Rational Expectations Equilibria

Kerry E. Back

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Abstract and Keywords
When differences in beliefs are due to differences in information, investors learn from prices. If there are no risk-sharing motives for trade, then differences in information do not lead to trade (the no-trade theorem). Equilibrium prices can fully reveal information, but then there is no incentive to gather information (the Grossman-Stiglitz paradox). Noisy trades or asset supplies facilitate partially revealing equilibria. In the Grossman-Stiglitz model and the Hellwig model, prices equal discounted expected values minus a risk premium term that depends on the average precision of investors’ information weighted by their risk tolerances. The chapter explains the mechanics of updating beliefs when fundamentals and signals are normally distributed.

Keywords: asymmetric information, rational expectations, no-trade theorem, Grossman-Stiglitz paradox, fully revealing equilibrium, partially revealing equilibrium, Grossman-Stiglitz model, Hellwig model, normal-normal updating

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This chapter discusses securities markets when investors have common (homogeneous) priors but heterogeneous information. Conditioning on different information causes investors to have different beliefs, but investors do not hold those beliefs dogmatically. Instead, they learn from security prices about the information of other investors, and they revise their beliefs in response. The concept that investors understand how prices depend on information, and hence can make correct inferences from prices, is called rational expectations.

The extent to which prices reveal information is a fundamental issue in finance and in economics in general. If prices are fully revealing, then markets are said to be strong-form efficient. Strong-form efficiency is paradoxical in that, if prices are fully revealing, then private information is of no benefit in equilibrium. Yet, it is presumably costly to acquire private information. Of course, if no one acquires information—because it is costly to do so and of no benefit in equilibrium—then there is no information for prices to reveal. This is called the Grossman-Stiglitz paradox. To avoid this paradox, prices must be less than fully revealing.

One reason for partial revelation is that demands for assets depend both on information that is relevant to asset payoffs and on other information that is irrelevant but which nonetheless causes investors to trade. Such motives for trading could include a need for cash or a surplus of cash due to some personal liquidity shock. Trading based on such motives is commonly called liquidity trading. In Section 22.1, we see that some non-information-based motive for trade—such as risk sharing—must be present for asymmetrically informed investors to trade.

22.1 No-Trade Theorem
Strictly risk-averse investors with common priors do not bet on events that are independent of asset payoffs and endowments (labor income), even if they have different information about the likelihoods of such events occurring. For example, they do not bet on sports events. The key to this conclusion is that the
terms of the bet are observable to both parties. If one party is agreeable to a bet, then the other party should learn enough from this fact that she will not agree to it. This applies to security trades as well. If there is no risk sharing benefit to a trade, then risk-averse investors with common priors will not make the trade, even if the trade appears profitable to both given their different private information.

Example
For a simple example, consider a binary random variable $\bar{x} \in \{0, 1\}$ on which two individuals might bet. The prior probabilities of the outcomes 0 and 1 are $1/2$ each. Suppose each individual observes a signal $\bar{s}_i$ prior to the bet. Conditional on the outcome of the event $\bar{x}$, the signals are independently distributed with $3/4$ probability that $\bar{s}_i = \bar{x}$ and $1/4$ probability that $\bar{s}_i = 1 - \bar{x}$. The posterior probabilities of the events conditional on a signal are calculated as, for example,

\[
\P(\bar{x} = 0 | \bar{s}_i = 0) = \frac{\P(\bar{x} = 0 | \bar{s}_i = 0)}{\P(\bar{s}_i = 0)} = \frac{\P(\bar{x} = 0)}{\P(\bar{s}_i = 0)} = \frac{3}{4}.
\]

Suppose the realizations of the signals observed by the two individuals are $\bar{s}_1 = 0$ and $\bar{s}_2 = 1$. So, the first individual believes there is a $3/4$ probability that $\bar{x} = 0$, and the second individual believes there is a $3/4$ probability that $\bar{x} = 1$. If this difference in beliefs were due to differences in priors, then the two individuals would bet. However, in the present example, the difference is due to differences in information, and the two individuals will not bet. Consider, for example, a bet in which the first individual receives $1 if $\bar{x} = 0$ and pays $2 if $\bar{x} = 1$. This bet is acceptable to both given their private information, provided they are not too risk averse. For the first individual, the expected gain from the bet is $\frac{3}{4} \times 1 - \frac{1}{4} \times 2 = \$0.25$. (p.571) For the second individual, the expected gain is $-\frac{1}{4} \times 1 + \frac{3}{4} \times 2 = \$1.25$. However, for this bet to take place, the first individual must realize that it is acceptable to the second individual, which is possible only if the second individual has observed $\bar{s}_2 = 1$. Given this information, the first individual realizes that the events $\bar{x} = 0$ and $\bar{x} = 1$ are equally likely, so the expected gain from the bet to her is actually $-\$0.50$. She will therefore refuse the bet.
Groucho Marx famously said that he would refuse to join any club that would admit him. There is a parallel here: An investor should refuse to accept any bet on $\hat{x}$ that the other investor is willing to make.
Information about Exogenous Risks

For a more general treatment of the above example, suppose two investors have signals $\hat{s}_h$ about the value of some random variable $\hat{x}$ that is independent of endowments and asset payoffs, both conditionally on the $\hat{s}_h$ and unconditionally.

Suppose for the sake of argument that $\hat{x}$ is traded at price $p(\hat{s}_1, \hat{s}_2)$ from investor 2 to investor 1, meaning that investor 1 adds $\hat{x} - p(\hat{s}_1, \hat{s}_2)$ to her terminal wealth, and investor 2 subtracts the same amount. The independence assumption implies that neither investor receives any hedging benefits from this trade. Given strict risk aversion, a necessary condition for the trade to be acceptable to both investors is that each investor view her gain from the trade as having a positive conditional expectation (Section 1.5). Each investor $h$ conditions on the signal $\hat{s}_h$ that she observes directly and on the price $p(\hat{s}_1, \hat{s}_2)$ of the trade. Conditioning on the price means that the investor knows how the price depends on signals—that is, she knows the function $p$. As remarked before, this is called rational expectations. Because of rational expectations, observing a realization of the price provides some information about the realized signal of the other investor. The positive conditional expectations are expressed as

$$\mathbb{E}[\hat{x} - p(\hat{s}_1, \hat{s}_2)|p(\hat{s}_1, \hat{s}_2)] > 0,$$

$$\mathbb{E}[p(\hat{s}_1, \hat{s}_2) - \hat{x} | p(\hat{s}_1, \hat{s}_2)] > 0.$$

They imply

$$\mathbb{E}[\hat{x} | p(\hat{s}_1, \hat{s}_2)] > p(\hat{s}_1, \hat{s}_2) > \mathbb{E}[\hat{x} | p(\hat{s}_1, \hat{s}_2)].$$

Take the expectation conditional on $p(\hat{s}_1, \hat{s}_2)$ throughout to obtain

$$\mathbb{E}[\hat{x} | p(\hat{s}_1, \hat{s}_2)] > p(\hat{s}_1, \hat{s}_2) > \mathbb{E}[\hat{x} | p(\hat{s}_1, \hat{s}_2)].$$

which is a contradiction. Thus, given rational expectations, there is no price function $p$ at which strictly risk-averse investors will trade a payoff-irrelevant random variable.

If investors are risk neutral, then they may make bets on events unrelated to endowments and payoffs of positive net supply assets, but they do not expect to gain from such bets. For risk-neutral investors, the strict inequalities in the previous paragraph are replaced by weak inequalities, leading to the conclusion

$$\mathbb{E}[\hat{x} | p(\hat{s}_1, \hat{s}_2)] = p(\hat{s}_1, \hat{s}_2) = \mathbb{E}[\hat{x} | p(\hat{s}_1, \hat{s}_2)].$$
The no-trade theorem also applies to trading on events that are relevant to endowments and asset payoffs, if we start from a Pareto optimum. Suppose the economy has reached a Pareto-optimal allocation of assets, perhaps through a round of trade in complete markets. Suppose investors then receive new private information about asset payoffs. Will they retrade the assets based on the new information? The answer is "no."

Even though, based solely on each investor’s private information, there may appear to be trades that would benefit each investor, the perceived benefits to some investors must disappear when the investors condition on the willingness of all parties to make the trades.

Normal-Normal Updating

Most of the models in the remainder of the chapter are based on what is called normal-normal updating. This is a special case of Bayes’ rule. We assume that a variable \( \tilde{x} \) to be estimated and a signal \( \tilde{s} \) are joint normally distributed.

Conditional Mean

Because of normality, the expectation of \( \tilde{x} \) conditional on \( \tilde{s} \) is the orthogonal projection of \( \tilde{x} \) on the space spanned by \( \tilde{s} \) and a constant. \(^2\) This projection is

\[
\mathbb{E}[\tilde{x}|\tilde{s}] = \mathbb{E}[\tilde{x}] + \beta (\tilde{s} - \mathbb{E}[\tilde{s}]).
\]

(22.2)

where

\[
\beta = \frac{\text{cov}(\tilde{x}, \tilde{s})}{\text{var}(\tilde{s})}.
\]

(22.3)

Conditional on \( \tilde{s} \), \( \tilde{x} \) is normally distributed with (22.2) as its mean.

Conditional Variance

Given \( \tilde{s} \), the unknown part of \( \tilde{x} \) is the residual from the projection (22.2). The variance of \( \tilde{x} \) conditional on \( \tilde{s} \) is the variance of the residual. The residual is \( \tilde{u} \) defined as \( \tilde{u} = \tilde{x} - \mathbb{E}[\tilde{x}|\tilde{s}] \).

Equivalently,

\[
\tilde{x} = \mathbb{E}[\tilde{x}] + \beta (\tilde{s} - \mathbb{E}[\tilde{s}]) + \tilde{u}.
\]

(22.4)

Because \( \tilde{u} \) and \( \tilde{s} \) are uncorrelated,
The squared correlation in this formula is the fraction of the variance of \( \bar{x} \) that is attributable to its correlation with \( \bar{s} \). It is called the \( R^2 \) of the projection of \( \bar{x} \) on \( \bar{s} \).

**Truth-Plus-Noise Signals**

There is no loss of generality in assuming that the signal is “truth plus noise,” meaning that \( \bar{s} = \bar{x} + \xi \), where \( \xi \) is normally distributed and independent of \( \bar{x} \). To see that there is no generality lost in this assumption, consider an arbitrary \( \tilde{s} \) that is joint normally distributed with \( \bar{x} \). Project \( \tilde{s} \) on \( \bar{x} \) as

\[
\tilde{s} = \alpha + \frac{\text{cov}(\bar{x}, \tilde{s})}{\text{var}(\bar{x})} \bar{x} + \tilde{\xi}
\]

In this projection, \( \tilde{\xi} \) is uncorrelated with—and therefore, because of normality, independent of—\( \bar{x} \). The same information is carried in the following affine transform of \( \tilde{s} \):

\[
\frac{\text{var}(\bar{x})}{\text{cov}(\bar{x}, \tilde{s})} (\tilde{s} - \alpha)
\]

(22.6)

The affine transform equals \( \bar{x} + \tilde{\xi} \), where

\[
\tilde{s} \triangleq \frac{\text{var}(\bar{x})}{\text{cov}(\bar{x}, \tilde{s})} \tilde{\xi}
\]

Thus, by working with the affine transform, we obtain a truth-plus-noise signal. Note that \( E[\tilde{s}] = E[\bar{x}] \) when the signal is truth plus noise, so the formula (22.2) for the conditional mean can be written as

\[
E[\tilde{s}|\bar{x}] = (1 - \beta) \bar{x} + \beta \tilde{x}
\]

(22.7)

**R and Beta for Truth-Plus-Noise Signals**

When the signal is truth plus noise, then \( \text{cov}(\bar{x}, \tilde{s}) = \text{var}(\bar{x}) \), so

\[
\beta = \frac{\text{var}(\bar{x})}{\text{var}(\tilde{s})}
\]

This implies further that

\[
\frac{\text{cov}(\bar{x}, \tilde{s})^2}{\text{var}(\bar{x}) \text{var}(\tilde{s})} = \beta
\]

In conjunction with (22.5), this produces the following formula for the conditional variance:
\[ \text{var}(\hat{s}) = (1 - \beta) \text{var}(\hat{x}), \]

(22.8)

where, because \( \hat{s} = \hat{x} + \hat{\varepsilon} \),

\[ \beta = \frac{\text{cov}(\hat{x}, \hat{s})}{\text{var}(\hat{s})} = \frac{\text{var}(\hat{x})}{\text{var}(\hat{x}) + \text{var}(\hat{\varepsilon})}. \]

(22.9)

Increase in Precision

The reciprocal of a variance is called a precision. For a truth-plus-noise signal, there is an alternative to formula (22.8) for the conditional variance that is often useful. The alternative formula is expressed in terms of precisions. First, note that (22.9) implies

\[ 1 - \beta = \frac{\text{var}(\hat{s})}{\text{var}(\hat{x}) + \text{var}(\hat{\varepsilon})}. \]

Substitute this and take reciprocals of both sides of (22.8) to obtain

\[ \frac{1}{\text{var}(\hat{s})} = \left(1 + \frac{\text{var}(\hat{s})}{\text{var}(\hat{x}) + \text{var}(\hat{\varepsilon})}\right) \frac{1}{\text{var}(\hat{x})} \]

(22.10)

Thus, the precision of the estimate of \( \hat{x} \) increases when the signal is observed, and the increase in the precision equals the precision of the signal’s noise term.

Multivariate Signals

If either \( \hat{x} \) or \( \hat{s} \) is a vector with \( (\hat{x}, \hat{s}) \) joint normal, then, using the notation of Section 3.5, the conditional expectation of \( \hat{x} \) given \( \hat{s} \) is given by the multivariate projection formula (3.32):

\[ \mathbf{E}[\hat{x}\mid\hat{s}] = \mathbf{E}[\hat{\varepsilon}] + \text{Cov}(\hat{x}, \hat{\varepsilon})\text{cov}(\hat{s})^{-1}(\hat{s} - \mathbf{E}[\hat{s}]). \]

(22.11)

Following the same reasoning leading to (22.5) yields the following formula for the conditional covariance matrix of \( \hat{x} \):

\[ \text{Cov}(\hat{x}\mid\hat{s}) = \text{Cov}(\hat{x}) - \text{Cov}(\hat{s}, \hat{x})\text{Cov}(\hat{s})^{-1}\text{cov}(\hat{s}, \hat{x}). \]

(22.12)

Sequential Projections

Suppose \( \hat{x} \) is a scalar, and there are signals \( \hat{s}_1, ..., \hat{s}_n \). An alternative to using the multivariate formulas (22.11) and (22.12) is to use sequential conditional projections. The projection of \( \hat{x} \) on a random variable \( \hat{s}_m \) conditional on other random variables \( \hat{s}_1, ..., \hat{s}_{m-1} \) is defined as in (22.2) except that means, variances, and covariances are replaced by means, variances, and covariances conditional on \( \hat{s}_1, ..., \hat{s}_{m-1} \).
Suppose the signals are truth-plus-noise $\tilde{s}_i = \tilde{x} + \tilde{\varepsilon}_i$ with independent noise terms $\tilde{\varepsilon}_i$. The formula (22.7) for the conditional mean generalizes as

$$\mathbb{E}(\tilde{s}_1, \ldots, \tilde{s}_n) = (1 - \beta_m)\mathbb{E}(\tilde{s}_1, \ldots, \tilde{s}_{m-1}) + \beta_m \tilde{s}_m$$

(22.13)

for $m \leq n$, where

$$\beta_m = \frac{\text{cov}(\tilde{x}, \tilde{s}_m|\tilde{s}_1, \ldots, \tilde{s}_{m-1})}{\text{var}(\tilde{s}_m|\tilde{s}_1, \ldots, \tilde{s}_{m-1})} = \frac{\text{var}(\tilde{x}|\tilde{s}_1, \ldots, \tilde{s}_{m-1})}{\text{var}(\tilde{s}_1, \ldots, \tilde{s}_{m-1}) + \text{var}(\tilde{\varepsilon}_m)}.$$  

(22.14)

Formula (22.13) states that the change in the conditional expectation when the signal $\tilde{s}_m$ is observed is proportional (with proportionality constant $\beta_m$) to the “innovation” in the signal, meaning the difference between $\tilde{s}_m$ and its conditional mean $\mathbb{E}(\tilde{s}_m|\tilde{s}_1, \ldots, \tilde{s}_{m-1}) = \mathbb{E}(\tilde{x}|\tilde{s}_1, \ldots, \tilde{s}_{m-1})$.

The formula (22.10) for the precision generalizes as

$$\frac{1}{\text{var}(\tilde{x}|\tilde{s}_1, \ldots, \tilde{s}_n)} = \frac{1}{\text{var}(\tilde{s}_1, \ldots, \tilde{s}_n)} + \frac{1}{\text{var}(\tilde{\varepsilon})}.$$  

for $m \leq n$. Successively substituting this formula yields

$$\frac{1}{\text{var}(\tilde{x}|\tilde{s}_1, \ldots, \tilde{s}_m)} = \frac{1}{\text{var}(\tilde{x})} + \sum_{i=1}^{m} \frac{1}{\text{var}(\tilde{\varepsilon})}.$$  

(22.15)

Thus, the precision of the estimate of $\tilde{x}$ increases by the precision of the signal noise term each time we observe an additional signal (under our assumption that the noise terms are independent).

We can use (22.14) and (22.15) to obtain a formula for $\beta_m$ directly in terms of the variances of $\tilde{x}$ and the $\tilde{\varepsilon}_i$. Namely,

$$\frac{1}{\beta_m} = 1 + \frac{\text{var}(\tilde{\varepsilon}_m)}{\text{var}(\tilde{s}_m|\tilde{s}_1, \ldots, \tilde{s}_{m-1})} = 1 + \frac{\text{var}(\tilde{\varepsilon}_m)}{\text{var}(\tilde{s}_m)} + \sum_{i=1}^{m-1} \frac{\text{var}(\tilde{\varepsilon}_m)}{\text{var}(\tilde{\varepsilon}_i)}.$$  

(22.14’)

Finally, divide both sides of (22.14’) by $\text{var}(\tilde{\varepsilon}_m)$, substitute (22.15), and take reciprocals of both sides to obtain the following alternative to (22.15):

$$\beta_m \text{var}(\tilde{\varepsilon}_m) = \text{var}(\tilde{x}|\tilde{s}_1, \ldots, \tilde{s}_m).$$  

(22.15’)

(p.577) 22.3 Fully Revealing Equilibria
This section presents an example of a fully revealing equilibrium. Consider a single-period market with a risk-free asset in zero net supply. Assume all investors have CARA utility. Let \( \bar{x} \) denote the vector of risky asset payoffs, and let \( \bar{s} = (s_1, \ldots, s_H) \) denote the vector of signals observed by investors before trade. Assume \( (\bar{x}, \bar{s}) \) has a joint normal distribution. The standard device for constructing a fully revealing equilibrium is to consider an artificial economy in which each investor observes the entire vector \( \bar{s} \). We will compute the equilibrium of this artificial economy and then show that equilibrium prices reveal all that investors need to know about \( \bar{s} \). Hence, these prices in the actual economy with rational expectations produce the same demands as in the artificial economy and are therefore equilibrium prices of the actual economy.

Because of joint normality, the distribution of \( \bar{x} \) conditional on \( \bar{s} \) is normal, and the covariance matrix of \( \bar{x} \) conditional on \( \bar{s} \) is constant. Let \( \mu(\bar{s}) \) denote \( \mathbb{E}[\bar{x}|\bar{s}] \), and let \( \Sigma \) denote the covariance matrix of \( \bar{x} \) conditional on \( \bar{s} \). The random vector \( \mu(\bar{s}) \) is a sufficient statistic for \( \bar{s} \) in terms of predicting \( \bar{x} \): The distribution of \( \bar{x} \) conditional on \( \bar{s} \) depends on \( \bar{s} \) only via \( \mu(\bar{s}) \).

Equilibrium of the Artificial Economy

In the artificial economy, equilibrium prices can be computed for each realization of \( \bar{s} \) by using \( \mu(\bar{s}) \) as the vector of expected payoffs and \( \Sigma \) as the covariance matrix of the payoffs in the model of Part I of this book. Thus, from Exercise 4.1, the risk-free return \( R(\bar{s}) \) and equilibrium price vector \( p(\bar{s}) \) are

\[
R(\bar{s}) = \frac{1}{\beta} \exp\left( \alpha \bar{s} - c_0 \right) - \frac{1}{2} \alpha \bar{s} \Sigma \bar{s},
\]

\[
p(\bar{s}) = \frac{1}{\sigma(\bar{s})} \mu(\bar{s}) - \alpha \bar{s},
\]

(22.16a)

where \( \bar{s} \) is a weighted geometric average of the investors’ discount factors, \( \alpha \) is the aggregate absolute risk aversion, \( \bar{s} \) is the vector of supplies of the risky assets, and \( c_0 \) is aggregate date-0 consumption.

Revelation

From the equilibrium prices (22.1), investors can compute \( \mu(\bar{s}) \) as

\[
\mu(\bar{s}) = \alpha \bar{s} + R(\bar{s})p(\bar{s}).
\]

(22.17)
Thus, equilibrium prices are fully revealing in the sense of revealing a sufficient statistic for predicting \( \bar{x} \). In particular, each investor can compute the portfolio of risky assets that would be optimal if \( \bar{x} \) were known (Exercise 2.2) simply by observing equilibrium prices; that is, each investor \( h \) can compute

\[
\frac{1}{\alpha} \varepsilon^\top [\mu(\bar{x}) - R(\bar{x})p(\bar{x})].
\]

(22.18)

Thus, the equilibrium in the artificial economy is a fully revealing equilibrium in the actual economy.

**Grossman-Stiglitz Paradox**

Fully revealing equilibria suffer from the Grossman-Stiglitz paradox as remarked before. Notice that no investor needs to use her private signal \( s_h \) to compute \( \mu(\bar{x}) \), because \( \mu(\bar{x}) \) is fully revealed by the equilibrium prices. Thus, no investor benefits from her private information in equilibrium.

**Diamond-Verrecchia Paradox**

A related paradox is that the equilibrium demand (22.18), when \( \mu(\bar{x}) \) is inferred from prices as in (22.17), is

\[
\frac{\hat{s}}{\alpha} \bar{\theta}.
\]

Thus, each investor’s demand is just a fraction of the aggregate supply and independent of all signals. Even if investors acquire information before trade, it is not clear how their information could get into prices when all investors express constant demands to the market.

**22.4 Grossman-Stiglitz Model**

One circumstance in which equilibrium prices are only partially revealing is when the date–0 supply of the risky assets is random. This would occur if there were traders other than the \( H \) investors being modeled who trade for exogenous reasons, perhaps due to liquidity shocks. Such traders are called noise traders or liquidity traders. We can also regard the noisy supply as being due to random endowments of the \( H \) investors being modeled. With CARA utility, the endowments do not affect demands (no wealth effects), so equilibrium prices are the same with random endowments as with liquidity trades.
Assume CARA utility and normal distributions as in the preceding section. Suppose $\bar{s}$ is a scalar instead of a vector, and adopt the following simplifying assumptions:

- There is a single risky asset.
- The supply of the risky asset is a normally distributed random variable $\bar{z}$ that is independent of $\bar{x}$ and $\bar{s}$.
- The equilibrium risk-free return $R_f$ is exogenously given (see the end-of-chapter notes for discussion).
- $H_I < H$ investors observe $\bar{s}$, and $H_U = H - H_I$ investors have no information other than the equilibrium price.
Linear Equilibrium and Revelation

We will look for an equilibrium in which \( p(\bar{s}, \bar{z}) = a_0 + a_\bar{s} + a_\bar{z} \) with \( a_1 \neq 0 \). By observing the price, uninformed investors can calculate

\[
\frac{\bar{p}(\bar{s}, \bar{z})}{\bar{a}} = \bar{s} + b \bar{z},
\]

where \( b = a_2/a_1 \). The solution for \( a_0, a_1 \), and \( a_2 \) is presented below.

Let \( a_1 \) denote the aggregate absolute risk aversion of the informed investors. As usual, this means the reciprocal of the aggregate risk tolerance of the informed investors. If the informed investors all have the same absolute risk aversion \( \alpha \), then \( a_1 = \alpha / H_\gamma \). A useful observation about this model is that in equilibrium observing \( p(\bar{s}, \bar{z}) \) is equivalent to observing

\[
\mu(\bar{s}) - \alpha_0 \sigma^2, \quad \text{where } \mu(\bar{s}) \text{ denotes the mean of } \bar{x} \text{ conditional on } \bar{s} \text{ as before, and } \sigma^2 \text{ denotes the variance of } \bar{x} \text{ conditional on } \bar{s}. \]

In the fully revealing equilibrium of the previous section, all investors infer \( \mu(\bar{s}) \) from equilibrium prices. Here, investors who are ex ante uninformed only observe \( \mu(\bar{s}) \) perturbed by noise. Thus, there is an advantage to being an informed investor in this model. This advantage can (p.580) be large enough to justify the acquisition of information, so the model does not suffer from the Grossman-Stiglitz paradox.

Note that the noise \(- \alpha_0 \sigma^2 \) disappears in the limit as \( \alpha_1 \to 0 \) or \( \sigma^2 \to 0 \). If the aggregate risk aversion of informed traders is small \(( \alpha_1 \to 0 \)\), informed traders push the price close to the expected discounted payoff, conveying this information to uninformed traders. The same is true if their information is nearly perfect \(( \sigma^2 \to 0 \)\), because they bear very little risk in that circumstance.

Risk Premium

The equilibrium price in this model is a weighted average of the informed and uninformed investors’ conditional expectations of \( \bar{x} / R_\gamma \) minus a risk premium term (see (22.20) below). The discount of the price for risk shown in (22.20) is

\[
\frac{\bar{z}}{(\tau_1 \phi_1 + \tau_2 \phi_2) R_\gamma,}
\]
where $r_i$ is the aggregate risk tolerance of investor class $i$, and $\phi_i$ is the precision of the information of investor class $i$ (the reciprocal of the conditional variance of $\tilde{x}$ given the information obtained in equilibrium). The unconditional expectation of the price is

$$\frac{E[x]}{R_f} \frac{E[z]}{(r_i\phi_i + r_U\phi_U)R_f}.$$ 

Assuming the expected supply $E[z]$ is positive, we have

$$\frac{E[z]}{(r_i\phi_i + r_U\phi_U)R_f} > \frac{E[z]}{(r_i + r_U)\phi_i R_f}.$$ 

(22.19)

The right-hand side of (22.19) would be the expected discount for risk if all investors observed $\tilde{x}$. Thus, on average, the price is lower and the expected return higher due to the presence of uninformed investors.

**Risk-Neutral Investors**

If the informed investors were risk neutral, then the only possible equilibrium would be fully revealing. This is suggested by the result for $\alpha_i \rightarrow 0$, but a more direct argument is based on the fact that such investors do not have optima unless

$$p(\tilde{x}, \tilde{z}) = \frac{E(z)}{R_f}.$$ 

(p.581) Thus, the price reveals $\mu(\tilde{x})$. Note that the random supply of the asset does not affect the price in this setting, because the risk-neutral investors are content to absorb whatever supply is offered when the asset is priced at its expected discounted value. Because the equilibrium is fully revealing, the Grossman-Stiglitz paradox does apply to the model with risk-neutral investors.

We will solve for an equilibrium price of the form

$$p(\tilde{x}, \tilde{z}) = a_0 + a_1\tilde{x} + a_2\tilde{z},$$

for constants $a_0$, $a_1 \neq 0$, and $a_2$. In such an equilibrium, each investor can calculate $\tilde{x} + b\tilde{z}$, where $b = a_2/a_1$.

Let $\sigma^2_i$ denote the variance of $\tilde{x}$ conditional on $\tilde{z}$. Denote the variance of $\tilde{x}$ conditional on $\tilde{x} + b\tilde{z}$ by $\sigma^2_i$. The number of shares demanded by an informed investor $i$ is
so the aggregate demand of informed investors is

\[
\frac{\mathbb{E}(\bar{s}) - R_p(s, z)}{\alpha_i \sigma_i^2},
\]

The number of shares demanded by an uninformed investor \( h \) is

\[
\frac{\mathbb{E}(\bar{s} + b z) - R_p(s, z)}{\alpha_U \sigma_U^2},
\]

so the aggregate demand of uninformed investors is

\[
\frac{\mathbb{E}(\bar{s} + b z) - R_p(s, z)}{\alpha_U \sigma_U^2} \left( \sum_{h \in \text{informed}} \frac{1}{\alpha_h} \right) = \frac{\mathbb{E}(\bar{s}) - R_p(s, z)}{\alpha_i \sigma_i^2}.
\]

The market clearing condition is

\[
\frac{\mathbb{E}(\bar{s} + b z) - R_p(s, z)}{\alpha_U \sigma_U^2} + \frac{\mathbb{E}(\bar{s} + b z) - R_p(s, z)}{\alpha_U \sigma_U^2} = \bar{z}.
\]

To express the solution of the market clearing condition in a simple form, define the risk tolerances \( \tau_i = 1/\alpha_i \) and \( \tau_U = 1/\alpha_U \) and the precisions \( \phi_i = 1/\sigma_i^2 \) and \( \phi_U = 1/\sigma_U^2 \). The market clearing condition can be written as

\[
\tau_i \phi_i \mathbb{E}(\bar{s}) + \tau_U \phi_U \mathbb{E}(\bar{s} + b z) - \bar{z} = (\tau_i \phi_i + \tau_U \phi_U) R_p(s, z),
\]

implying

\[
\begin{align*}
\left( \frac{\tau_i \phi_i}{\tau_i \phi_i + \tau_U \phi_U} \right) \mathbb{E}(\bar{s}) + \left( \frac{\tau_U \phi_U}{\tau_i \phi_i + \tau_U \phi_U} \right) \mathbb{E}(\bar{s} + b z) - \bar{z} &= R_p(s, z) \\
\end{align*}
\]

(22.20)

(p.582) To solve for \( b \), let \( \mu_x \) denote the unconditional mean of \( \bar{x} \), \( \mu_s \) the unconditional mean of \( \bar{s} \), and \( \mu_z \) the unconditional mean of \( \bar{z} \). The normal-normal updating rule (22.2) produces

\[
\begin{align*}
\mathbb{E}(\bar{s}) &= \mu_x + \beta (\bar{s} - \mu_s), \\
\mathbb{E}(\bar{s} + b z) &= \mu_x + \kappa (\bar{s} - \mu_s + b \bar{z} - b \mu_z),
\end{align*}
\]

where

\[
\beta = \frac{\text{cov}(\bar{s}, \bar{z})}{\text{var}(\bar{x})},
\]

\[
\kappa = \frac{\text{cov}(\bar{s}, \bar{z})}{\text{var}(\bar{s}) + \text{var}(\bar{z})}.
\]

Substitute these into (22.20) to obtain

\[
p(\bar{s}, \bar{z}) = \frac{\tau_i \phi_i \mu_x + \beta \beta \kappa (\bar{s} - \mu_s)}{\frac{\tau_i \phi_i}{\tau_i \phi_i + \tau_U \phi_U} + \frac{\tau_U \phi_U}{\tau_i \phi_i + \tau_U \phi_U} \beta \kappa} - \frac{\bar{z}}{\frac{\tau_i \phi_i}{\tau_i \phi_i + \tau_U \phi_U} + \frac{\tau_U \phi_U}{\tau_i \phi_i + \tau_U \phi_U} \beta \kappa}.
\]

This equals \( a_0 + a_2 \bar{z} + a_3 \bar{s} \) if and only if
\[ a_0 = \frac{\mu_x}{\beta} - \frac{\sum_{i=1}^{n} \int \phi_i \beta^{-1} \gamma \phi_i \mu_x}{\sum_{i=1}^{n} \int \phi_i \beta^{-1} \gamma \phi_i \mu_x}, \]
\[ a_1 = \frac{\sum_{i=1}^{n} \int \phi_i \beta^{-1} \gamma \phi_i \gamma}{\sum_{i=1}^{n} \int \phi_i \beta^{-1} \gamma \phi_i \gamma}, \]
\[ a_2 = \frac{\sum_{i=1}^{n} \int \phi_i \beta^{-1} \gamma \phi_i \gamma}{\sum_{i=1}^{n} \int \phi_i \beta^{-1} \gamma \phi_i \gamma}. \]

Note that \( a_1 \neq 0 \). The last two equations imply
\[ b \equiv \frac{a_2}{a_1} = -\frac{1}{\gamma} = -\frac{a_1^2}{\beta}. \]

To obtain explicit formulas for \( a_0, a_1, \) and \( a_2 \), substitute this formula for \( b \) into \( \kappa \) and
\[ \sigma^2_\hat{\mu} = \text{var}(\hat{x}) \left( \frac{\text{cov}(\hat{x}, \hat{z})^2}{\text{var}(\hat{x}) + \beta \text{var}(\hat{z})} \right). \]

For this last fact, see (22.5).

Notice that observing \( \hat{s} + \beta \hat{z} \) is equivalent to observing
\[ \mu_x = \beta \mu_z + \beta (\hat{s} + \beta \hat{z}) = \mu(\hat{s}) - a \sigma^2 \beta. \]

Therefore, the information revealed by prices is \( \mu(\hat{s}) \) perturbed by noise as stated above.

**22.5 Hellwig Model**

In some circumstances, rational expectations equilibria (whether fully or partially revealing) suffer from what Hellwig (1980) terms “schizophrenia.” If the equilibrium price reveals something that is observed by only a single investor, and the investor understands that the price at least partially reveals her information (that is, if the investor has rational expectations), then the investor should also understand that her trades must have affected the price. To be a price taker when formulating demands, as assumed in competitive models, and to simultaneously recognize the dependence of the price on one’s private information is “schizophrenic.”

The schizophrenia issue does not arise when there are multiple investors with identical information. An investor in such an economy can reasonably assume that the trades of others with the same information affect the price but that her own trades have negligible influence. The model in the previous section is a model of that type.

The schizophrenia issue also does not arise when the price is independent of an investor’s signal. For the equilibrium price to be independent of each investor’s signal, each investor’s signal must be irrelevant for forecasting, given the signals of
others. This can be true only if there is a large number (more precisely, an infinite number) of investors. This section presents such a model, due to Hellwig (1980).

Model

Consider the model of the previous section, but suppose all investors have the same absolute risk aversion $\alpha$ and change the signal structure as follows. Suppose each investor $h$ observes $\tilde{s}_h = \tilde{x} + \tilde{\varepsilon}_h$, where the $\tilde{\varepsilon}_h$ are IID zero-mean normal random variables that are independent of $\tilde{x}$. To make clear that the variance of $\tilde{s}_h$ is the same for each $h$, denote the variance by $\text{var}(\varepsilon)$. We are going to consider the limit as $H \to \infty$. Assume the supply of the asset is $\tilde{z} = H\tilde{y}$, where $\tilde{y}$ is normally distributed and independent of $\tilde{x}$ and the $\tilde{\varepsilon}_h$. The random variable $\tilde{y}$ is the supply per capita ($\tilde{z}/H$), and it will be held fixed as $H$ is increased. By the strong law of large numbers, $\frac{1}{H} \sum_{h=1}^{H} \tilde{s}_h \to \tilde{x}$ as $H \to \infty$. We are going to work in the limit economy, taking $H = \infty$, so $\tilde{x}$ would be known if we had access to all of the signals observed by investors.\(^4\)

**(p.584) Revelation**

We show below that there is a partially revealing equilibrium in this limit economy in which the equilibrium price $p(\tilde{x}, \tilde{y})$ reveals

$$\tilde{x} - \alpha \text{var}(\varepsilon) \tilde{y}.$$  

Because $\tilde{x}$ can be computed from the signals of any infinite subset of investors—in particular, from the set excluding investor $h$ for any $h$—no investor’s private information can be seen in $p(\tilde{x}, \tilde{y})$. Thus, the behavior of investors is not schizophrenic. On the other hand, each investor’s private information $\tilde{s}_h$ is useful in equilibrium, because the equilibrium price reveals neither $\tilde{x}$ nor $\tilde{s}_h$. Thus, the Grossman-Stiglitz paradox is avoided. Also, each investor’s equilibrium demand depends on her private signal $\tilde{s}_h$, so the Diamond-Verrecchia paradox is avoided.

Notice that the equilibrium price reveals more about $\tilde{x}$ when risk aversion $\alpha$ is smaller or when individual signals are more precise ($\text{var}(\varepsilon)$ is smaller). This is similar to the results of the previous section.

Risk Premium
The equilibrium price is

$$p(x, y) = \frac{1}{R_f} \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \mathbb{E}[x + b\bar{y}, \bar{s}_h] - \frac{\alpha^2 \bar{y}}{R_f},$$

(22.21)

where $\sigma^2$ denotes the variance of $\bar{x}$ conditional on $x + b\bar{y}$ and $\bar{s}_h$ and where $b = -\alpha \text{var}(\bar{e})$. The first term on the right-hand side of (22.21) is the average conditional expectation of the discounted asset value, conditional on the information obtained in equilibrium. The last term, $-\alpha^2 \bar{y}/R_f$, is a risk premium term, depending on risk aversion, the conditional risk, and the supply of the asset, also as in the previous section.

**Continuum of Investors**

This model can be solved when investors differ with regard to risk aversion and signal quality. It is easiest to express such a model in a continuum of investors framework, with investors indexed by $h \in [0, 1]$. The equilibrium price in the model reveals

$$\bar{x} - \bar{y} \int_0^1 \frac{1}{\alpha \text{var}(\bar{e}_h)} dh$$

See Exercise 22.2.

(p.585)

We will solve for an equilibrium price of the form

$$p(x, y) = a_0 + a_1 x + a_2 y$$

for constants $a_0$, $a_1 \neq 0$, and $a_2$. In such an equilibrium, each investor can calculate $x + b\bar{y}$, where $b = a_2/a_1$, so she has that information in addition to her private signal $\bar{s}_h$.

The number of shares of the risky asset demanded by investor $h$ is

$$\frac{\mathbb{E}[x + b\bar{y}, \bar{s}_h] - R_f p(x, y)}{\alpha \sigma^2},$$

where $\sigma^2$ denotes the variance of $x$ conditional on $x + b\bar{y}$ and $\bar{s}_h$.

The market clearing condition, in per capita terms, is

$$\lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \mathbb{E}[x + b\bar{y}, \bar{s}_h] - \frac{\alpha^2 \bar{y}}{R_f} = \bar{y},$$

which we can rearrange as (22.21).

Formulas (22.13), (22.14'), and (22.15') for sequential conditional expectations, conditioning first on $x + b\bar{y}$ and then on $\bar{s}_h$, imply
\begin{align}
\mathbb{E}[\bar{x} + b\bar{y}] &= \bar{x} + \beta(\bar{x} - \bar{x} + b\bar{y} - b\bar{y}) \\
(22.22a) \\
\text{and} \\
\mathbb{E}[\bar{x} + b\bar{y}, \bar{s}] &= \mathbb{E}[\bar{x} + b\bar{y}] + \kappa(\bar{x} - \mathbb{E}[\bar{x} + b\bar{y}]), \\
(22.22b) \\
\text{where} \\
\frac{1}{\beta} &= 1 + \frac{b^2\text{var}(\bar{y})}{\text{var}(\bar{x})}, \\
(22.22c) \\
\frac{1}{\kappa} &= 1 + \frac{\text{var}(\bar{x})}{\text{var}(\bar{x})} + \frac{\text{var}(\bar{y})}{b^2\text{var}(\bar{y})}, \\
(22.22d) \\
\sigma^2 &= \kappa\text{var}(\bar{x}). \\
(22.22e) \\
\text{By the strong law of large numbers and (22.22b),} \\
\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[\bar{x} + b\bar{y}, \bar{s}] &= \mathbb{E}[\bar{x} + b\bar{y}] + \kappa(\bar{x} - \mathbb{E}[\bar{x} + b\bar{y}]) = \kappa\bar{x} + (1 - \kappa)\mathbb{E}[\bar{x} + b\bar{y}]. \\
\text{Thus, the market clearing condition (22.21) is equivalent to} \\
p(\bar{x}, \bar{y}) &= \frac{\kappa\bar{x} + (1 - \kappa)[\bar{x} + \beta(\bar{x} - \bar{x} + b\bar{y} - b\bar{y})] - \alpha a^2 \bar{y}}{Rt}. \\
(22.21') \\
\text{(p.586) This equals } a_0 + a_1\bar{x} + a_2\bar{y} \text{ if and only if} \\
a_0 &= \frac{(1 - \kappa)[\beta(\bar{x} - \bar{x} + b\bar{y})]}{Rt}, \\
a_1 &= \frac{\kappa(\bar{x} - \bar{x})}{Rt}, \\
a_2 &= \frac{(1 - \kappa)[\alpha^2 \bar{y}]}{Rt}. \\
\text{Note that } a_1 \neq 0. \text{ The last two equations imply} \\
bdet &a_1 a_2 \frac{\alpha^2 \kappa}{Rt} \\
\text{and now (22.22e) implies } b &= -\alpha \text{var}(\bar{e}). \text{ Thus, the equilibrium price reveals } \bar{x} - \alpha \text{var}(\bar{e}) \bar{y} \text{ as claimed.} \\
\text{22.6 Notes and References} \\
\text{For the history of the rational expectations hypothesis, see Grossman (1981). The weak, semi-strong, and strong forms of the efficient markets hypothesis appear in Fama (1970), who attributes the weak- and strong-form terminology to Harry Roberts.} \\
\text{The effect of short sales constraints on prices is very different when differences in beliefs are due to asymmetric information rather than to heterogeneous priors. In rational expectations equilibria with homogeneous priors, investors understand that short sales constraints preclude the expression of negative}
opinions, and they account for the possibility that such unexpressed opinions may exist when they attempt to infer the asset value from the price. The consequence of short sales constraints in such a setting is that prices may be less informative but are not biased. This is demonstrated by Diamond and Verrecchia (1987) using a variation of the Glosten-Milgrom model (Section 24.1).

The no-trade theorem is due to Milgrom and Stokey (1982) and Tirole (1982). The proof given in the text follows Tirole (1982). Milgrom and Stokey consider more general economic mechanisms (not just rational expectations equilibria) and show that there is no trade if the initial allocation is Pareto optimal and if investors have “concordant beliefs.” Investors have concordant beliefs if they perceive the same conditional distribution of signals given payoffs. They could have different beliefs regarding the marginal distribution of payoffs, so concordant beliefs is a weaker assumption than common priors. A result closely (p.587) related to the no-trade theorem is the fact that individuals with common priors cannot “agree to disagree” (Aumann, 1976; Rubinstein and Wolinsky, 1990).

The conditional projections discussed in Section 22.2 are frequently applied in dynamic models. Suppose that \( \tilde{x}, \tilde{s}_1, \ldots, \tilde{s}_T \) are joint normal, and the information at date \( t \) is \( \tilde{s}_t \). Then,

\[
E[\tilde{x}|t] = E_x[\tilde{x}t] + \frac{\text{cov}_{xt}(\tilde{x}, \tilde{s}_t)}{\text{var}_t(\tilde{s}_t)}(\tilde{s}_t - E_x[\tilde{s}_t])
\]

(22.23)

We call \( \tilde{s}_t - E_x[\tilde{s}_t] \) an innovation. Formula (22.23) is the discrete-time Kalman filtering formula, and it states that the conditional mean of \( \tilde{x} \) is revised in proportion to the innovation. Chapter 23 presents the Kalman filter in continuous time.


The model of a partially revealing equilibrium in Section 22.4 is due to Grossman (1976) and Grossman and Stiglitz (1976). For a continuous-time version of that model, see Wang (1993). Diamond and Verrecchia (1981) solve a variation of the model in Section 22.4 in which each investor observes a private
They also discuss the issue that Section 22.3 calls the Diamond-Verrecchia paradox. The model of a large number of investors is from Hellwig (1980). Hellwig also presents convergence results, as the number of investors converges to infinity. Breon-Drish (2015) solves the Grossman-Stiglitz model—and an extension in which there are multiple types of investors who receive different signals—without assuming that the asset value is normally distributed. Instead, he assumes that the conditional distribution of the payoff given signals has an exponential distribution. This includes, for example, the case in which the asset value has a Bernoulli (two-point) distribution.

The Grossman-Stiglitz and Hellwig models are a bit unsatisfactory, because they assume a single risky asset and risk-averse investors. We do not really want to assume the single risky asset is the aggregate market, because there is unlikely to be asymmetric information about the market as a whole. On the other hand, we cannot easily regard the models as being about a single asset in a multiasset market, because demands by risk-averse investors depend on correlations between assets, which are not modeled. One possibility is to assume the value of the asset being modeled is independent of all other assets, so demands in the CARA-normal world depend only on the mean and variance of the risky asset and the risk-free return (Section 2.4). In this setting, it is natural to assume that the risk-free return is given exogenously (as in Sections 22.4 and 22.5), because demands for an individual asset should have a negligible effect on the risk-free return. Admati (1985) develops a version of the Hellwig model with multiple assets.

The reason we want to assume the risk-free return is exogenously given is that the inference problem is complicated when the risk-free return depends on signals and asset supplies. For example, \( \log R_t \) in (22.16a) is quadratic rather than affine in \( \bar{\theta} \) and \( \bar{z} \). One justification that is commonly given for taking the risk-free return to be exogenous is that we can take the risk-free asset to be the numeraire. However, when the risk-free asset is the numeraire, the price of the date-0 consumption good in units of the risk-free asset should reveal exactly the same information that would be revealed by the price of the risk-free asset if the consumption good were the numeraire. A different justification that could be given is that
there exists a risk-free production technology that is perfectly elastic. However, investors in aggregate short the risk-free asset in these models with positive probability (due to linearity of demands in normally distributed signals), so this assumption would require that it be possible to run the production technology in reverse, borrowing in aggregate from the future.

Grundy and McNichols (1989), Brown and Jennings (1989), and Brennan and Cao (1996) study dynamic (finite-horizon, discrete-time) versions of the Hellwig model. In those models, investors trade at each date even if there is no additional information other than the asset price, because of updating of expectations from the asset price. This is another possible explanation (in addition to heterogeneous priors) of the large volume of trade observed in actual markets. Grundy and McNichols (1989) show that there may be an equilibrium in which prices in a second round of trade fully reveal the asset value. However, equilibrium prices do not depend on demands in that equilibrium (that is, the equilibrium is paradoxical as discussed at the end of Section 22.3). There is also an equilibrium in which prices at each date reveal additional information about the asset value but are not fully revealing. In such an equilibrium, the current price at any date is not a sufficient statistic for predicting the asset value, so investors condition on the history of prices. This could be interpreted as technical analysis (Brown and Jennings, 1989). Brennan and Cao (1996) show that better-informed investors act as contrarians, selling when the asset price rises, and lesser-informed investors act as momentum traders, buying when the asset price rises. Wang (1993) obtains the same result, for some parameter values. Brennan and Cao (1996) also show that investors would not trade after date 0 (assuming there are no liquidity trades after date 0) if there were a derivative asset the payoff of which is quadratic in the underlying asset value. This is because such an asset is sufficient for implementing the Pareto-optimal sharing rules, similar to Exercise 21.1.

Whether information asymmetry affects a firm’s cost of capital (the expected return required by shareholders) is a topic that has received considerable attention, particularly in the accounting literature. Easley and O’Hara (2004) observe that making some private information public will reduce the risk premium of a stock. They conclude that information...
asymmetry matters for the cost of capital of a firm. However, Lambert, Leuz, and Verrecchia (2011) point out that the shift in information described by Easley and O’Hara (2004) has the effect of increasing the average precision of investors’ information. Lambert et al. also point out that, given the average precision, the distribution of information across investors is irrelevant for the risk premium. Thus, it is the average amount of information, not the asymmetry of information, that affects the cost of capital.

Exercises

22.1. In the economy of Section 22.4, assume the uninformed investors are risk neutral. Find a fully revealing equilibrium, a partially revealing equilibria in which the price reveals \( \bar{z} + b\bar{z} \) for any \( b \), and a completely unrevealing equilibrium (an equilibrium in which the price is constant rather than depending on \( \bar{z} \) and/or \( \bar{z} \)).

22.2. Consider the model of Section 22.5, but assume there is a continuum of investors indexed by \( h \in [0, 1] \) with possibly differing risk-aversion coefficients \( \alpha_h \) and possibly differing error variances \( \text{var}(\varepsilon_h) \). Suppose, for some \( b \), that each investor observes \( \bar{x} + b\bar{y} \) in addition to her private signal \( \bar{z}_h \). The market clearing condition is

\[
\int_{0}^{1} \theta_h(x + b\bar{y}, \bar{z}_h) dh = y,
\]

where \( \theta_h \) is the number of shares demanded by investor \( h \). Let \( \sigma_h^2 \) denote the variance of \( \bar{x} + b\bar{y} \) and \( \bar{z}_h \). Set \( \phi_h = 1/\sigma_h^2 \). Define

\[
\tau = \int_{0}^{1} \tau_h dh \quad \text{and} \quad \phi = \int_{0}^{1} \phi_h dh,
\]

where \( \tau_h \) is the risk tolerance of investor \( h \).

(a) Show that the equilibrium price is a discounted weighted average of the conditional expectations of \( \bar{x} \) minus a risk premium term, where the weight on investor \( h \) is \( \tau_h\phi_h/(\tau\phi) \).

(b) Define

\[
\beta = \frac{\text{var}(\varepsilon)}{\text{var}(\varepsilon) + \text{var}(\bar{y})} \quad \text{and} \quad \kappa_h = \frac{(1-b)\text{var}(\varepsilon)}{\text{var}(\varepsilon) + \text{var}(\bar{y})}.
\]

Show that

\[
\tau_b\phi_h\kappa_h = \frac{1}{\alpha_h\text{var}(\varepsilon_h)}.
\]
(p.590) (c) Assume the strong law of large numbers holds in the sense that
\[ \int_0^1 \theta_k \kappa_k \kappa'_k dh = 0. \]
Define
\[ \kappa = \frac{1}{\theta} \int_0^1 \tau_k \phi_k \kappa'_k dh. \]
Show that the equilibrium price equals \( a_0 + a_1 (\bar{x} + b \bar{y}) \) if and only if
\[
\begin{align*}
 a_0 &= \frac{(1 - x)(b - x b y)}{\theta}, \\
 a_1 &= \frac{x - y}{\theta}, \\
 b &= -1 \int_0^1 \frac{1}{\theta} \phi_k \kappa'_k dh.
\end{align*}
\]

Notes:
(1.) To derive (22.1), start from
\[ \mathbb{E}[\bar{x} | \bar{s}_p, p(\bar{s}_p, \bar{s}_j)] \geq \mathbb{E}[\bar{s}_p, \bar{s}_j] \]
which is possible only if
\[ \mathbb{E}[\bar{x} | \bar{s}_p, \bar{s}_j] = \mathbb{E}[\bar{s}_p, \bar{s}_j]. \]
If two random variables \( \bar{y} \) and \( \bar{z} \) satisfy \( \bar{y} \geq \bar{z} \) and \( \mathbb{E}[\bar{y}] = \mathbb{E}[\bar{z}] \), then it must be that \( \bar{y} = \bar{z} \) with probability 1. Apply this fact to \( \bar{y} = \mathbb{E}[\bar{x} | \bar{s}_p, p(\bar{s}_p, \bar{s}_j)] \) and \( \bar{z} = \mathbb{E}[\bar{s}_p, \bar{s}_j] \) to expectation conditional on \( p(\bar{s}_p, \bar{s}_j) \) to obtain the first equality in (22.1). The second equality is derived by the same reasoning.

(2.) Normality implies that the residual \( \bar{z} \) in the projection (which is always uncorrelated with \( \bar{z} \)) is independent of \( \bar{s} \), hence mean independent of \( \bar{s} \). The fact that \( \mathbb{E}[\bar{x} | \bar{s}_p] = 0 \) implies that \( \mathbb{E}[\bar{x} | \bar{s}_p] \) equals the projection.

(3.) If the matrix \( \text{Cov}(\bar{s}) \) is nonsingular, then the covariance matrix of \( \bar{x} \) conditional on \( \bar{s} \) is shown in (22.12) to be
\[ \Sigma = \text{Cov}(\bar{x}) - \text{Cov}(\bar{s}, \bar{x}) \text{Cov}(\bar{s})^{-1} \text{Cov}(\bar{s}, \bar{x}). \]

(4.) For a more formal model, we could take the set of investors to be a continuum, subject to the issues discussed in Section 11.6 regarding the law of large numbers for a continuum of IID random variables, or we could take \( H = \infty \) with the size of any set of investors being defined by a purely finitely additive measure, as is also discussed in Section 11.6.