Abstract and Keywords

The chapter describes some asymmetric information models of liquidity. In these models, trades move prices because of the possibility that the trades are based on information not known to the market. A strategic trader is one who takes into consideration that her trades move prices. The chapter describes the Glosten-Milgrom model of the bid-ask spread, the Kyle model of market depth, the Glosten model of limit-order markets, and models of auctions. Except for the auction models, prices are set in these models by uninformed market makers who face adverse selection from informed traders. In the auction models, prices are set by informed individuals bidding against one another. The winner’s curse, the revenue equivalence theorem, and related aspects of auctions are explained.

Keywords: asymmetric information, liquidity, Glosten-Milgrom model, Kyle model, Glosten model, auctions, strategic trading, winner’s curse, revenue equivalence theorem

Contents
Traders measure their transaction costs in four buckets: commissions, bid-ask spreads, price impacts, and opportunity costs. Commissions are payments to securities brokers for executing trades. The bid-ask spread is the difference between the ask quote and the bid quote, the former being the price at which a trader can buy with a market order (an order to trade that requests execution at the market price) and the latter being the price at which a trader can sell with a market order. For traders who want to trade large quantities, price impacts and opportunity costs are the largest costs. Price impact is how much the market has moved between the time a decision to trade is made and a trade is executed. Opportunity costs are lost profits on trades that were never executed because the market moved too far and the trader decided to cancel her remaining plans to trade.

Traders who are concerned with the price impacts of their trades are called strategic. They are not price takers. Studying strategic traders takes us outside the competitive paradigm that has been employed in all previous parts of this book. Large investors in actual markets are well aware that their trades move prices, and they exert a great deal of effort in attempting to minimize adverse price impacts. So, strategic trading is an important topic.

1 The magnitudes of bid-ask spreads, price impacts, and opportunity costs depend on the liquidity of the security being traded. Or, rather, they are the same thing as liquidity. If a security is very liquid, then it can be traded at low cost. If it is illiquid, costs are high. This is effectively a definition of liquidity. One cause of illiquidity is asymmetric information. Asymmetric information gives rise to adverse selection. If you offer to trade at a given price, your offer is more likely to be accepted (selected) if other investors have information that the price is favorable to them and consequently unfavorable (adverse) to you.
This chapter is about asymmetric information models of liquidity. It consists of elaborations of the explanation of liquidity given by Treynor (1971)—published under the pseudonym Walter Bagehot—who observes that “the liquidity of a market ... is inversely related to the average rate of flow of new information ... and directly related to the volume of liquidity-motivated transactions.” As elsewhere in this part of the book, we only try to describe some basic models. These models are part of the branch of finance called market microstructure. However, many issues in market microstructure (prominently including high-frequency trading, dark pools, make-or-take fees, over-the-counter markets, etc.) are entirely omitted from this chapter.

Section 24.4 on auctions differs a bit from the rest of the chapter. In the models in the rest of the chapter, prices are set by uninformed traders (market makers) who face adverse selection from informed traders. Section 24.4 considers a trade in which the price is set by informed individuals bidding against one another. Competition by informed traders who have common values produces the winner’s curse, which is a phenomenon with which any student of markets should be familiar. Furthermore, auctions are widely used to sell new issues of securities (for example, government bonds) and are an important prototype for other models in which price-quoting agents are informed and compete (for example, dealer markets).

24.1 Glosten-Milgrom Model
Glosten and Milgrom (1985) study a model in which risk-neutral competitive market makers set bid and ask quotes for transacting a single unit of an asset with a trader who submits a market order. The market-order trader may be informed or she may be trading for other reasons. If trading for other reasons—for example, selling the asset because she is in need of cash for some other purpose—then the trader is commonly called a liquidity trader, consistent with Treynor’s reference to liquidity-motivated transactions.

In the simplest setting, the asset has only two possible values, $L < H$, and the informed trader knows the value and buys if the value is $H$ and sells if the value is $L$, and the liquidity trader buys or sells with equal probabilities regardless of the quotes
(her demands are price inelastic). Adopt the following notation:

\[ p = \text{prob}(H), \]
\[ \mu = \text{prob}(\text{Informed Order}), \]
\[ n_b = \text{prob}(\text{Buy Order}) = p\mu + (1 - \mu)/2 \]
\[ n_s = \text{prob}(\text{Sell Order}) = (1 - p)\mu + (1 - \mu)/2 \]

\[ p_b = \text{prob}(t|\text{Buy Order}) = \frac{\text{prob}(t|\text{Buy Order})\text{prob}(t)}{\text{prob}(\text{Buy Order})} = \frac{\mu(1-p)^2}{n_b} \]

\[ p_s = \text{prob}(t|\text{Sell Order}) = \frac{\text{prob}(t|\text{Sell Order})\text{prob}(t)}{\text{prob}(\text{Sell Order})} = \frac{(1-p)^2}{n_s}. \]

Note that beliefs are revised in the directions of orders in the sense that the probability of \( H \) increases with a buy order and the probability of \( L \) increases with a sell order—that is, \( p_b > p > p_s \).

The expected value of the asset conditional on a buy order is

\[ A \overset{\text{def}}{=} \mathbb{E}[v|\text{Buy Order}] = p_bH + (1 - p_b)L. \] (24.1)

The expected value conditional on a sell order is

\[ B \overset{\text{def}}{=} \mathbb{E}[v|\text{Sell Order}] = p_sH + (1 - p_s)L. \] (24.2)

Competition between risk-neutral market makers forces the quotes to the expected value of the asset conditional on the order. Thus, (24.1) is the ask price (the price that dealers ask in exchange for selling the asset) and (24.2) is the bid price (the price that dealers bid for the asset). The bid and the ask are regret-free quotes, in the sense that they anticipate the adverse selection of informed market order traders. Each quote already incorporates the information provided by a market order that hits it.

The difference between the bid and the ask is the bid-ask spread \( A - B = (p_b - p_s)(H - L) \). The spread is larger in this model when the high and low values \( H \) and \( L \) are further apart and when orders are more informative in the sense that the posterior probabilities \( p_b \) and \( p_s \) are further apart. Thus, the market is less liquid \( (p.616) \) when private information is larger and/or orders are more informative. Notice that

\[ \pi_1A + \pi_2B = pH + (1 - p)L. \] (24.3)

The left-hand side is the expected transaction price (an expectation of a conditional expectation of the asset value),
and the right-hand side is the unconditional expectation of the asset value. Thus, (24.3) is an example of the law of iterated expectations.

In a dynamic version of the model, transaction prices form a martingale, because the transaction price (ask or bid) is always the conditional expectation of the asset value, given the information at the time, and a sequence of conditional expectations is always a martingale. Because transaction prices form a martingale, changes in transaction prices are a martingale difference series and hence are uncorrelated. Changes in transactions prices are correlated in some other models of the bid-ask spread. See the end-of-chapter notes.

24.2 Kyle Model
In the Kyle model, there is a strategic informed trader. Illiquidity in the model is measured by what is called Kyle’s lambda. Kyle’s lambda depends in a very simple way on model parameters that measure private information and liquidity trading. The market is more liquid when there is less private information and/or more liquidity trading.

Model
There are two dates, 0 and 1. The asset is traded with asymmetric information at date 0, and the asset value \( \tilde{y} \) is realized at date 1. Assume \( \tilde{v} \) is normally distributed and observed by a single risk-neutral informed trader prior to trade at date 0. After observing \( \tilde{v} \), the informed trader submits a market order \( \tilde{x} \). There are also uninformed (liquidity) trades represented by a random variable \( \tilde{z} \) that is normally distributed and independent of \( \tilde{v} \). No generality is lost in assuming the mean of \( \tilde{z} \) is zero.

Market makers observe \( \tilde{y} \) defined as \( \tilde{x} + \tilde{z} \). This is the aggregate order. If \( \tilde{x} \) and \( \tilde{z} \) have opposite signs (one is buying and the other selling), then the interpretation is that they trade with each other and only the residual is seen by market makers. However, all trades take place at the same price, which is the price set by market makers after observing \( \tilde{x} + \tilde{z} \). Use a risk-free asset as the numeraire, and assume market makers are risk neutral and compete in a Bertrand fashion to fill the aggregate order. Denote the equilibrium price by \( p(\tilde{y}) \).

Equilibrium
An equilibrium of this model is an informed order $\bar{x}$ depending on $\bar{v}$ and a price function $p$ satisfying

\[ p(y) = E[\psi] \]

(24.4a)

\[ \bar{x} \in \arg\max_x x(\bar{v} - E[p(x + \bar{z})]). \]

(24.4b)

The first condition states that the price equals the expected asset value conditional on the information in the aggregate order. The second condition states that the informed trader maximizes her conditional expected gain from trade, understanding that her order affects the price.

An equilibrium is said to be linear if there are constants $\delta$, $\lambda$, $\alpha$, and $\beta$ such that $p(y) = \delta + \lambda y$ and $\bar{x} = \alpha + \beta \bar{v}$. Denote the standard deviations of $\bar{v}$, $\bar{z}$, and $\bar{y}$ by $\sigma_v$, $\sigma_z$, and $\sigma_y$ respectively. Denote $E[\bar{v}]$ by $\bar{v}$. There is a unique linear equilibrium given by

\[ \delta = v, \]

(24.5a)

\[ \lambda = \frac{\sigma_y}{2 \sigma_z}, \]

(24.5b)

\[ \alpha = -\delta \beta, \]

(24.5c)

\[ \beta = \frac{1}{2 \lambda}. \]

(24.5d)

We verify that this is the unique linear equilibrium at the end of the section.

**Kyle’s Lambda**

The parameter $\lambda$ is universally denoted by this symbol. In fact, it is universally known as Kyle’s lambda. It measures the impact on the equilibrium price of a unit order. Its reciprocal is the size of the trade that can be made with a unit impact on the price. A market in which large trades can be made with only a small price impact is called a deep (or liquid) market, so $1/\lambda$ measures the depth of the market: If $1/\lambda$ is larger, then the market is deeper. Note that a market is deeper if there is less private information in the sense of $\sigma_v$ being smaller or if there is more liquidity trading in the sense of $\sigma_z$ being larger. Thus, the formula for lambda is consistent with Treynor’s description of liquidity quoted in the introduction to the chapter.

**Information Revelation**
The variance of $\tilde{y}$ measures the ex ante informational advantage of the informed trader. For example, if $\sigma_0$ is large, then it will frequently be the case that the informed trader has an important informational advantage, in the sense that her estimate $\tilde{v}$ of the asset value is quite far from the value $\tilde{v}$ perceived ex ante by market makers. The information revealed to market makers by the order flow $\tilde{y}$ in the linear equilibrium is very simply described: The variance of $\tilde{y}$ conditional on $\tilde{y}$ is half of the unconditional variance. Moreover, the equilibrium price, because it is affine in $\tilde{y}$, reveals the same information. Thus, the market at large learns half of the private information of the informed trader. This is verified below.

**Value of Private Information**

Notice that the equilibrium strategy of the informed trader is $\tilde{x} = \beta(\tilde{v} - \tilde{v})$. The unconditional expected gain of the informed trader is

$$
\mathbb{E}[\tilde{x}(\tilde{v} - p(\tilde{x} + \tilde{v}))] = \beta \mathbb{E}[\tilde{v} - \tilde{v} - \lambda(\tilde{v} - \tilde{v})]
$$

$$
= \beta(1 - \lambda)\sigma_0^2.
$$

Thus, the informed trader’s expected profit is higher when she has more private information or when there is more liquidity trading. The expected gain of market makers is zero, because the price at which they trade is the conditional expected \textit{value} of the asset. Thus, the expected profits of the informed traders are expected losses for the liquidity traders. In fact, the expected gain of liquidity traders is

$$
\mathbb{E}[\tilde{y}(\tilde{v} - p(\tilde{x} + \tilde{v}))] = -\lambda \sigma_0^2 = -\frac{\alpha \sigma_0^2}{2}.
$$

The liquidity traders are presumably willing to accept these losses, due to their unmodeled motives for trading.

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We want to establish that (24.1) is the unique linear equilibrium, and we want to verify the statements made above about information revelation. Suppose the informed trade is $\tilde{x} = \alpha + \beta \tilde{v}$ for some $\alpha$ and $\beta$. Then,

$$
\mathbb{E}[\tilde{x}(\tilde{v} + \tilde{z})] = \mathbb{E}[\tilde{v}(\tilde{v} - \tilde{v}) + \tilde{z}]
$$

$$
= \tilde{v} + \frac{\text{cov}(\tilde{v}, \tilde{v})}{\text{var}(\tilde{v})}(\beta \tilde{v} - \beta \tilde{v} + \tilde{z})
$$

$$
= \tilde{v} - \frac{\sigma_0^2}{\beta^2 \sigma_0^2} (\alpha + \beta \tilde{v}) + \frac{\sigma_0^2}{\beta^2 \sigma_0^2} (\alpha + \beta \tilde{v} + \tilde{z}).
$$

Thus, in a linear equilibrium, we must have
\[ \lambda = \frac{\beta \sigma_v^2}{\beta^2 \sigma_x^2 + \sigma_v^2}, \]
\[ \delta = \nu - \lambda (\alpha + \beta \nu). \]

(24.6a)
(24.6b)

On the other hand, if \( p(y) = \delta + \lambda y \) for any \( \delta \) and \( \lambda \), then the informed trader’s optimization problem is to maximize

\[ \bar{\nu} - \bar{x}[\delta + \lambda \bar{x} + \lambda \mathbb{E}[x]] = (\bar{\nu} - \delta) \bar{x} - \lambda \bar{x}^2. \]

There is a solution to this problem only if \( \lambda > 0 \), and, in that case, the solution is

\[ \bar{x} = \frac{\bar{\nu} - \delta}{2\lambda}. \]

Thus, in a linear equilibrium, we must have

\[ \lambda > 0, \]

(24.6c)
\[ \alpha = -\frac{\delta}{2\lambda}, \]

(24.6d)
\[ \beta = \frac{1}{\lambda}. \]

(24.6e)

We will show that the unique solution of the system (24.3) is (24.2).

(p.620) Substitute (24.6a) into (24.6e) to obtain

\[ \beta = \frac{1}{\lambda} \left( \frac{\sigma_v^2 - \sigma_x^2}{\beta \sigma_v^2} \right), \]

so \( \beta^2 = \sigma_x^2 / \sigma_v^2 \). From \( \lambda > 0 \) and (24.6e), we have \( \beta > 0 \). Hence, \( \beta = \sigma_x / \sigma_v \) as claimed in (24.5d). Now, substitute \( \beta \) into (24.6e) to obtain the formula claimed for \( \lambda \), and substitute \( \beta \) and \( \lambda \) into (24.6b) and (24.6d) to obtain the formulas claimed for \( \alpha \) and \( \delta \).

The remaining task is to compute the variance of \( \tilde{\nu} \) conditional on \( \tilde{y} \). According to (22.5), the conditional variance is

\[ [1 - \text{corr}(\tilde{\nu}, \tilde{y})] \sigma_{\tilde{\nu}}^2. \]

The correlation is

\[ \frac{\sigma_{\tilde{\nu}}(\tilde{\nu}, \tilde{y})}{\sigma_{\tilde{\nu}} \sigma_{\tilde{y}}} = \frac{\sigma_{\tilde{\nu}}}{\sigma_{\tilde{y}} \sqrt{\sigma_{y}^2 - \sigma_{\tilde{y}}^2}} = \frac{1}{\sqrt{\lambda}}. \]

Therefore, the conditional variance is \( \sigma_{\tilde{\nu}}^2 / 2 \).
24.3 Glosten Model of Limit Order Markets

Most modern exchanges are organized as limit order markets. A buy limit order is an order to buy at a price not exceeding the limit price specified in the order, and a sell limit order is an order to sell at a price not less than the limit price specified in the order. The collection of limit orders residing at an exchange is called the exchange’s book of limit orders. Exchanges respect price priority—a bid to buy at a higher price executes before a bid to buy at a lower price, and an offer to sell at a lower price executes before an offer to sell at a higher price. Exchanges also respect time priority—among limit orders at the same price, the first to arrive executes first.  

If an order arrives that can execute immediately against another order or orders already in the book, then it is called a marketable limit order. The execution price is the limit price of the order that is already in the book. To illustrate the concept, Table 24-1 presents a simple example of a possible book in a stock. This stock is trading at $49.96 bid, $49.97 ask.

<table>
<thead>
<tr>
<th>Price</th>
<th>Orders to Buy</th>
<th>Orders to Sell</th>
</tr>
</thead>
<tbody>
<tr>
<td>$50.00</td>
<td>2,000</td>
<td></td>
</tr>
<tr>
<td>$49.99</td>
<td>2,500</td>
<td></td>
</tr>
<tr>
<td>$49.97</td>
<td>500</td>
<td></td>
</tr>
<tr>
<td>$49.96</td>
<td>1,000</td>
<td></td>
</tr>
<tr>
<td>$49.95</td>
<td>500</td>
<td></td>
</tr>
<tr>
<td>$49.93</td>
<td>2,000</td>
<td></td>
</tr>
</tbody>
</table>

If a limit order to buy 1,500 shares at $49.99 arrives, then it will execute against the shares offered at $49.97 and against shares offered at $49.99, with 500 shares trading at $49.97 and 1,000 shares trading at $49.99. After execution of the order, the new book will be as shown in Table 24-2.

Table 24-2 The Limit Order Book after Execution of a Marketable Buy Order.
<table>
<thead>
<tr>
<th>Price</th>
<th>Orders to Buy</th>
<th>Orders to Sell</th>
</tr>
</thead>
<tbody>
<tr>
<td>$50.00</td>
<td>2,000</td>
<td>2,000</td>
</tr>
<tr>
<td>$49.99</td>
<td>1,500</td>
<td>1,500</td>
</tr>
<tr>
<td>$49.96</td>
<td>1,000</td>
<td>1,000</td>
</tr>
<tr>
<td>$49.95</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>$49.93</td>
<td>2,000</td>
<td>2,000</td>
</tr>
</tbody>
</table>

The stock is now trading at $49.96 bid, $49.99 ask. The now larger gap between the bid and ask price may attract new orders to the market.

Glosten (1994) analyzes a limit order market in which there is perfect competition among uninformed market makers in posting buy and sell limit orders and then a marketable limit order arrives that may or may not be informed. This is similar to the Glosten-Milgrom model except that there is only one possible order size in the Glosten-Milgrom model, so only the limit orders at the best quotes (the bid and ask) are relevant. In contrast, the Glosten (1994) model allows for multiple order sizes and solves for the entire book of limit orders.

A trader who submits a marketable order into the limit order book faces nonlinear pricing. Consider a buy order. As in the simple example above, the order will “walk up the book,” executing at successively higher prices. The cost of the 1,500 share buy order in the example is

\[
\text{(p.622)}
\]

where

\[
P(x) = \begin{cases} 
50.00 & \text{for } 3,000 < x \leq 5,000, \\
49.99 & \text{for } 500 < x < 3,000, \\
49.97 & \text{for } 0 < x \leq 500.
\end{cases}
\]

We say that \( P(x) \) is the limit price at depth \( x \). In this example, the cost of the order is the amount of money transferred from the trader who submitted the marketable order to the traders whose limit orders were already in the book. The transfer function for market buy orders is defined in general as

\[
T(q) = \int_{0}^{q} P(x) dx.
\]
The same definition of the transfer function applies for marketable sell orders, which we regard as negative in sign. The monetary transfer from the trader who submits the marketable order is also negative, representing a payment to the trader. Consider a sell order for 2,000 shares in the example above. We can calculate the associated transfer as

\[
T(-2000) = \int_{0}^{-2000} P(x) \, dx = - \int_{0}^{2000} P(x) \, dx = - (500 \times 49.93 + 500 \times 49.95 + 1,000 \times 49.96),
\]

where we extend the definition of \( P \) as

\[
P(x) = \begin{cases} 
49.96 & \text{for } -1,000 \leq x < 0, \\
49.95 & \text{for } -1,500 \leq x < -1,000, \\
49.93 & \text{for } -3,500 \leq x < -1,500.
\end{cases}
\]

For simplicity, assume the marketable order is actually a market order (its limit price is \( +\infty \) if it is a buy order and \( -\infty \) if it is a sell order, so the limit price is never constraining). Let \( \hat{x} \) denote the size of the market order, with \( \hat{x} < 0 \) denoting a sell order as before, and let \( \hat{v} \) denote the value of the asset. The equilibrium condition in the Glosten model is that, for each \( x > 0, \)

\[
P(x) = \mathbb{E}[\hat{v} | \hat{x} \leq x].
\]

The conditional expectation here is called an upper-tail expectation. The reason for taking the upper-tail expectation is that the limit sell order at depth \( x \) will execute against any market buy order that is of size \( x \) or larger. For \( x < 0 \), the equilibrium condition is that

\[
(p.623)
\]

\[
P(x) = \mathbb{E}[\hat{v} | \hat{x} \leq x].
\]

This is motivated by the fact that a limit buy order at depth \( x \) will execute against any market sell order of size \( |x| \) or larger.

An important feature of the Glosten model is that there is a small trade spread. The inside quotes (best bid and ask) are

\[
\text{BID} = \lim_{\hat{x} \to 0} P(\hat{x}) = \mathbb{E}[\hat{v} | \hat{x} < 0], \quad \text{ASK} = \lim_{\hat{x} \to 0} P(\hat{x}) = \mathbb{E}[\hat{v} | \hat{x} > 0].
\]

If there is any information at all in the market order \( \hat{x} \), then the bid is less than the ask, so there is a discontinuity in \( P(\cdot) \) at 0. The small trade spread \( \text{ASK} - \text{BID} \) does not stem from small trades being informative. Instead, it stems from the fact that the inside limit orders execute against market orders of any
size, including larger orders that are informative. This is another example of adverse selection.

As an example, suppose the market order comes from an investor with constant absolute risk aversion $\alpha$ who has a random endowment $\tilde{w}$ of the asset—known to the investor but not to the market—and a private signal $\tilde{s}$ about the asset value $\tilde{v}$. Define $\tilde{z} = E[\tilde{v} | \tilde{s}]$, and set $\tilde{z} = \tilde{v} - \tilde{z}$. Because the investor knows $\tilde{w}$ and $\tilde{s}$, the risk she faces is $\tilde{\epsilon}$. Assume $\tilde{\epsilon}$ is normally distributed. Then, as always in a CARA-normal model, the investor will maximize the mean-variance certainty equivalent. Given an order $x$, the certainty equivalent is

$$ (x + \tilde{w})z - \frac{1}{2} \alpha (x + \tilde{w}) \text{var}(\tilde{\epsilon}) - T(x). $$

(24.7)

The definition of the transfer function $T$ implies that $T(x) = P(x)$. Consequently, the first-order condition for the optimal order $\hat{x}$ is

$$ z - \alpha (x + \tilde{w}) \text{var}(\tilde{\epsilon}) = P(\hat{x}). $$

The left-hand side here is the marginal benefit of buying a little more (or selling a little less) of the asset, and $P(\hat{x})$ is the marginal cost. Define

$$ \hat{\theta} = z - \alpha \text{var}(\tilde{\epsilon}) \tilde{w}. $$

We can rewrite the first-order condition as

$$ \hat{\theta} = P(x) + \alpha \text{var}(\tilde{\epsilon}) x. $$

(24.8)

The first-order condition (24.8) depends on the investor’s information only via $\hat{\theta}$, so different investors with the same $\hat{\theta}$ make the same choices. We call $\hat{\theta}$ the investor’s type. To compute the tail expectations, note that, for any $x$,

$$ x \geq z \iff P(x) + \alpha \text{var}(\tilde{\epsilon}) x \geq P(x) + \alpha \text{var}(\tilde{\epsilon}) x \iff \hat{\theta} \geq P(x) + \alpha \text{var}(\tilde{\epsilon}) x. $$

(p.624)

Thus, the equilibrium condition is that

$$ P(x) = \begin{cases} E[\tilde{v} | \hat{\theta} \geq P(x) + \alpha \text{var}(\tilde{\epsilon}) x] & \text{if } x > 0, \\ E[\tilde{v} | \hat{\theta} \leq P(x) + \alpha \text{var}(\tilde{\epsilon}) x] & \text{if } x < 0. \end{cases} $$

(24.9)

This is an equation that is to be solved in $P(x)$ for each $x \neq 0$. There are cases in which a solution does not exist, but a solution exists if $\tilde{v}$, $\tilde{s}$, and $\tilde{w}$ are normally distributed. An illustration is provided in Figure 24.1.
24.4 Auctions

Many securities transactions are organized as auctions. In fact, any market in which a request to trade is made and other traders compete to take the opposite side, such as the Kyle model, can be considered an auction. Those who compete to fill the order (for example, the market makers in the Kyle model) are bidders in an auction. Auctions are also used by the Treasury (of the U.S. and many other countries) to sell bonds and bills, are used to sell entire companies when there are competing bidders in M&A transactions, have been used to sell shares in initial public offerings and seasoned offerings, and are used in many other financial settings. Even when a market is not explicitly organized as an auction, the economic theory of auctions is likely to provide insight into the working of the market and perhaps also into why the market is organized as it is.

![Figure 24.1 Example of the Glosten model](image)

In this example, \( \tilde{w} \) has a zero mean and is independent of \( \tilde{v} \) and \( \bar{\delta} \). Also, \( \mathbb{E}[\tilde{v}] = 4 \), \( \text{stdev}(\tilde{\theta}) = 1 \), \( \text{cov}(\tilde{v}, \tilde{\theta}) / \text{var}(\tilde{\theta}) = 0.5 \), and \( \text{avar}(\tilde{\theta}) = 1 \). The left panel illustrates the solution of (24.9) for \( P(\chi) \) when \( x = 1 \). The dashed line is the 45° line, and the solid line is the upper-tail expectation of \( \tilde{v} \) on the right-hand side of (24.9). The solution is \( P(\chi) = 5.34 \). The right panel plots the limit order book; that is, it plots the solution \( P(\chi) \) of (24.9). The inside quotes are \( \text{BID} = 3.39 \) and \( \text{ASK} = 4.61 \).
section describes a few important issues in auctions, including the winner’s curse and the linkage principle.

Assume there is a seller offering one unit of an asset for sale and there are multiple potential buyers. If the buyers all have the same information and the same value for the asset, and the seller asks them to bid for the asset, then the game is a Bertrand pricing game, and the equilibrium is for all buyers to bid their value. This is the assumption made in the Kyle model. Here, we consider the case in which the bidders have different information and possibly different values.

**Model**

Assume the bidders are risk neutral and each bidder $i = 1, \ldots, n$ observes a signal $\tilde{s}_i$. Assume for convenience that the signals are continuously distributed and there is a minimum possible signal $s_0$; that is, assume the support of each $\tilde{s}_i$ is the interval $[s, \infty)$, or, for some $s$, assume the support of each $\tilde{s}_i$ is the interval $[s_i, \infty)$. (p.625)

Let $\bar{x}_i$ denote the value of the object to bidder $i$ in monetary units, and set $\bar{v}_i = \mathbb{E}[\bar{x}_i|\tilde{s}_1, \ldots, \tilde{s}_n]$. Because of risk neutrality, bidders would act the same if their true values were $\bar{v}_i$ rather than $\tilde{x}_i$, so we henceforth call $\bar{v}_i$ the value of bidder $i$.

The definition as a conditional expectation implies that $\bar{v}_i = g(\tilde{s}_1, \ldots, \tilde{s}_n)$ for some function $g$. Assume the bidders are symmetric in the sense that

(p.626) • the joint distribution of $(\tilde{s}_1, \ldots, \tilde{s}_n)$ is invariant with respect to a permutation of the indices,

• $g(s_1, \ldots, s_n) = g(s, s_1, \ldots, s_n)$ for some function $g$, where $s_i$ denotes the list of signals of bidders $j \neq i$,

• $g$ is a symmetric function of the components of $s_i$.

Assume further that $g$ is a nondecreasing function in each of its arguments. Finally, assume that the signals are affiliated.

Each bidder chooses a bid $\tilde{b}_i$ depending on her signal. The allocation of the asset and the monetary transfers between the buyers and seller are functions of the vector of bids $(b_1, \ldots, b_n)$. In a symmetric equilibrium, there is a function $\mathcal{F}(\cdot)$ such that
each bidder \( i \) bids \( b_i = b(s_i) \). Given that the asset value is monotonically related to signals, the function \( b \) will be monotone.

Common Values and Independent Private Values

There are two important special cases of the auction model. In the common values model, all bidders have the same value. This is the special case of the model in which \( g \) is a symmetric function of all of its arguments, so the value depends on the collection of signals but not on who gets which signal. The second important special case is the independent private values (IPV) model, which assumes that each bidder’s value depends only on her own signal—that is, the function \( g \) depends only on \( s_i \)—and the signals are IID. In that case, we can define the signal to be the value (taking \( g(s) = s \)). In a financial setting, the possibility of reselling an asset after it is auctioned creates a common value aspect for the bidders. Of the two special cases, the common values model is usually more appropriate for financial auctions.

**Auction Formats**

Auctions are organized in a variety of ways. Common formats include ascending price, descending price, first-price sealed bid, and second-price sealed bid. In an ascending price auction, the price rises continuously and bidders drop out successively until only one bidder remains. The winning bidder pays the price at which the second-highest bidder dropped out. In a descending price auction, the price starts high and falls until some bidder accepts it. In sealed bid auctions, bidders submit bids without observing other bidders’ actions. The highest bidder wins the asset and pays her bid in a first-price auction or pays the second-highest bid in a second-price auction. Each of these auction formats is called a standard auction, meaning an auction in which the asset is awarded to the highest bidder, and only the highest bidder makes a payment to the seller.

The descending price auction is used in Amsterdam to sell flowers and is called a Dutch auction. In a descending price auction, each bidder chooses a price at which to step in and claim the asset. When a bidder does so, other bidders have not stepped in, so nothing has been learned about the choices of
other bidders. Consequently, a descending price auction is strategically equivalent to a first-price sealed bid auction.

The ascending price auction is a simple model of auctions such as art auctions, though it omits some details (the choice of a bidder to bid only slightly higher or much higher than the outstanding high bid, for example). In an ascending price auction, each bidder chooses a price at which to drop out. An ascending price auction is strategically equivalent to a second-price sealed bid auction if bidders in the ascending price auction do not learn anything from the prices at which other bidders drop out. This is the case when there are only two bidders (because the auction ends when one drops out) and in the IPV model (because then the exit prices of other bidders are irrelevant for each bidder’s value).

**Dominant Strategy Equilibria in the IPV Model**

In the IPV model, it is a dominant strategy in a second-price sealed bid auction for each bidder to bid her own value and a dominant strategy in an ascending price auction to drop out when the price reaches her value. If a bidder were to bid higher (or to drop out later), then she would change the outcome (winning when she would otherwise have lost) only when the price she has to pay exceeds her value. If a bidder were to bid lower (or drop out earlier), then she would change the outcome (losing when she would otherwise have won) only when her value exceeds the price she would have paid. Thus, neither deviation is ever profitable.

**(p.628)** **Winner’s Curse with Common Values**

In a standard auction that has a common value aspect, a bidder evaluates different possible bids by computing her expected gain (expected value minus expected payment) conditional on her signal and conditional on winning. Conditioning on winning is important, because there is important information in the fact of winning. A bidder who wins with a bid $x$ knows that the other bidders’ signals lie below $b^\gamma(x)$. This information about an upper bound on others’ signals causes the bidder (except in the IPV model) to revise her estimate of the asset value downward. This phenomenon is called the winner’s curse. It is both good news and bad news for a bidder to win in an auction. The good news is of course that she won; the bad news is that the asset is probably worth
less than she thought. A rational bidder recognizes that this “bad news” aspect of winning is inevitable, and she takes it into account before bidding when calculating her expected gain and in determining her optimal bid.

Equilibrium of an Auction

Consider a standard auction, and let \( p(b_y, \ldots, b_n) \) denote the amount paid by the winning bidder. Assume \( p \) is symmetric in its arguments, meaning that the payment depends on bids but not on bidder names. We want to characterize a symmetric equilibrium bidding strategy \( b(\cdot) \). Define

\[
h(s_1, \ldots, s_n) = p(b(s_1), \ldots, b(s_n))
\]

so that \( h \) is the equilibrium amount paid by the winning bidder as a function of signals. Define

\[
T(s, z) = \mathbb{E} \left[ h(\tilde{s}_1, \ldots, \tilde{s}_n, z, \tilde{s}_{n+1}, \ldots, \tilde{s}_n) | \tilde{s}_i = s, \max_{j \neq i} \tilde{s}_j < z \right]
\]

Thus, \( T(s, z) \) denotes the expected payment—conditional on winning—by a bidder with signal \( s \) who submits a bid as if she had signal \( z \). Similarly, denote the expected asset value conditional on winning when the bidder’s signal is \( s \) and she acts as if it were \( z \) by

\[
V(s, z) = \mathbb{E}(\tilde{a} | \tilde{s}_i = s, \max_{j \neq i} \tilde{s}_j < z).
\]

Denote the conditional probability of winning in this circumstance by

\[
P(s, z) = \operatorname{Pr}(\max_{j \neq i} \tilde{s}_j < z | \tilde{s}_i = s).
\]

(p.629) The bidder’s expected gain with a signal of \( s \) and a bid of \( b(z) \) is

\[
(V(s, z) - T(s, z))P(s, z).
\]

(24.10)

In equilibrium, this function of \( z \) must be maximized at \( z = s \), for each \( s \). The first-order condition at \( z = s \) is

\[
[V(s, s) - T(s, s)]P(s, s) + [V(s, s) - T(s, s)]P(s, s) = 0.
\]

(24.11)

Another necessary condition for equilibrium is that bids never exceed expected values conditional on winning. This means that \( b(s) \leq V(s, s) \). In particular, \( b(s) \leq V(s, s) \).

Example
Consider two bidders who have a common value $\tilde{v} = \tilde{s}_1 + \tilde{s}_2$. Assume $\tilde{s}_1$ and $\tilde{s}_2$ are independently uniformly distributed on $[0, 1]$. The equilibrium of a first-price auction is calculated as follows: Let $b(\cdot)$ be a candidate strictly increasing equilibrium bid function. The probability of winning conditional on a signal of $s$ and a bid of $b(z)$ is the probability that the other bidder’s signal lies below $z$, which is $z$. Thus, $P(s, z) = z$. The payment conditional on winning with a bid of $b$ is $b$. Thus, $T(s, z) = b(z)$. The expected value conditional on winning with a signal of $s$ and a bid of $z$ is

$$V(s, z) = s + E[s_1 = s, \tilde{s}_2 < z] = s + E[\tilde{s}_2 < z] = s + \frac{z}{2}.$$  

Thus, the first-order condition (24.11) is

$$\left[\frac{1}{2} - b(s)\right]s + \left[\frac{3s}{2} - b(s)\right] = 0.$$

(24.12)

The condition $b(s) \leq V(s, s)$ means that $b(0) \leq 0$. The solution of the ODE (24.12) with initial condition $b(0) = 0$ is the equilibrium. The solution is $b(s) = s$. 
Linkage Principle and Revenue Rankings

The linkage principle states that expected revenue is higher in a standard auction if the expected payment conditional on winning is linked to signals in addition to being linked to bids. Consider two different standard auctions. They differ in having different payment functions $T$. Let $T^A$ and $T^B$ denote the payment functions in two auctions $A$ and $B$. The linkage principle states that if $T^A(s, \cdot) \geq T^B(s, \cdot)$ for all $s$, where the subscript 1 denotes the partial derivative with respect (p. 630) to the first argument, then the expected revenue in auction $A$ is at least as great as the expected revenue in auction $B$. The linkage principle is proven below. The linkage principle means that higher average revenue is obtained if the payment is linked to a bidder’s signal (the first argument of $T$), rather than just to her action (the second argument of $T$). An auction can link the payment to bidder $i$’s signal by linking the payment to the bids of bidders $j \neq i$, which depend on the signals of bidders $j \neq i$, which are typically correlated with the signal of bidder $i$. Linkage can also be created by tying the payment price to the future realized value of the asset (for example, through royalties on production if the asset is an oil lease) or by tying the payment to information possessed by the seller via disclosure of the seller’s information prior to bidding—one implication of the linkage principle is that a seller should commit to truthfully disclosing any information she possesses prior to soliciting bids, if such a commitment is feasible.

The linkage principle implies that the expected revenue from a second-price auction is at least as large as that from a first-price auction. This may seem somewhat counterintuitive, because, for a given vector of bids, the payment is obviously higher in the first-price auction. However, the equilibrium bids are higher in the second-price auction. To deduce the result from the linkage principle, observe that, in a first-price auction, we have $T(s, z) = b(z)$, so $T^1(s, z) = 0$. In a second-price auction,

$$T(s, z) = \mathbb{E}\left[ \max_{\mu \in \mathcal{S}} \mathbb{E}[Y | \mathcal{F}_T] | s, \max_{\mu \in \mathcal{S}} \mu \mu \mu \mu_{i} < z] \right].$$

Because the signals are affiliated, $T(s, z)$ is a nondecreasing function of $s$, implying $T^1(s, z) \geq 0$. Therefore, the linkage principle together with affiliation implies that the expected
revenue from a second-price auction is at least as large as that from a first-price auction.

Another implication of the linkage principle is that the expected revenue from a second-price auction is the same as the expected revenue from a first-price auction if bidders’ signals are independent. In fact, every standard auction produces the same expected revenue when signals are independent. This is true even if values are common. This is called the revenue equivalence theorem. It follows from the linkage principle and the fact that $T(s, z)$ does not depend on $s$ in a standard auction when signals are independent; thus, $T_i(s, s) = 0$ for every standard auction when signals are independent.

Assume $T_i^A(s, s) \geq T_i^B(s, s)$ for all $s$. We will show that this implies

$$T_i^Y(s, s) - T_i^Y(s, s) \geq T_i^Y(s, s) - T_i^Y(s, s) = 0$$

(24.13) (p.631) for each $s$. Because of symmetry, the expected revenue in the auction is the expected payment by any buyer conditional on winning; that is, the expected revenue is $E[T(s, s)]$ for any $i$. Therefore, (24.13) implies that the expected revenue from auction $A$ is at least as high as that from auction $B$.

We begin by establishing the equality in (24.13). Consider either of the auctions and drop the superscript. A bidder who bids as if she had the minimum possible signal has a zero probability of winning, so $P(s, s) = 0$ for all $s$. Therefore, the value of the objective function (24.10) is zero at $z = s = s$. For truthful bidding to be optimal for a bidder with signal $s$, we must consequently have

$$[V(s, s + \epsilon) - T(s, s + \epsilon)] P(s, s + \epsilon) \leq 0$$

for $\epsilon > 0$. Thus,

$$V(s, s + \epsilon) \leq T(s, s + \epsilon),$$

and taking the limit as $\epsilon \to 0$ yields

$$V(s, s) \leq T(s, s).$$

On the other hand, for truthful bidding to be optimal for a bidder with signal $s + \epsilon$, we must have
the equality following again from the fact that \(P(s,w)=0\) for all \(s\). Thus,

\[
V(s+\varepsilon, s+\varepsilon) \geq V(s+\varepsilon, s+\varepsilon),
\]

and taking the limit as \(\varepsilon \to 0\) yields

\[
V(s, s) \geq T(s, s).
\]

Therefore,

\[
T(s, s) = V(s, s).
\]

The function \(V\) is the same for both auctions, so this equality implies the equality in (24.13).

Set \(\Delta(s) = T_A^A(s, s) - T_B^B(s, s)\). Then, using subscripts to denote partial derivatives,

\[
\Delta(s) = T_A^A(s, s) - T_B^B(s, s) + T_A^A(s, s) - T_B^B(s, s).
\]

(24.14)

The first-order condition for \(z=s\) to maximize (24.10) is

\[
[V(s, s) - T_A^A(s, s)]P_A(s, s) + [V(s, s) - T_B^B(s, s)]P_B(s, s) = 0.
\]

This equation must hold for all \(s\) in both auctions \(A\) and \(B\). The functions \(V\) and \(P\) are the same in the two auctions, so subtracting (24.15) for auction \(A\) from (24.15) for auction \(B\) produces

\[
[T_A^A(s, s) - T_B^B(s, s)]P_A(s, s) + [T_A^A(s, s) - T_B^B(s, s)]P_B(s, s) = 0.
\]

Therefore,

\[
T_A^A(s, s) - T_B^B(s, s) = -\frac{[T_A^A(s, s) - T_B^B(s, s)]P_B(s, s)}{P_A(s, s)} = -\frac{\Delta(s)P_B(s, s)}{P_A(s, s)}.
\]

Substitute this into (24.14) to obtain

\[
\Delta(s) = T_A^A(s, s) - T_B^B(s, s) - \frac{\Delta(s)P_B(s, s)}{P_A(s, s)} \geq -\frac{\Delta(s)P_B(s, s)}{P_A(s, s)}.
\]

(24.16)

using the assumption \(T_A^A(s, s) \geq T_B^B(s, s)\) for the inequality. Note that \(P_A(s, s)\) is the marginal effect on the probability of winning from increasing the bid and is therefore positive. Thus, the inequality (24.16) shows that if \(\Delta(s) < 0\) for any \(s\), then \(\Delta(s) > 0\).

We have shown that \(\Delta\) is a function that starts at 0 at \(s=s\) and which has the property that its derivative is positive whenever its value is negative. Such a function can never be negative. Therefore, \(\Delta(s) \geq 0\) for all \(s\), which is the inequality in (24.13). [To
see that $\Delta$ can never be negative, suppose to the contrary that it is negative at some $s$. Denote this value of $s$ by $s_2$. Define

$$s_1 = \max(s \leq s_2 | \Delta(s) \geq 0).$$

By continuity, $\Delta(s_1) = 0$. Furthermore, $\Delta(s) < 0$ for $s$ between $s_1$ and $s_2$. Consequently, $\Delta(s) > 0$ for $s$ between $s_1$ and $s_2$. Therefore,

$$\Delta(s_2) = \Delta(s_1) + \int_{s_1}^{s_2} \Delta(s) ds > 0,$$

which is a contradiction.

24.5 Continuous-Time Kyle Model
A trader who recognizes that her trades affect the market price will typically want to execute a trade in small pieces.\footnote{The single-period Kyle model described in Section 24.2 gives the informed trader only one opportunity to trade, which is an unnatural constraint. Relaxing this constraint enables us to examine how the (p.633) dynamic optimization of the informed trader affects the evolution of liquidity and the informativeness of prices over time. The continuous-time model affords the informed trader the maximum flexibility in timing her trades. It is also more tractable than a discrete-time dynamic model.}

Model
An asset with a terminal value of $\bar{v}$ is traded over a finite time interval. Use the time interval as the unit in which time is measured, so the interval is $[0, 1]$. Suppose $\bar{v}$ is normally distributed, and the single informed trader observes $\bar{v}$ at date 0. Denote the mean and standard deviation of $\bar{v}$ by $\bar{v}$ and $\sigma_v$, respectively.

Let $Z_t$ denote the number of shares held by liquidity traders at date $t \in [0, 1]$, and take $Z_0 = 0$. Assume $dZ = \sigma_v dB$, where $B$ is a Brownian motion and $\sigma_v$ is a constant.\footnote{The cumulative liquidity trade during the period $[0, 1]$ is $Z_{\bar{v}}$, which is normally distributed with mean zero and variance $\sigma_v^2$. To compare the continuous-time model to the single-period model, we will interpret $Z_t$ as corresponding to $Z_{\bar{v}}$ in the single-period model. Let $\chi_t$ denote the number of shares the informed trader purchases by date $t$.}
Market makers observe the stochastic process $Y$ defined by $Y_t = X_t + Z_t$ and set the price $P_t$ to be the expected value of $\bar{v}$, conditional on the information in $Y$ through date $t$. The interpretation of this model is that market makers see at each instant the aggregate order $dX_t + dZ_t$ and revise the price based on the information in the order.

Similar to the construction of fully and partially revealing equilibria in Chapter 22, we can assume the informed trader observes $Z$ and then justify the assumption by showing that the equilibrium price reveals $Y$ to the informed trader and hence, given knowledge of $X$, reveals $Z$. So, require $\chi$ to be adapted to $\bar{v}$ and $Z$. In fact, assume $dX_t = \theta_t dt$ for some stochastic process $\theta$ adapted to $\bar{v}$ and $Z$. \textbf{(p.634)} We discuss this assumption at the end of the section. Given these assumptions, the profit of the informed trader is the mispricing of the asset times the number of shares purchased, added up over time; that is, the profit of the informed trader is

$$\int_0^1 (\bar{v} - P_t) \theta_t dt.$$  

(24.17)

We need to rule out doubling strategies, similar to the issue discussed in Section 12.2. It suffices to require that $\theta$ be such that

$$E \int_0^1 P_t^2 dt < \infty$$

(24.18)

The role of this condition is explained below.

**Equilibrium**

An equilibrium in this model is defined in very much the same way as in the single-period Kyle model. The price at each time must equal the conditional expected value of the asset, given the information in orders, and the informed trader’s strategy must maximize her expected profit. The informed trader optimizes with the understanding that her trades affect prices, taking as given the manner in which the price at each time $t$ depends on the history of $Y$ through time $t$. It is shown below that there is an equilibrium in which

$$p_0 = \bar{v}, dP_t = \lambda dY_t, \text{ and } \theta_t = \frac{\bar{v} - P_t}{(1 - \lambda) \lambda}, \text{ where } \lambda = \frac{\sigma_v}{\sigma_t}.$$  

(24.19)
In this equilibrium, market depth $1/\lambda$ is constant and only half the depth in the single-period model. Other important properties of the model are

(a) All of the private information is eventually incorporated into the price (the price converges to $\tilde{v}$ by date 1).

(b) The conditional variance of $\tilde{v}$ at date $t$ given the market maker’s information is $(1 - \eta)\sigma^2_t$. Thus, information is transmitted at a constant rate.

(c) The equilibrium price process is a Brownian motion with zero drift and standard deviation $\sigma_v$, given the market makers’ information. The standard deviation does not depend on the level of liquidity trading $\sigma_L$.

(d) The expected profit of the informed trader is $\sigma_v\sigma_t$. Thus, the informed trader’s expected profit is twice what it is when she is constrained to trade only once. This implies that the expected losses of liquidity traders are also twice what they are in the single-period model.

Filtering Method

There have been numerous extensions of the dynamic Kyle model. The extensions rely on one of two alternative proof techniques. One method is to guess the forms of the price adjustment rule and informed trading strategy, use filtering theory to compute $\hat{\nu}_t = \mathbb{E}[\tilde{v} | (Y_s)_{s \leq t}]$ and then solve the equation

$$dP_t = \lambda dY_v,$$

(24.20a)

$$dY_v = \frac{\tilde{v} - P_t}{(1 - \theta)\lambda} dt + \sigma_v dB_t,$$

(24.20b)

The first step is to define the observation and innovation processes. Set $Y_0 = 0$ and

$$dY_v^* = \frac{1}{\lambda} \left( \frac{\tilde{v}}{(1 - \theta)\lambda} dt + dB_t \right),$$

(24.20c)

\[ dY_v = S dY_v^* = \frac{\tilde{v}}{(1 - \theta)\lambda} dt + dB_t. \]
Because $\gamma$ and hence $P$ are observable to market makers, $Y^*$ is also observable. The process $Y^*$ is an observation process as defined in Section 23.1. The corresponding innovation process is given by

$$dZ_t^r = \frac{\bar{\nu} - \hat{\nu}_t}{(1 - \gamma)\sigma_r} dt + dB_r,$$

where $\hat{\nu}_t$ denotes $E[\bar{\nu}|(Y_{s})_{s < t}]$. The filtering equation (23.7) is

$$d\hat{\nu}_t = \frac{1}{(1 - \gamma)\sigma_r} \sum_j (\gamma_j) dZ^r_t,$$

(24.21)

where $\Sigma(t)$ is the conditional variance of $\nu$ and is given in (23.3) as

$$\sum_j (\sigma_j^2) = \frac{1}{\sigma_r^2} + \int_0^t \frac{1}{(1 - s)^2} \sigma_s^2 ds = \frac{1}{\sigma_r^2} - \frac{t}{(1 - \gamma)\sigma_r^2} = \frac{1}{(1 - \gamma)\sigma_r^2}.$$

(24.22)

Therefore, $\Sigma(t) = (1 - \gamma)\sigma_r^2$, and the filtering equation is

$$d\hat{\nu}_t = \sigma_r dZ^r_t = \frac{\bar{\nu} - \hat{\nu}_t}{1 - \gamma} dt + \sigma_r dB_r.$$

(24.23)

Notice that (24.20) implies

$$dP_t = \frac{\bar{\nu} - P_t}{1 - \gamma} dt + \sigma_r dB_r.$$

Thus, $\nu$ and $P$ satisfy the same stochastic differential equation (SDE). Because $P_0 = \bar{\nu}$, the uniqueness of solutions to SDEs (satisfying Lipschitz conditions) implies $P = \hat{\nu}$. Thus, we have shown that $P_t = E[\bar{\nu}|(Y_s)_{s < t}]$, as desired.

One other fact will be useful later. Notice that $\Sigma(t) \rightarrow 0$ as $t \rightarrow 1$. Thus, $P_t \rightarrow \nu$, as claimed in the previous subsection.

Brownian Bridge

An alternative to filtering theory is to guess that the price at each date $t$ depends only on cumulative orders $Y_t$ rather than on the history of orders—that is, $P_t = f(t, Y_t)$ for some function $f$.

Condition (a) above is a necessary condition for equilibrium, because, if the price differs from $\nu$ at $t = 1$, then the informed trader had profitable trades remaining that she should have taken before $t = 1$ (buying if $P_1 < \nu$ and selling if $P_1 > \nu$). Thus, we guess that $P_t = f(t, Y_t) \rightarrow \nu$ as $t \rightarrow 1$, suggesting that $Y_t$ converges to some function of $\nu$—that is, $Y_t \rightarrow f(1, \cdot)^{-1}(\nu)$. This suggests that
market makers can estimate \( \tilde{y} \) by estimating where \( y \) will end up at date 1.

In the basic dynamic Kyle model, \( y \) is a Brownian motion relative to market makers’ information and a Brownian bridge relative to the informed trader’s information. We will explain and demonstrate that fact here. From (24.4), we have \( P_t = \tilde{v} + \lambda Y_t \), so

\[
dY_t = \frac{\tilde{v} - \lambda Y_t}{1 - t} dt + \sigma dB_t
\]

Now, define \( Y^* = Y / \sigma_2 \), so we have

\[
dY^*_t = \frac{(\tilde{v} - Y^*)/\sigma_2 - Y^*_t}{1 - t} dt + dB_t
\]

(24.24)

The process \( Y^* \) is called a Brownian bridge (see, for example, Karatzas and Shreve, 2004). It satisfies \( Y^*_t \rightarrow (\tilde{v} - \tilde{v})/\sigma_2 \) with probability 1 as \( t \rightarrow 1 \). This is equivalent to \( Y_t \rightarrow (\tilde{v} - \tilde{v})/\lambda \), which is equivalent to \( P_t \rightarrow \tilde{v} \).

The distribution of a Brownian bridge is that of a Brownian motion conditional on knowledge of the terminal value, in this case \( (\tilde{v} - \tilde{v})/\sigma_2 \). If the terminal value is unknown and regarded at date 0 as a standard normal, as is the case for (p.637) the market makers here due to the fact that \( (\tilde{v} - \tilde{v})/\sigma_2 \) is a standard normal, then the distribution is that of a Brownian motion. Thus, \( Y^* \) is a standard Brownian motion and \( Y \) is a Brownian motion with standard deviation \( \sigma_2 \) relative to market makers’ information.

Market makers estimate \( \tilde{y} \) by forecasting where \( y \) will end up, knowing that \( y \) ends at

\[
y_1 = \sigma_2 Y_1 = \frac{\sigma_2 (\tilde{y} - \tilde{v})}{\sigma_2} \Rightarrow \tilde{y} = \tilde{v} + \sigma_2 Y_1 / \sigma_2
\]

and regarding \( Y \) as a Brownian motion with zero drift. Because it is a Brownian motion with zero drift, the best estimate of where it will end is its current value, so the market makers’ best estimate of \( \tilde{y} \) is \( \tilde{v} + \sigma_2 Y_1 / \sigma_2 = \tilde{v} + \lambda Y_t \). This again verifies that the pricing rule in (24.19) is an equilibrium pricing rule.

Optimality of the Informed Trading Strategy
Having shown by either the filtering or Brownian bridge argument that $P_t = \hat{\nu}_t$, it remains to show that the informed trader’s strategy is optimal. This is fairly simple, because it turns out that any strategy is optimal if it implies $P_t \to \hat{\nu}$ as $t \to 1$. The economic explanation of that fact is that, in continuous time, the informed trader can move continuously along the residual supply curve $\Delta P = \lambda \Delta Y$ defined by the market makers’ pricing rule. She can buy shares and then resell them at the same price that she bought on average, because liquidity trades do not change the price on average (the effects of liquidity trades on prices is a martingale component $\lambda dZ$). So, it does not matter when she buys shares or sells shares or how many she trades prior to $t = 1$. The only requirement for optimality is that the informed trader should eventually exploit all mispricing, buying or selling enough to ensure that there is no gap between the final price and the asset value $\hat{\nu}$.

We verify the statements about the informed trader’s optimal strategies by using dynamic programming. Fix a realization $\nu$ of $\hat{\nu}$. We regard $\theta$ as the control chosen by the informed trader, $P$ as the state variable, and $(\nu - P)\theta$ as the instantaneous utility of the trader. The state variable evolves as $dP = \lambda \theta dt + \sigma P dB$. The value function is

$$J(t, P, \nu) = \max_{\theta} \left\{ \int \left( (\nu - P)\theta + \lambda \theta \frac{dP}{dP} - \frac{\sigma^2}{2} \right)^2 \right\}.$$ 

Keeping in mind that $\nu$ is regarded as fixed (known to the informed trader), the HJB equation is

$$0 = \max_{\theta} \left\{ (\nu - P)\theta + \lambda \theta P + \frac{1}{2} \sigma^2 J_{PP} \right\}.$$

(Note that the objective function in the HJB equation is linear in $\theta$, so the maximum can be zero only if the coefficient of $\theta$ is zero. In this case, the HJB equation states that the remaining terms add to zero. Thus, the HJB equation is equivalent to the pair of equations

$$(24.25a) \quad J_P = \frac{P - \nu}{\lambda},$$

$$0 = J_t + \frac{1}{2} \sigma^2 J_{PP}.$$ 

Because the coefficient of $\theta$ is 0 in the HJB equation, we can guess that any $\theta$ is locally optimal. The second equation implies that $J$ will have zero drift (and consequently be a local}
martingale) if we evaluate it at \((t, \bar{v} + \lambda Z_t)\) instead of at \((t, p)\). Note that \(v + \lambda Z_t\) would be the price process if \(\theta = 0\). Thus, it appears that \(\theta = 0\) is optimal and

\[
J(t, p, v) = \mathbb{E}[f_1(p + \lambda Z_t, v) | Z_t]
\]

If so, we can compute the value function from the function \(p \mapsto J(1, p, v)\). To guess this function, consider trading at time 1 (or very close to time 1) to move the price from \(p\) to \(v\). Given the linear relation \(\Delta p = \lambda \Delta Y\), the gain from doing so is \((v - p)^2/(2\lambda)\).

Thus, we guess that \(J(1, p, v) = (v - p)^2/(2\lambda)\) and

\[
J(t, p, v) = \mathbb{E}\left[\frac{(v - p - \lambda Z_t)^2}{2\lambda} \left| Z_t\right.\right] = \frac{(v - p)^2}{2\lambda} + \frac{1 - t}{\lambda} \sigma_t^2.
\]

(24.26)

This function \(J\) satisfies the HJB equation (24.5).

We prove a verification theorem using the value function guessed above. For any trading strategy \(\theta\),

\[
J(1, P_0, v) = J(0, P_0, v) + \int_0^1 \left[ \frac{1}{2} \sigma_t^2 \int_0^t \lambda Z_s^2 ds + \frac{1}{2} \int_0^t \lambda Y_s^2 dt \right] dt
\]

\[
+ \int_0^1 \left[ \int_0^t \lambda Y_s^2 ds + \frac{1}{2} \int_0^t \lambda Y_s^2 dt \right] dt.
\]

\[
= J(0, P_0, v) + \int_0^1 \frac{1}{2} \sigma_t^2 \int_0^t \lambda Z_s^2 ds + \frac{1}{2} \int_0^t \lambda Y_s^2 dt
\]

\[
+ \int_0^1 \left[ \int_0^t \lambda Y_s^2 ds + \frac{1}{2} \int_0^t \lambda Y_s^2 dt \right] dt.
\]

Thus, we guess that \(J(t, p, v) = (v - p)^2/(2\lambda)\) and

\[
J(t, p, v) = \mathbb{E}\left[\frac{(v - p - \lambda Z_t)^2}{2\lambda} \left| Z_t\right.\right] = \frac{(v - p)^2}{2\lambda} + \frac{1 - t}{\lambda} \sigma_t^2.
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\[
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\]

\[
+ \int_0^1 \left[ \int_0^t \lambda Y_s^2 ds + \frac{1}{2} \int_0^t \lambda Y_s^2 dt \right] dt.
\]

(24.26)

The left-hand side is the profit of the informed trader, and the right-hand side is bounded above by

\[
J(0, P_0, v) - \sigma_t \int_0^1 (v - P_t) dB_t
\]

\[
(24.27)
\]

due to the nonnegativity of \(J(1, p|v)\).

The no-doubling-strategies condition (24.18) implies that the stochastic integral in (24.27) has a zero expectation. Therefore,

\[
\mathbb{E}\left[\int_0^1 (v - P_t) dB_t \right] \leq J(0, P_0, v)
\]

(24.28)
with equality if and only if $J(1, P, v) = 0$, which is equivalent to $P_1 = v$. Thus, $J(0, P, v)$ is an upper bound on the informed trader's expected profit, conditional on $\bar{\nu} = v$, and the upper bound is realized—and the corresponding strategy is consequently optimal—if and only if $P_1 = v$. We have already shown that the trading strategy in (24.19) implies $P_i \rightarrow \bar{\nu}$ with probability 1. It follows that the strategy is optimal, provided that it is feasible. To be feasible, it must satisfy the “no doubling strategies” condition (24.18). As discussed previously, the strategy implies that $Y$ is a Brownian motion with standard deviation $\sigma$, relative to market makers’ information. Consequently, it implies that $P_t = \bar{\nu} + \lambda Y_t$ is a Brownian motion with standard deviation $\sigma$, relative to market makers’ information. Thus,

$$E_0^t \left[ \int_0^t \nu^2 \, dt \right] = \int_0^t \left[ \int_\Theta \nu^2 \, v \, du \right] \, dt = \int_0^t \nu^2 + \text{var}(P_t) \, dt = \nu^2 + \sigma^2 \int_0^t \sigma^2 \, dt < \infty.$$  

Why the Informed Trader Trades Slowly

To this point, we have only considered informed trading strategies $dX_t = \theta_t \, dt$ for stochastic processes $\theta$. This is a departure from the standard model of portfolio choice. For example, consider the model of Chapter 14 with a single risky asset. Suppose the dividend-reinvested price $S$ is a geometric Brownian motion with expected return $\mu$ and volatility $\sigma$.

Assume there is a constant risk-free rate $r$ and consider an investor with an infinite horizon and CRRA utility with risk aversion $\rho$. The investor’s optimal portfolio is

$$\pi = -\frac{\mu - r}{\rho \sigma^2}.$$  

This is the fraction of wealth invested in the risky asset. The number of shares held is $X_t = \pi W_t / P_t$, which has dynamics

$$\frac{dX}{X} = \frac{dW}{W} - \frac{dP}{P} - \left( \frac{dW}{W} \right) \frac{dP}{P} + \frac{dP^2}{P^2}.$$  

(\textit{p.640}) Assume the asset pays dividends continuously, so $S_t = \exp(\int_0^t \frac{dP}{P} - \frac{dW}{W})$. Then, the stochastic part of $dP / P$ is the same as that of $dS / S$, which equals $\sigma \, dB$. This implies that the stochastic part of $dX / X$ is $(\pi - 1) \sigma \, dB$. The presence of this stochastic part reflects the investor’s instantaneous rebalancing. If $\pi < 1$, then the investor sells shares whenever the stock price rises and buys shares whenever it falls. If $\pi > 1$, then she makes the reverse trades. This instantaneous rebalancing causes the number of shares $X$ to have infinite variation.
In contrast, in the Kyle model, the investor’s change in the number of shares she owns is of order $dr$. As usual, we think of $dr$ as being of smaller order than $dB$, which means that the investor trades more slowly in the Kyle model than in the standard model. There are two differences between the Kyle model and the standard model of portfolio choice that contribute to this difference in the speed of trade. First, the investor is risk neutral in the Kyle model, so she does not rebalance in response to price changes to manage the risk of her portfolio. Second, the investor in the Kyle model recognizes that her trades affect prices. It is costly for the informed trader in the Kyle model to trade excessively.

To see the costs of trading excessively, we need to modify the model to accommodate trading that is of order $\sqrt{dr}$. To obtain the correct formula for the informed trader’s profits, we return to the intertemporal budget equation in the standard model of portfolio choice. With no interim consumption, no interim dividends, and a risk-free rate of 0, the intertemporal budget equation is simply $dW = X dP$, which says that the change in wealth is the number of shares owned multiplied by the change in price. The investor’s total profit is

$$W_1 - W_0 = \int_0^1 X dP_t$$

From Itô’s formula (integration by parts), we have

$$d(XP) = X dP + P dX + (dX)(dP)$$

Thus,

$$X dP - d(XP) = P dX + (dX)(dP)$$

(p.641) and the investor’s profit is

$$W_1 - W_0 = X_0 P_1 - X_0 P_0 - \int_0^1 P dX_t - \int_0^1 (dX_t)(dP_t)$$

Substitute $X_0 P_1 = P_0 (X_1 - X_0) + P_1 X_0$ to obtain

$$W_1 - W_0 = (P_1 - P_0)X_0 + \int_0^1 (P_1 - P_0) dX_t - \int_0^1 (dX_t)(dP_t)$$

(24.29)

This formula is valid in both the Kyle model and in the standard portfolio choice model (with no interim consumption, no interim dividends, and a risk-free rate of 0).

There are three differences between (24.17) and (24.29), two of which are immaterial. The term $(P_1 - P_0)X_0$ in (24.29) is irrelevant, because it is independent of the investor’s actions and the investor is risk neutral. We have $P_1 - P_0$ in (24.29), but
appears in (24.17); however, we have already seen that $P_t = \bar{v}$ in equilibrium.¹⁰ The important difference between (24.17) and (24.29) is the last term in (24.29), which is the integral of trades multiplied by price changes. We can combine the two integrals in (24.29) as

$$\int_0^t (P_t - P_i - dP_i) dX_i$$

This is the sum of the mispricing multiplied by shares purchased as in (24.17), but here we recognize that the price at which the asset is traded is $P_t + dP_t$. This is natural, because $P_t + dP_t$ is the price after market makers respond to the market order $dY_t$. The difference between trading at $P_t$ and trading at $P_t + dP_t$ is negligible if the investor trades slowly, because $(dP)(dt) = 0$. But, it is nonnegligible if the investor’s trades are of order $\sqrt{dt}$.

As noted before, the formula (24.29) is valid in both the Kyle model and in the standard portfolio choice model. However, it leads to slower trading in the Kyle model than in the standard model. The reason is that in the Kyle model the investor considers the effect of her trades on the price. In the Kyle model,

$$(dP)(dx) = \lambda(dx)(dx) = \lambda(dx)^2 + \lambda(dx)(dz).$$

There is no term in the standard model that is analogous to $\lambda(dx)^2$, because the investor is a price taker in the standard model.
Liquidity Trader Losses

The gain (loss) of liquidity traders is also calculated according to the formula (24.29), but with \( Z \) replacing \( X \). Taking \( Z_0 = 0 \) and using the fact that

\[
\mathbb{E}\int_0^1 (P_1 - P) dZ_t = 0,
\]

we see that expected liquidity trader losses are

\[
\mathbb{E}\int_0^1 (dZ_t)(dP_t) = \lambda \mathbb{E}\int_0^1 (dZ_t)^2 = \lambda \sigma_p^2 = \sigma_x \sigma_p.
\]

This matches the expected gain of the informed trader, which, from (24.26), is

\[
\mathbb{E}[\mathcal{J}(0, \psi)] = \frac{2 \sigma_p^2}{2\lambda} = \sigma_x \sigma_p.
\]

In this model, as in the Glosten-Milgrom model and the single-period Kyle model, market makers lose to informed traders and recoup their losses from liquidity traders, who trade at unfavorable prices because market makers surmise their trades may be informed whereas in reality they are not. The unfavorable prices are captured by the bid-ask spread in the Glosten-Milgrom model, by the product of orders with price changes \( \mathbb{E}[\mathcal{Z} \cdot \Delta P] = \lambda \mathbb{E}[\mathcal{Z}^2] \) in the single-period Kyle model, and in the sum of orders multiplied by price changes \((dZ)(dP)\) in the continuous-time Kyle model.

24.6 Notes and References

In the models of this chapter, liquidity affects the asset price only indirectly through its effect on optimal trading and the information conveyed by trades. It is generally accepted that liquidity also affects asset prices through its effect on the return required by the marginal investor.\(^{11} \) The magnitude of this effect is debated. In a model in which illiquidity is manifested in a proportional transactions cost, Constantinides (1986) shows that illiquidity has a first-order effect on volume but only a second-order effect on utility, due to investors optimally reacting to illiquidity by trading less frequently. This implies that the effect of illiquidity on asset prices should be second order. Constantinides assumes IID returns and infinitely \((p.643)\) lived investors. Jang, Koo, Liu, and Lowenstein (2007) show that if returns are not IID, then investors want to trade more to adjust to time-varying risks and expected returns; consequently, the effects of illiquidity on utility and asset prices are larger. Amihud and Mendelson (1986) analyze a model in which investors have random finite
horizons at which they must liquidate their positions. They also conclude that the effect of transaction costs on asset prices is significant.

There are several possible explanations for the bid-ask spread. The Glosten-Milgrom model derives the spread from adverse selection considerations. Another possible explanation is that market makers are risk averse and must earn a risk premium to compensate for the risk of holding inventory. A third is that there are fixed costs of securities dealing that market makers cover in equilibrium from bid-ask spreads. A fourth is that market makers have monopoly power. Explanations other than adverse selection typically imply that transaction price changes are serially correlated. To explain this phenomenon, suppose the bid and ask prices are fixed at $B$ and $A$ respectively and each market order is equally likely to be a buy or a sell order. Then, there are eight equally likely configurations for a sequence of three transactions: (buy, buy, buy), (buy, buy, sell), .... Three transactions produce two price changes, each of which has an unconditional mean of zero. In this model, there cannot be any continuations of price changes. For example a positive price change from $B$ to $A$ must be followed by a zero change (another $A$) or a negative change (from $A$ to $B$). When a reversal occurs, the product of price changes is $-(A-B)^2$, and reversals occur with probability $1/4$ ($ABA$ and $BAB$ are the two of the eight possible sequences of prices for which there are reversals). Thus, the covariance between the pair of price changes is $-(A-B)^2/4$. Roll (1984) uses this fact to estimate the bid-ask spread from transaction prices. The martingale property in the Glosten-Milgrom model implies that price changes have zero serial correlation. Inventory control models can imply positive serial correlation (Exercise 24.1).

Glosten and Milgrom (1985) analyze a version of the model presented in Section 24.1 in which the asset value has a general distribution (not just $H$ and $L$), informed traders may be only partially informed about the asset value and may learn more over time, the probability $\mu$ of an informed trade may be a general stochastic process, and the demands of uninformed traders may be price elastic. One observation they make is that the market can break down if there is too much private information and uninformed demands are too elastic. In such cases, no matter how wide the bid-ask spread is set, market makers cannot break even on average, because uninformed
traders drop out when the spread widens, exacerbating the adverse selection problem. A major result of Glosten and Milgrom is that, if the market does not break down and the horizon $T$ is sufficiently far away, then the beliefs of market makers and informed traders must converge over time.

(p.644) There have been many extensions and applications of the Glosten-Milgrom model. Two notable extensions are the study of short sales constraints by Diamond and Verrecchia (1987) and the PIN (Probability of INformed trading) model of Easley, Kiefer, O’Hara, and Paperman (1996). Diamond and Verrecchia (1987) modify the Glosten-Milgrom model by assuming there are some traders who cannot sell short or who find it costly to sell short. When the asset value is low, informed traders who face short sales restrictions may choose not to trade. Diamond and Verrecchia show that this slows the incorporation of negative news into the market, but it does not bias prices. The reason there is no bias is that market makers update their beliefs with the recognition that traders with negative news may have chosen not to trade. This contrasts with heterogeneous prior models in which short sales restrictions cause prices to be biased upward (Section 21.4).

Easley, Kiefer, O’Hara, and Paperman (1996) work in continuous time and assume trades arrive as Poisson processes. At the beginning of the model, there may be an information event, in which case there is an informed trader who knows the asset value in $(L, H)$, or there may be no information event, in which case there are only uninformed trades. Thus, the number of buys and the number of sells are each drawn from a mixture distribution, mixing over whether there was an information event. They suggest applying the model empirically by assuming each day is a new iteration of the model and using a sample of daily buys and sells to fit the model parameters. They define PIN to be the expected rate of informed trades (the probability of an information event multiplied by the arrival rate of informed trades) divided by the expected rate of total trades. This model has been widely applied. However, Venter and de Jongh (2006) and Duarte and Young (2009) argue that the unconditional distribution of trades implied by the model does not fit the empirical distribution of trades. There are also some questions about whether empirical estimates of PIN actually measure the
Two notable extensions of the single-period Kyle model are to multiple assets (Caballé and Krishnan, 1994) and to a risk-averse informed trader (Subrahmanyam, 1991, and Exercise 24.2). There have been numerous other extensions and applications of the single-period Kyle model. A different model by Kyle has also been applied many times, though certainly not as many times as the 1985 model has been applied. Kyle (1989) analyzes a model in which informed and uninformed strategic investors with CARA utility submit demand curves for an asset. There are also normally distributed price-inelastic demands from liquidity traders. The equilibrium price is determined by market clearing. Thus, the role of the competitive market makers in the Kyle (1985) model is played by the uninformed strategic traders in Kyle (1989). Because traders submit demand curves (multiple orders at different prices), this is sometimes mistakenly called a limit order model. However, execution in a limit order model involves nonlinear pricing (walking up the book), and in the Kyle (1989) model all orders are executed at the market clearing price. Solving for equilibria in demand curves is difficult in general, because the space of functions is an infinite-dimensional space. However, the model has the property that each trader can compute her optimum one point at a time, solving for the optimum for each possible supply curve the other traders’ actions might present her and then compiling these various optima into an optimal demand curve. The same phenomenon appears in Klemperer and Meyer (1989), who describe the curves as passing through ex post optimal points.

The Glosten model assumes an infinite number of risk-neutral uninformed limit order traders. Berhnardt and Hughson (1997) show that, if there is only a finite number of limit order traders, then they will make positive expected profits in equilibrium. Thus, competition in limit orders is different from Bertrand competition in prices. Biais, Martimort, and Rochet (2000) derive a differential equation that may define an equilibrium in a limit order market with a finite number of limit order traders. Back and Baruch (2013) and Biais, Martimort, and Rochet (2013) provide sufficient conditions for the differential equation to define an equilibrium, correcting a misstatement in Biais, Martimort, and Rochet (2000). Dynamic
models of limit order markets are developed by Goettler, Parlour, and Rajan (2005, 2009) and Rosu (2009). In these models, as in many models in market microstructure theory, each market order trader wants to buy only a single unit of the asset.

Back and Baruch (2007) show that there should be a small-trade spread as in the Glosten model even in markets that are not organized as limit order markets. The reason is that, as discussed in footnote 7, traders who want to trade large quantities (including informed traders) should split their orders to minimize price impact costs. This is the phenomenon that occurs in the dynamic Kyle model. With order splitting, market makers cannot condition on the full size of a trade when they set the price for a piece of a trade. Thus, they should compute expected values as tail expectations exactly as in the Glosten model.

There is a large literature on auctions, originating with the seminal paper of Vickrey (1961). The revenue rankings described in Section 24.4 for models with common value aspects are due to Milgrom and Weber (1982). The discussion of the linkage principle and its proof in Section 24.4 are based on Milgrom (1989). For the definition and properties of affiliated random variables, see Milgrom and Weber (1982) or Krishna (2009). A topic not discussed in this chapter is the design of an optimal auction. The seminal paper on that topic is Myerson (1981). For surveys of auction theory, see Klemperer (1999, 2004) and Krishna (2009). The “half of her value” result in Part (a) of Exercise 24.3 is an example of a more general result: In first-price auctions with independent private values, it is an equilibrium to bid the expected value of the maximum of the other bidders’ values conditional on the maximum being less than the bidder’s value (Krishna, 2009, Proposition 2.2). The “twice her signal” result in Exercise 24.4 is also an example of a more general result: In second-price auctions with common values and symmetric signals, it is an equilibrium for each bidder to bid what her value would be if all other bidders had the same signal as her (Krishna, 2009, Proposition 6.1).

Section 24.4 describes the sale of a single unit of an asset. Financial auctions like Treasury auctions are typically for many units, and bidders can submit multiple bids for different quantities at different prices. These are called divisible-good
auctions. The two main divisible-good auction formats are uniform price (all winning bidders pay the lowest winning price) and discriminatory (all winning bidders pay the price they bid). There is some analogy between uniform-price divisible-good auctions and second-price single-good auctions and between discriminatory divisible-good auctions and first-price single-unit auctions. However, the revenue ranking theorem for single-unit auctions does not extend to divisible-good auctions. In fact, Wilson (1979) shows that uniform-price auctions can have equilibria that are very bad for the seller. In these equilibria, bidders bid steep demand curves that create a large discrepancy between price and marginal cost for other bidders, encouraging all bidders to bid low for the quantity they expect to win. Correcting a misstatement in Wilson (1979), Back and Zender (1993) show that discriminatory auctions can be strictly superior to uniform-price auctions.

Kyle (1985) solves the discrete-time and continuous-time versions of his model and proves convergence of the equilibria as the length of the time periods in the discrete-time model goes to zero. Thus, the discrete-time model with small time periods and the continuous-time model have equilibria that are approximately the same. There are a few variations of the model in which this is not true. In each of the three cases listed below, the discrete-time model has an approximate strong-form efficiency property when the period length is small. In the limit, there is exact strong-form efficiency, but the strong-form efficient outcome is not an equilibrium of the continuous-time model.

- Holden and Subrahmanyam (1992) consider a model with multiple informed traders who have identical information. When the time periods are short, the competition between the traders is so aggressive that information is revealed almost immediately—the market is approximately strong-form efficient. In the continuous-time version of the model, there is no equilibrium (Back, Cao, and Willard, 2000). The extremely aggressive competition is a consequence of the informed (p.647) traders having identical information. When they have heterogeneous (correlated but not identical) information, the discrete-time equilibria converge to a continuous-time equilibrium—see Foster and Viswanathan (1996) for the discrete-time model and Back, Cao, and Willard (2000) for continuous time.
• When (i) the asset value evolves over time, (ii) the informed trader learns about the asset value over time, and (iii) the announcement date is random, Caldentey and Stacchetti (2010) show that the discrete-time equilibria converge to something that is not a continuous-time equilibrium. In the limit, there is a finite date $T$ such that, if the announcement does not occur prior to $T$, then at $T$ and after the market knows everything the informed trader knows. The new information the trader learns after $T$ is communicated to the market instantly. Thus, the market is strong-form efficient after $T$. This is approximately true in discrete time when the period length is small; however, the continuous-time limit is not an equilibrium of the continuous-time version of the model. Related work includes Back and Pedersen (1998), who derive an equilibrium in a continuous-time model with the informed trader learning about the asset value over time (and in which the asset value can evolve over time) but in which the announcement date is fixed. Their existence result depends on there being a sufficiently high amount of information asymmetry at the beginning of the model. The amount of information asymmetry evidently falls below the required level when the announcement date is random and the announcement does not occur soon enough.

• Chau and Vayanos (2008) study a model similar to that of Caldentey and Stacchetti (2010) but with no announcement date and with an asset that pays dividends each period. In the Chau-Vayanos model, market makers observe a public signal that reveals part of the informed trader’s information. Chau and Vayanos analyze the steady state of their model. They show that, as the period length becomes small, the steady states converge to strong-form efficiency. However, informed profits do not converge to zero.

Back and Baruch (2004) analyze a hybrid of the Glosten-Milgrom and dynamic Kyle models. In their model, liquidity trades arrive as Poisson processes and there is a fixed order size. Market makers quote bid and ask prices, as in the Glosten-Milgrom model. Their model also has a random announcement date and a binary (high or low) distribution for the asset value. They show that, as the order size goes to zero and the arrival rates of liquidity trades go to infinity, equilibria converge to the equilibrium of a continuous-time Kyle model. One feature of (p.648) their model is that, under some
circumstances, the informed trader trades with positive probability in the opposite direction of her information (buying when the asset value is low or selling when it is high). This feature of “bluffing” also occurs in a dynamic Kyle model when there is a requirement that trades be disclosed ex post (Huddart, Hughes, and Levine, 2001).

The solution of the continuous-time Kyle model in Section 24.5 using the Brownian bridge argument is based on Back (1992), who solves the model for general continuous—not necessarily normal—asset value distributions (Exercise 24.5). Other extensions of the continuous-time Kyle model include trading an underlying asset and a derivative security (Back, 1993; Back and Crottty, 2015); risk aversion on the part of the informed trader (Baruch, 2002); atoms in the distribution of $\nu$ (Back and Baruch, 2004; Çetin and Xing, 2013; Back and Crotty, 2015; Back, Crotty, and Li, 2015); private information about the default time of a bond (Campi and Çetin, 2007; Campi, Çetin, and Danilova, 2013); a stochastic process for the standard deviation of liquidity trades (Collin-Dufresne and Fos, 2014); an asset value that can be influenced by the large trader with costly effort (Back, Collin-Dufresne, Fos, Li, and Ljungqvist, 2016).

Large traders (who anticipate how their trades will affect prices) have also been studied in models with symmetric information. Bertsimas and Lo (1998) study how a large trader should split her orders over time to minimize the expected cost of executing a predetermined trade within a prespecified amount of time. They describe the optimum as splitting trades evenly over time plus an adjustment for changing market circumstances. Almgren and Chriss (2000) and Huberman and Stanzl (2005) study how an investor with mean-variance preferences should do the same thing. A risk-averse investor trades more quickly to minimize price risk. The price impact functions in these models are exogenously specified. They include permanent and transitory effects. Huberman and Stanzl (2004) show that the permanent effect must be linear in order to avoid arbitrage opportunities for the large trader. Obizhaeva and Wang (2013) analyze how to minimize expected execution costs when trading into a limit order book when the evolution of the book depends on trades. Brunnermeier and Pedersen (2005), Carlin, Lobo, and Viswanathan (2007), and Teguia (2015) study equilibrium trading by large traders when
it becomes known that another trader needs to make a large trade (is distressed). In equilibrium, other traders try to front-run the distressed trader (predatory trading) and then buy to cover their positions after the distressed trader’s sale has depressed prices.

Vayanos (2001) studies a model similar to the dynamic Kyle model, except that the large trader’s private information is about her endowment shocks rather than about the fundamental value of the asset. This captures the fact that private information of a large trader is frequently about her own trade intentions, which have price consequences when the market is less than perfectly liquid. The (p.649) illiquidity in the Vayanos (2001) model arises not from adverse selection but because market makers are risk averse. Consequently, the asset’s risk premium depends on the inventory that the market makers hold and hence depends on the trades of the large trader. Choi, Larsen, and Seppi (2015) study a dynamic Kyle model in which there are two strategic traders. One has long-lived private information as in the standard Kyle model and the other is constrained to trade a prespecified number of shares known only to her. Thus, Choi, Larsen, and Seppi solve the Kyle model with (somewhat) endogenous liquidity trades, and they also solve the optimal execution problem with endogenous price impacts.

Exercises

24.1. Suppose there is a representative market maker with constant absolute risk aversion $\alpha$, and competition forces the bid and ask to the prices that make the market maker indifferent about trade. Suppose there is no information in market orders, which are of a unit size. Assume that the future value of a unit of the asset is normally distributed with mean $\mu$ and variance $\sigma^2$.

(a) Let $\theta$ denote the number of shares owned by the market maker before trade. Compute the ask and bid prices. Show that, even though the bid and ask depend on $\theta$, the bid-ask spread does not.

(b) Assume buys and sells are equally likely. Consider all possible sequences of three transactions. Compute the transaction price for each transaction in each
sequence, and show that the transaction price changes are positively serially correlated.

24.2. In the single-period Kyle model, assume the informed investor has CARA utility. There is a linear equilibrium. Derive an expression for \( \lambda \) as a root of a fifth-order polynomial.

24.3. Assume there are two buyers in an auction who have independent private values. Assume the value of each buyer is uniformly distributed on \([0, 1]\). Each buyer knows her own value but does not know the value of the other buyer.

(a) Assume the auction is conducted as a first-price auction. Show that it is an equilibrium for each buyer to bid one-half of her value. Compute the expected revenue for the seller.

(b) Assume the auction is conducted as a second-price auction. Show that it is an equilibrium for each buyer to bid her signal. Compute the expected revenue for the seller.

24.4. Assume there are two buyers in an auction who have a common value. Assume the buyers receive signals that are independently uniformly distributed on \([0, 1]\), and assume the value is the sum of the signals. Each buyer knows her own signal but does not know the signal of the other buyer. Assume the auction is conducted as a second-price auction. Show that it is an equilibrium for each buyer to bid twice her signal. Compute the expected revenue for the seller and compare it to the expected revenue from the first-price auction solved as an example in Section 24.4.

24.5. In the continuous-time Kyle model, assume \( \log \tilde{v} \) is normally distributed instead of \( \tilde{v} \) being normally distributed. Denote the mean of \( \log \tilde{v} \) by \( \mu \) and the variance of \( \log \tilde{v} \) by \( \sigma^2 \). Set \( \lambda = \sigma_v / \sigma_\zeta \). Show that the strategies

\[
\begin{align*}
P_t &= e^{\mu \frac{1}{2} \tilde{v}^2} \\
\frac{dP_t}{P_t} &= \lambda P_t dY_t \\
\frac{dX_t}{X_t} &= (\log \tilde{v}) \frac{\lambda \tilde{v}}{\tilde{v}^2} dt
\end{align*}
\]

form an equilibrium by showing the following:

(a) Define \( W_t = Y_t / \sigma_\zeta \). Show that, conditional on \( \tilde{v} \), \( W \) is a Brownian bridge on \([0, 1]\) with terminal value \( \frac{\log \tilde{v} - \mu}{\sigma_v} \).
Use this fact to show that \( P \) satisfies \( P_1 = \bar{v} \) and is a martingale relative to the market makers’ information.

(b) For \( \nu > 0 \) and \( p > 0, \) define

\[
J(t, p) = \frac{\nu p + \nu(\log v - \log p)}{\lambda} + \frac{1}{2} d \sigma (1 - t) v.
\]

Prove the verification theorem. (The intuition for this \( J \) is the same as that described in Section 24.5—take \( \theta = 0 \) and then trade at the end to the point that \( p = v. \))

Notes:

(1.) Liquidity has been described as being similar to pornography in the sense that, as famously stated by U.S. Supreme Court Justice Potter Stewart, it is hard to define but we know it when we see it (O’Hara, 1995).

(2.) Given dates \( s < t < u \) and a martingale \( M, \)

\[
\text{cov}(M_u - M_s, M_t - M_s) = \mathbb{E}(M_u - M_s)(M_t - M_s) = \mathbb{E}(M_t - M_s)\mathbb{E}[M_u - M_s] = 0.
\]

(3.) Alternatively, we can assume the informed trader observes a signal \( \bar{s} \) that is joint normally distributed with \( \bar{v}. \) Because of risk neutrality, the residual risk \( \bar{v} - \mathbb{E}[\bar{v}|\bar{s}] \) is irrelevant for prices, and all of the results of this section hold if we substitute \( \mathbb{E}[\bar{v}|\bar{s}] \) for \( \bar{v} \) and substitute the variance of \( \mathbb{E}[\bar{v}|\bar{s}] \) for the variance of \( \bar{v}. \)

(4.) If the mean \( \bar{z} \) of \( \bar{z} \) is nonzero, then all of the results in this section hold when we replace \( \bar{z} \) by \( \bar{z} - \bar{z}. \) The irrelevance of the mean of \( \bar{z} \) is a consequence of market makers being risk neutral. This is in contrast to models studied in Chapter 22, in which all investors are risk averse and the mean of liquidity trades affects prices, because it affects the amount of risk other investors must bear in equilibrium.

(5.) Exceptions to time priority may be made based on order display. If an exchange allows hidden orders or iceberg orders (where only part of the order is made visible), then it might stipulate that orders that are publicly displayed are executed before hidden or iceberg orders, regardless of the time of arrival.
(6.) Affiliation is a relationship among random variables that is in the spirit of positive correlation. It means that if some subset of the random variables are all large, then it is more likely that the others are also large. Suppose the signals have a joint density function \( f \). Given two signal vectors \( s = (s_1, \ldots, s_n) \) and \( s' = (s'_1, \ldots, s'_n) \), let \( s \lor s' \) denote the vector with elements \( \max(s_i, s'_i) \), and let \( s \land s' \) denote the vector with elements \( \min(s_i, s'_i) \). The signals are affiliated if

\[
(\forall s, s') \quad f(s \lor s') f(s \land s) \geq f(s)f(s').
\]

Roughly speaking, this means that it is more likely that the signals are all large or all small than that they are mixed. The property of affiliation is used in this section in the discussion of the winner’s curse and the discussion of revenue rankings.

(7.) For example, if the price is \( p_0 + \lambda x \) for a constant \( \lambda \), where \( x \) denotes the cumulative amount purchased, then the cost of buying a quantity \( A \) in a single trade is \( A(p_0 + \lambda A) = p_0 A + \lambda A^2 \), whereas the cost of buying the same quantity via a series of infinitesimal trades is

\[
\int_0^A (p_0 + \lambda x) dx = p_0 A + \frac{1}{2} \lambda A^2.
\]

This distinction between a single purchase and a series of infinitesimal purchases is the distinction between a monopsonist and a perfectly discriminating monopsonist.

(8.) Choosing the unit of time so that the trading period is \([0, 1]\) has an effect on \( \sigma_T \). Suppose instead that we measure time in years, the standard deviation per year of liquidity trading is \( \sigma_T^* \), and trading is over a time interval \([0, T]\). When we rescale time so that a unit of time is \( T \) years, making the trading interval \([0, 1]\), the standard deviation of liquidity trading per unit of time becomes \( \sigma_T = \sqrt{T} \sigma_T^* \). We can rewrite all of the formulas in this section in terms of the number of years \( T \) and the standard deviation \( \sigma_T^* \) of liquidity trading per year by substituting \( \sigma_T = \sqrt{T} \sigma_T^* \).
(9.) An investor whose optimal allocation $\pi$ is less than 1 trades as a contrarian, and an investor whose optimal allocation $\pi$ is greater than 1 trades as a momentum trader. If $\pi < 1$, then the investor becomes overexposed to the risky asset when it rises, because the fraction of her wealth in the risky asset rises above $\pi$ when the asset price rises. Thus, she sells shares when the stock price rises. Symmetrically, she buys shares when it falls. On the other hand, if $\pi > 1$, then the investor is levered—she has borrowed to buy the risky asset. In this case, her wealth changes by a greater percentage than the asset price changes when the asset price rises or falls. When the asset rises, she becomes underexposed, so she buys more. Symmetrically, she sells shares when the asset price falls.

(10.) Furthermore, even out of equilibrium, it is correct to replace $P_1$ with $\bar{v}$ in (24.29), because if we do not have $P_1 \to \bar{v}$, then the price will jump to $\bar{v}$ at the end of the model.

(11.) In most of the models of this chapter, the marginal investor is a risk-neutral market maker who is not restricted by margin requirements and hence can bear any or all of the risk of an asset. In reality, market makers attempt to maintain low inventories, and the asset must be held primarily by other investors, who are adversely affected by illiquidity.