_m), x_{ij}]
\end{align*}

and given that $\mathbb{E}[\phi(x_m)] = 1$, we have

\begin{equation}
F_j = \mathbb{E}(x_j) + \text{cov}[\phi(x_m), x_{ij}]
\end{equation}

It follows that the behaviour of $\phi$, in particular its covariance with the cash flow $x_{ij}$, determines the risk premium for the asset, which is represented by the excess of the expected value of the cash flow over its forward price. In most cases, as we will see in Chapter 2, it turns out that $\phi(x_m)$ is negatively correlated with $x_{ij}$, in which case the risk premium is positive.

1.5 The Capital Asset Pricing Model

In this section, we illustrate the generality of the pricing kernel approach by deriving a version of the CAPM. The CAPM can be derived either by assuming that the pricing kernel is a linear function of $x_m$, or by assuming that the firm's cash flow
ASSET PRICES IN A SINGLE-PERIOD MODEL

and the aggregate market cash flow are joint-normally distributed. Here we take the latter approach.

Assume that the function \( \varphi(x_m) \) is differentiable with \( \varphi'(x_m) < 0 \), as in Fig. 1.3 and that \( x_j, x_m \) are joint-normally distributed. It then follows from Stein’s lemma (see appendix at the end of the book) that:

\[
\begin{align*}
(1.4) \\
F_j &= E(x_j) E[\varphi(x_m)] - \kappa \text{cov}(x_j, x_m) \\
&= E(x_j) - \kappa \text{cov}(x_j, x_m).
\end{align*}
\]

(p.8) since 
\( E[\varphi(x_m)] = 1 \), where 
\( \kappa = -E[\varphi(x_m)] \)
This is a cash flow version of the well-known CAPM.

The more familiar rate of return version of the CAPM follows in a few steps from (1.4).

First, we apply the model to find the forward price of the market cash flow, \( x_m \). This is given by

\[
(1.5) \\
F_m = E(x_m) - \kappa \text{var}(x_m),
\]
where \( F_m \) is the forward price of the market portfolio cash flow, \( x_m \). Rearranging (1.5), we get the market price of risk,

\[
(1.6) \\
\kappa = -\frac{F_m + E(x_m)}{\text{var}(x_m)}.
\]

The forward price of asset \( j \) is then, substituting in (1.4),

\[
F_j = E(x_j) - \left[ -\frac{F_m + E(x_m)}{\text{var}(x_m)} \right] \text{cov}(x_m, x_j).
\]

Dividing both sides by the forward price, \( F_j \), we obtain

\[
1 = \frac{E(x_j)}{F_j} - \left[ \frac{E(x_m) - F_m}{\text{var}(x_m)} \right] \text{cov}(x_m, x_j/F_j).
\]

Rearranging this equation gives

\[
(1.7)
\]
Finally, if we denote β_j = \frac{\text{cov}(x_m/F_m, x_j/F_j)}{\text{var}(x_m/F_m)} as the beta of x_j with respect to the market portfolio, then,

\[
\frac{E(x_j) - F_j}{F_j} = \beta_j \left( \frac{E(x_m) - F_m}{F_m} \right)
\]

This is a forward version of the standard CAPM. It says that the risk premium of a stock is the beta of the stock times the risk premium on the market.

Now, in order to derive the more familiar spot version of the CAPM, substitute the forward price \( F_j = S_j (1 + r_f) \), using spot-forward parity, where \( B_{j,t-1} = (1 + r_f) \) and let \( S_m \) be the spot (p.9) value of the market portfolio. Then, we have:

\[
\frac{E(x_j)}{S_j} - (1 + r_f) = \beta_j \left( \frac{E(x_m)}{S_m} - (1 + r_f) \right)
\]

Finally, denoting the returns on \( S_j \) and \( S_m \) as \( r_j \) and \( r_m \), respectively, we find that

\[
E(r_j) - r_f = \beta_j [E(r_m) - r_f]
\]

This is the more commonly seen spot version of the CAPM.

1.6 The Arbitrage Pricing Theory

In this section we apply the pricing kernel approach to derive a version of the arbitrage pricing theory (APT). We assume that the cash flow of firm \( j \), \( x_j \) is a linear function of a set of factors. For example, we assume there are \( K \) factors and for factor \( f_k \) the factor loading is \( \beta_{jk} \). In this case

\[
x_j = a_j + \sum_{k=1}^{K} \beta_{jk} f_k + \varepsilon_j
\]

where \( \varepsilon_j \) is independent of \( f_k \). We then have:

\[
\text{cov}[x_j, \varphi(x_m)] = \sum_{k=1}^{K} \beta_{jk} \text{cov}[f_k, \varphi(x_m)] + \text{cov}[\varepsilon_j, \varphi(x_m)]
\]

Note that this is merely an expansion of the covariance term into \( K \) covariances with the underlying factors, plus a residual covariance. However, if one of the following conditions holds, an economically meaningful decomposition follows. The conditions are:

(i) there is no idiosyncratic risk, \( \varepsilon_j = 0 \); or
(ii) the idiosyncratic risk is not related to the pricing kernel, \( \text{cov}[\varepsilon_j, \varphi(x_m)] = 0 \).

If \( \varepsilon_j = 0 \) or \( \text{cov}[\varepsilon_j, \varphi(x_m)] = 0 \), then
Here, the risk premium $E(x_j) - F_j$ is the sum of $K$ risk premia.

This is a version of the APT model. Note that the APT is a paradigm which is somewhat different from the CAPM. For the CAPM we need either quadratic utility or the joint-normal distribution. These assumptions are not required for the APT provided that $\text{cov}[\epsilon_j, \phi(x_m)] = 0$ or $\epsilon_j = 0$. In other words; (i) idiosyncratic risk is not priced; or (ii) $x_j$ is a fully diversified portfolio.

1.7 Risk Aversion and the Pricing Kernel in an Equilibrium Model

So far we have worked with the pricing kernel $\phi_i$, with no underlying model of the determinants of this crucial variable. We now derive one such model. Equilibrium models assume that investors maximise expected utility and derive an equilibrium in which markets clear, i.e. there is zero excess demand for all assets. In this section, we simplify the model somewhat, by assuming that there is only one investor in the economy. An alternative, equivalent assumption is that the market acts as if there is just one investor with ‘average’ characteristics. This is often referred to as the ‘representative agent’ assumption.7

Let $w_{t+T,i}$ be the wealth of the investor in the state $i$ at time $t + T$. Assume that the investor is endowed with investible wealth $w_t$ at time $t$, in the form of cash. The investor can purchase state-contingent claims which pay $1$, if and only if the state $i$ occurs at time $t + T$. The price of the claims are $q_i$ for $i = 1, 2, \ldots, I$. The investor’s problem is to choose a set of state-contingent claims paying $w_{t+T,i}$, given a budget allocation of cash, $w_t$.

We make the following additional assumptions:

1. The investor maximises the expected value of a utility function $u(w_{t+T})$. Hence the investors problem is:

$$\max_{w_{t+T,i}} \mathbb{E}[u(w_{t+T})] = \sum_i p_i u(w_{t+T,i})$$

subject to

$$\sum_i w_{t+T,i} \beta_i = w_t$$

$$\text{cov}[x_i, \phi(x_m)] = \sum_{i=1}^N \beta_i \text{cov}[f_i, \phi(x_m)]$$
2. The utility function has the properties $u'(w_{t+T}) > 0$ (non-satiation) and $u'(w_{t+T}) < 0$ (risk aversion).

**p.11** The first assumption follows from the more basic assumption of rational choice. The second assumption guarantees that fulfilling the first-order conditions leads to an optimal and unique solution. Note that the discount factor enters the budget constraint because the $q_i$ are forward prices, whereas the given cash wealth $w_t$ is a time $t$ allocation.

We solve the optimisation problem by forming the Lagrangian:

$$L = \sum_i p_i u(w_{t,T}) + \lambda (w_t B_{i,t} - \sum_i q_i w_{t,T})$$

Then the first-order conditions for a maximum are:

$$\frac{\partial L}{\partial w_{t,T}} = p_i u(w_{t,T}) - q_i \lambda = 0. \quad (1.9)$$

Summing equation (1.9) over the states $i$ we then find

$$\sum_i p_i u(w_{t,T}) = \sum_i q_i,$$

or

$$E[u(w_{t,T})] = \lambda,$$

since $\sum_i q_i = 1$. Now, substituting for $\lambda$ in (1.9), the first-order condition becomes

$$\frac{p_i u(w_{t,T})}{E[u(w_{t,T})]} = q_i,$$

or

$$q_i = \frac{p_i}{E[u(w_{t,T})]} u(w_{t,T}).$$

In this model, a condition for the investor’s expected utility to be maximised is that the pricing kernel equals the ratio of marginal utility in a state to the expected marginal utility. To complete the model, we need to determine the investor’s wealth at time $t + T$, in each state. However, in equilibrium the single investor’s demand for state-contingent claims must equal the available supply. Hence $w_{t+T,i}$ must equal $x_{m,i}$, the aggregate market cash flow in, state $i$. Substituting in the expression for the pricing kernel, we conclude that

$$\phi_i = \frac{u(x_{m,i})}{E[u(x_{m})]},$$

for all $i$. Hence, we have

$$\phi = \phi(x_{m}),$$

as assumed earlier in the chapter. Since marginal utility is a positive function of $x_m$ and we may assume $u'(x_m) < 0$, it follows that $\phi(x_m)$ is a declining function of $x_m$ as assumed in Fig. 1.3.
1.8 Examples

1.8.1 Case 1: Risk Neutrality

A risk-neutral investor is one who has a linear utility function

\[ u(w_{r_{t+T}}) = a + bw_{r_{t+T}} \]

where \( a \) and \( b \) are constants. Then differentiating the utility function

\[ u(w_{r_{t+T}}) = E[u(w_{r_{t+T}})] = b \]

and the pricing kernel is therefore

\[ \phi(w_{r_{t+T}}) = \frac{u(w_{r_{t+T}})}{E[u(w_{r_{t+T}})]} = 1. \]

In this case, the forward price is

\[ F_{t,t+T} = E(w_{x}) = E(1 \times x) = E(F_{t+T,t+T}). \]

In this example of risk neutrality, \( F_{t,t+T} = E(F_{t+T,t+T}) \) has the martingale property.

1.8.2 Case 2: Utility is Quadratic

Assume utility is given by:

\[ u(w_{r_{t+T}}) = a + bw_{r_{t+T}} + \delta w_{r_{t+T}}^2, \]

where \( b > 0 \), \( \delta < 0 \). In this case, marginal utility is

\[ u(w_{r_{t+T}}) = b + 2\delta w_{r_{t+T}}, u > 0 \]

(p.13) and the pricing kernel is given by

\[ \phi(w_{r_{t+T}}) = \frac{u(w_{r_{t+T}})}{E[u(w_{r_{t+T}})]} = \frac{b + 2\delta w_{r_{t+T}}}{b + 2\delta E(w_{r_{t+T}})}. \]

We then have

\[ \text{cov}\{\phi(w_{r_{t+T}}), x_{j,t+T}\} = \frac{2\delta \text{cov}(w_{r_{t+T}}, x_{j,t+T})}{b + 2\delta E(w_{r_{t+T}})}, \]

and the forward price of \( x_{j,t+T} \) is

(1.10)

\[ F_{j,t+T} = E(x_{j,t+T}) + \kappa \text{cov}(w_{r_{t+T}}, x_{j,t+T}), \]

where

\[ \kappa = \frac{2\delta}{b + 2\delta E(w_{r_{t+T}})} \]

is a constant. Equation (1.10) is a version of the CAPM. Earlier, the CAPM was derived under the assumption that wealth and asset prices were joint normal. This example illustrates that quadratic utility is an alternative sufficient condition. In this case, the pricing kernel is linear in wealth.

1.9 A Note on the Equivalent Martingale Measure

We noted above that the set of forward state prices \( \{q_i\} \) is a probability measure. In the literature it is often referred to as the Equivalent Martingale Measure, or simply EMM. Since this measure will be used extensively in later chapters, we now include a brief explanation of this terminology.
Let \( P = \{ p_i \} \) and \( Q = \{ q_i \} \) be two probability measures. \( P \) and \( Q \) are equivalent if \( q_i > 0 \) if and only if \( p_i > 0 \). Let \( E^P(.) \) and \( E^Q(.) \) be expectations under the probability measures, \( P \) and \( Q \), respectively. From equation (1.2), dropping the \( j \) subscript:

\[
F = \sum q_i x_i = E^Q(x)
= \sum p_i \varphi(x) = E^P(\varphi(x)).
\]

**p.14** Now rewrite the forward price, \( F \), as \( F_{t,T} \). Also, note that the time \( t + T \) spot price, \( x \), can be expressed as \( F_{t+T,t+T} \). This is because the forward price at \( t + T \) for immediate delivery, is simply the spot price at \( t + T \). Hence

\[
F_{t,T} = E^Q(F_{t+T,t+T}).
\]

If such a relationship holds, the variable is said to have the Martingale property, and \( Q \) is therefore referred to as the EMM.\(^9\) In the literature, \( E^Q \) is often used loosely as the risk-neutral measure, since it has the same property that the true measure would have under risk neutrality, the case discussed in Section 1.8.1.

1.10 A Note on the Asset Specific Pricing Kernel, \( \psi(x_j) \)

The asset specific pricing kernel was introduced by Brennan (1979) and is important in the analysis of option pricing. It is the expected value of the pricing kernel \( \varphi(x_m) \) given the outcome of \( x_j \). For asset \( j \), we can write the forward price

\[
F_j = E_{\varphi}[\psi(x_j)x_j] = E_{\varphi}[E_{\varphi}[\psi(x_m)]x_j].
\]

where the notation \( \varphi \) indicates expectation over values of \( x \).

Defining \( \psi(x) = E_{\varphi}[\psi(x_m)] \), we have

\[
F_j = E_{\varphi}[\psi(x_j)].
\]

Two properties of the asset-specific pricing kernel are important and will be used in later chapters. First, it follows directly from \( \varphi(x_m) > 0 \), that \( \psi(x_j) > 0 \). Second, \( E[\psi(x_j)] = 1 \). This property follows from the fact that the forward price of a non-stochastic cash flow, must be the the cash flow itself.

Alternatively, directly we have

\[
E_{\varphi}[E_{\varphi}[\psi(x_m)]x_j] = E[\psi(x_m)] = 1.
\]

**p.15** 1.11 Conclusions

We have used a complete markets approach to derive asset forward prices in a one-period model. Readers can compare our approach to that of other well-known texts on financial theory. It is closely related to the model developed by Huang and Litzenberger (1988). However, for the most part our approach does not require an equilibrium, although such an
equilibrium could be one possible way for the pricing kernel to be determined. The CAPM derived in this chapter is incomplete in one respect. Under the assumption of joint normality we have derived a linear relationship between the risk premium on the stock and the risk premium on the market. However, unless we have an equilibrium model, we do not know that the risk premium on the market is positive.

Our use of the pricing kernel is similar in many respects to the use of a ‘stochastic discount factor’ in Cochrane (2001). However, as in Huang and Litzenberger (1988), the emphasis in Cochrane is on a consumption based CAPM. Our approach is closer to that used by Pliska (1997). The difference is that Pliska starts at the more basic level of a no-arbitrage economy. His more mathematical treatment is targeted at deriving prices in incomplete markets.

**Exercises**

1.1. (a) Use the definition of covariance to show that

\[ E(xy) = E(x)E(y) + \text{cov}(x, y) \]

(b) Show, using a numerical example, that

\[ E(xy) = E[xE(y|y)] \]

1.2. Assume there are 3 states with probabilities \(0.3, 0.4, 0.3\). Suppose the corresponding state prices are \((0.36, 0.38, 0.26)\). Illustrate the probability distribution of the pricing kernel.

1.3. Assume the following joint probability distribution of \((x, \varphi)\):
<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>0.6</th>
<th>0.8</th>
<th>1.2</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.4</td>
<td>0.1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0.1</td>
<td>0.4</td>
</tr>
</tbody>
</table>
Compute the forward price of the asset using \( F = E(\varphi x) \) and using \( F = E[xE(\varphi | x)] \).

1.4. Show that the pricing kernel in an economy of risk-neutral investors is always 1 and in this economy \( E^P = E^Q \).

1.5. Assume that utility is cubic:
\[
u(w_{i,t}) = a + bw_{i,t} + \delta w_{i,t}^2 + \gamma w_{i,t}^3\]
Compute \( \varphi \) and \( \text{cov}(\varphi, x) \), and derive a CAPM relating the forward price, \( F \), to the expected payoff, \( E(x) \).

1.6. Explain the significance of each of the assumptions made in section 1.1. Which of the assumptions is strong in the sense of ‘not being likely to be true in practice’. How could the theory be generalised by relaxing these assumptions?

1.7. Assume that three firms produce cash flows: \( x_1, x_2, x_3 \) each with possible outcomes \((1, 0)\), with probability 0.5 and assume the cash flows are independent. Assume that the pricing kernel, \( \varphi(x_m) \) has the following values, \( \varphi(0) = 1.8, \varphi(1) = 1.2, \varphi(2) = 0.8, \varphi(3) = 0.2: \)

(a) Show the distribution of \( q_i \).
(b) Show that \( E(\varphi) = 1 \).
(c) Compute the forward price of cash flow \( x_1 \).

1.8. Assume that an investor has power utility with
\[
u(w) = \left( \frac{w}{1 - \gamma} \right)^{-\gamma-1},
\]
with \( \gamma < 1, \gamma \neq 0 \).
Assume that there are only three states of the world, \( i = 1,2,3 \). Write out the maximisation of expected utility problem for the investor. Show the first-order conditions for a maximum.

Notes:
(1) In practice, it may not be possible to directly purchase such state-contingent claims. However, if put and call option contracts on an asset can be purchased at all strike prices, then effectively a complete market exists for claims on the asset. Portfolios of puts and call can be formed to replicate the contingent-claim payoffs.
(2) The assumption of a finite state space could be relaxed to permit an infinite state space, while retaining the complete markets assumption. For such a generalisation, see the proof in Nachman (1982), where he assumes digital options are traded at all strike prices.

(3) This is an implication of what is known as the Law of One Price, see for example Cochrane (2001), chapter 4, or Pliska (1997), chapter 1.

(4) For further reading on risk-neutral measures, see Williams (1991).

(5) If deflated state prices depend upon the marginal utility for consumption in a state, as in the equilibrium model derived later, then they will depend on aggregate market cash flows rather than on the composition of the aggregate cash flow.

(6) See Exercise 1(a). The covariance between two variables $X$ and $Y$ is given by

$$\text{cov}(X, Y) = E[(X - E(X))(Y - E(Y))]$$

(7) See, for example, Huang and Litzenberger (1988), Chapter 5.

(8) This follows from the Von Neuman–Morgenstern expected utility theorem, see Fama and Miller (1972). Basically, it states that if the investor behaves according to five axioms of choice under uncertainty, then maximising expected utility should always lead to maximising utility and hence to an optimal investment choice. The five axioms govern the comparability, transivity, independence, certainty equivalence, and ranking of choices.

(9) The concept of EMM and the use of this probability measure were made popular by Harrison and Kreps (1979). For a detailed discussion see Williams (1991).
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