Heterogeneous Beliefs

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Abstract and Keywords

There is a representative investor in a complete single-period market if all investors have log utility or if all investors have CARA utility, even if investors have different beliefs. This extends to dynamic markets for log utility but not for CARA utility. With CARA and other LRT utility, the concept of a representative investor can be extended to include a random discounting factor that is either a supermartingale or a submartingale. If there are short sales constraints, then assets may be overpriced relative to average beliefs, because pessimistic investors are constrained from trading on their beliefs. The overpricing is an increasing function of the dispersion of beliefs. In a dynamic market with short sales constraints, prices can exceed even the values of optimistic investors (a speculative bubble).

Keywords: heterogeneous beliefs, heterogeneous priors, representative investor, short sales constraints, Miller’s model, dispersion of beliefs, speculative trading, speculative bubble

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This part of the book provides brief introductions to several important extensions of the theory. There is a large literature on each of the topics covered. Rather than trying to survey the literatures, we try to provide careful treatments of some foundational issues and models within each literature. The goal is to provide enough background to make the literatures accessible.

This chapter relaxes the assumption made in most previous chapters that investors have identical beliefs. We assume in this chapter that investors hold their beliefs dogmatically, in the sense that they are not persuaded to revise their beliefs when they realize that others hold different beliefs. It is very natural to assume different individuals have different beliefs and hold them dogmatically if we take the subjectivist view of probability (Ramsey, 1931; Savage, 1954). Chapter 22 examines differences in beliefs arising from differences in information. In that case, each investor learns something about other investors’ information from the terms of trade they are willing to accept, and they regard this information as useful, revising their own beliefs in consequence. Differences in beliefs due to differences in information are differences in posterior beliefs—that is, after observing information. In this chapter, we study differences in prior beliefs.

The two main topics discussed in this chapter are aggregation (existence of a representative investor) and equilibria in markets with short sales constraints. (p.554) If all investors have log utility, then there is a representative investor in either a single-period model or a dynamic model. If all investors have CARA utility, then there is a representative investor in a single-period model.

If investors agree on which events have zero probability—in the sense that if an investor \( h \) assesses the probability of any event to be zero, then so do all other investors—then their beliefs are said to be mutually absolutely continuous. Mutual absolute continuity is necessary for the existence of a Pareto optimum, because if investor \( h \) assesses the probability of an event \( A \) to be zero and investor \( j \) does not, then adding \( 1_A \) to the consumption of investor \( j \) and subtracting \( 1_A \) from the consumption of investor \( h \) leads to a Pareto improvement. Furthermore, (p.555) if investors’ beliefs are not mutually absolutely continuous and there are no constraints on
portfolios, then a competitive equilibrium typically does not exist. If some investor regards an event as having zero probability and another investor regards it as having positive probability and an Arrow security for the event exists, then the investor who views it as having zero probability will want to sell an infinite amount of the Arrow security to the other investor. Short sales constraints make equilibria possible in this circumstance.

21.1 State-Dependent Utility Formulation
When beliefs are heterogeneous, it is frequently useful to transform the model to one with homogeneous beliefs and state-dependent utility. Let \( P_h \) denote the beliefs of investor \( h \), for \( h = 1, \ldots, H \), and let \( E_h \) denote the corresponding expectation operator. Define \( P \) to be the average beliefs; that is, for each event \( A \), set

\[
P(A) = \frac{1}{H} \sum_{h=1}^{H} P_h(A)
\]

Let \( E \) denote expectation with respect to \( P \). For each \( h \), \( P_h \) is absolutely continuous with respect to \( P \) (meaning that if \( P(A) = 0 \), then \( P_h(A) = 0 \), for each event \( A \)). Hence, there exists a nonnegative random variable \( \tilde{z}_h \) (the Radon-Nikodym derivative of \( P_h \) with respect to \( P \)—see Appendix A.10) such that, for each random variable \( \tilde{x} \),

\[
E[\tilde{x}] = E[\tilde{z}_h \tilde{x}]
\]

In particular, for a random wealth \( \tilde{w} \) and utility function \( u_{h'} \),

\[
E[ u_{h'}(\tilde{w}) ] = E[ \tilde{z}_h u_{h'}(\tilde{w}) ]
\]

Therefore, we can interpret all investors as having the same beliefs (the average beliefs) and \( \tilde{z}_h u_{h'}(\tilde{w}) \) as being the state-dependent utility of wealth of investor \( h \). If investors’ beliefs are mutually absolutely continuous, then \( P \) is absolutely continuous with respect to each \( P_h \) and each \( \tilde{z}_h \) is strictly positive. An example of \( \tilde{z}_h \) in the case of normal distributions is given in Exercise 21.1.

If there are only finitely many states of the world, then the definitions of the previous paragraph are very simple: For each state \( \omega \),

\[
P(\omega) = \frac{1}{H} \sum_{h=1}^{H} P_h(\omega)
\]
and, for each $h$,

$$Z_h(\omega) = \frac{P_h(\omega)}{P(\omega)}.$$  

Note that the set of possible states of the world should be defined as the union of the sets regarded as possible by the various investors.

The first-order condition for an investor in a single-period model is, as always, that marginal utility evaluated at the optimal wealth is proportional to an SDF: $u_h(\tilde{w}_h) = \gamma_h \tilde{m}_h$, where $u_h$ is the utility function of investor $h$, $\tilde{w}_h$ is her optimal wealth, $\gamma_h$ is a constant, and $\tilde{m}_h$ is an SDF. The meaning of an SDF is the same as before—equation (3.1)—but now specifying that the expectation operator is $E_\mu$; that is, for all assets $i$, $E_\mu(\tilde{m}_h \tilde{x}_i) = p_i$. The expectation operator is $E_\mu$ because it is with respect to $E_\mu$ that investor $h$ maximizes her expected utility. Transforming to the average beliefs, we have

$$\frac{1}{\gamma_h} E_\mu(u_h(\tilde{w}_h) \tilde{x}_i) = p_i = \frac{1}{\gamma_h} E_\mu(\tilde{m}_h u_h(\tilde{w}_h) \tilde{x}_i) = p_i.$$  

Thus, at the optimal wealth $\tilde{w}_h$ of investor $h$, $\tilde{m}_h u_h(\tilde{w}_h)$ is proportional to an SDF relative to the average beliefs.

### 21.2 Aggregation in Single-Period Markets

This section establishes that there is a representative investor in a complete single-period market if all investors have log utility or if all investors have CARA utility. Investors can differ in beliefs and in endowments. Assume the beliefs of investors are mutually absolutely continuous. Except for the heterogeneity of beliefs, we model the securities market as in Chapter 4. In particular, we assume short sales are unconstrained. We assume the market is at a competitive equilibrium, and we exploit the Pareto optimality of competitive equilibria in complete markets, that is, the First Welfare Theorem. The First Welfare Theorem is true under very general assumptions on preferences—it is not necessary that investors’ preferences be represented by expected utilities nor, if they are, that the beliefs underlying the expectations be the same.

*(p.556)* With heterogeneous beliefs in a single-period model, the social planner’s objective function (4.2) is replaced by

$$\sum_{h=1}^H \lambda_h E_\mu(u_h(\tilde{w}_h)) = \sum_{h=1}^H \lambda_h B \tilde{m}_h u_h(\tilde{w}_h).$$
We can interpret the social planner as a representative investor having the average beliefs and a state-dependent utility function, but the state dependence renders the concept much less useful, compared to the model with homogeneous beliefs. With either log utility or CARA utility, there is a representative investor with a state-independent utility function.

Log Utility
Suppose each investor $h$ has logarithmic utility. A competitive equilibrium in a complete market maximizes the social planner’s utility function for some positive weights $\lambda_h$. Take the weights $\lambda_1, \ldots, \lambda_H$ to sum to 1 (which we can always do by dividing each weight by the sum). The solution of the social planning problem

$$\max \sum_{h=1}^{H} \lambda_h \mathbb{E}[Z_h \log \bar{w}_h] \quad \text{subject to} \quad \sum_{h=1}^{H} \bar{w}_h = \bar{w}_m$$

is

$$\bar{w}_h = \frac{\lambda_h \bar{w}_m}{\sum_{h=1}^{H} \lambda_h \bar{w}_h}.$$

Substitute this into the objective function in (21.2) to see that the social planner’s utility is

$$\sum_{h=1}^{H} \lambda_h \mathbb{E}[Z_h \log \bar{w}_h] = \mathbb{E} \left[ \left( \sum_{h=1}^{H} \lambda_h \bar{w}_h \right) \log \bar{w}_m \right] + \mathbb{E} \left[ \sum_{h=1}^{H} \lambda_h \bar{w}_h \log \left( \frac{\lambda_h \bar{w}_h}{\sum_{h=1}^{H} \lambda_h \bar{w}_h} \right) \right].$$

Define the weighted-average beliefs

$$P_m(A) = \sum_{h=1}^{H} \lambda_h P_h(A)$$

(p.557) The expectation operator $\mathbb{E}_m$ corresponding to $P_m$ satisfies

$$\mathbb{E}_m[x] = \sum_{h=1}^{H} \lambda_h \mathbb{E}_h[x] = \mathbb{E} \left[ \left( \sum_{h=1}^{H} \lambda_h \bar{w}_h \right) \bar{w}_m \right].$$

Therefore, the social planner’s utility (21.4a) can be written as

$$\mathbb{E}_m[\log \bar{w}_m] + \mathbb{E} \left[ \sum_{h=1}^{H} \lambda_h \bar{w}_h \log \left( \frac{\lambda_h \bar{w}_h}{\sum_{h=1}^{H} \lambda_h \bar{w}_h} \right) \right].$$
(21.4) Because the second term in (21.4’) is a constant that does not depend on the \( \tilde{w}_h \), (21.4’) implies that there is a representative investor with log utility and beliefs \( P_m \) (Section 7.1).

Even with log utility, there is an important distinction between homogeneous beliefs and heterogeneous beliefs. The sharing rules (21.3) typically cannot be implemented unless markets are complete, because of the state dependence introduced through the \( z_h \). This is in contrast to log utility with homogeneous beliefs, in which case the sharing rules are affine, and, in the absence of labor income, an equilibrium allocation in incomplete markets is Pareto optimal.

Constant Absolute Risk Aversion

Assume now that each investor \( h \) has utility \( u_h(w) = -e^{-\alpha w} \). Set
\[
\tau_h = 1/\alpha_h \quad \tau = \sum_{h=1}^H \tau_h \quad \alpha = 1/\tau,
\]
so \( \alpha \) is the aggregate absolute risk aversion. The solution of the social planning problem
\[
\max -\sum_{h=1}^H \lambda_h \mathbb{E}[z_h e^{-\alpha \tilde{w}_h}] \text{subject to } \sum_{h=1}^H \tilde{w}_h = \tilde{w}_m
\]
(21.5)
is
\[
\tilde{w}_h = -\tau_h \tilde{w}_m + \tau_h \left[ \log(\alpha_h \tilde{w}_h) - \sum_{j=1}^H \log(\alpha_j \tilde{z}_j) \right].
\]
(21.6)
Substitute this into the objective function in (21.5) to see that the social planner’s utility is
\[
\frac{1}{\tau} \prod_{j=1}^H (\tilde{z}_j)^{\gamma_j} \mathbb{E}\left[e^{-\alpha \tilde{w}_m - \sum_{j=1}^H \gamma_j \log(\alpha_j \tilde{z}_j)} \right].
\]
(21.7)

(p.558) Define
\[
\tilde{z} = \frac{\prod_{j=1}^H \tilde{z}_j^{\gamma_j}}{\mathbb{E}\left[\prod_{j=1}^H \tilde{z}_j^{\gamma_j}\right]}.
\]

Note that the numerator in the definition of \( \tilde{z} \) is a weighted geometric average of the \( \tilde{z}_j \). Each \( \tilde{z}_j \) has mean equal to 1 under the average beliefs, but the geometric average does not. This motivates the division by the mean in the definition of \( \tilde{z} \). The random variable \( \tilde{z} \) is nonnegative and has an expected value equal to 1, so it can serve as a Radon-Nikodym derivative. For each event \( A_j \), define
where $1_A$ is the random variable equal to 1 when the state of the world is in $A$ and equal to 0 otherwise. Let $E_m$ denote expectation with respect to $P_m$. The social planner's utility (21.4) is proportional to

$$\mathbb{E}[e^{-\sigma_w^2}] = E_m[e^{-\sigma_w}]$$

Therefore, there is a representative investor with constant absolute risk aversion $\sigma$ and beliefs $P_m$.

As in the case of log utility, markets typically must be complete in order for the sharing rules (21.6) to be implementable in the securities market and for the existence of a representative investor. However, in one special case, the sharing rule (21.6) is affine and hence can be implemented (if there is a risk-free asset and no labor income). This special case is the case of investors who agree that aggregate wealth $\tilde{w}_m$ is normally distributed and agree on its variance. If investors disagree about the variance, then it is sufficient to have an asset the payoff of which is a quadratic function of $\tilde{w}_m$ (Exercise 21.1).

### 21.3 Aggregation in Dynamic Markets

There is a representative investor with log utility in a dynamic complete market if all investors have log utility and discount utility at the same rate. Investors can differ in beliefs and endowments. However, aggregation in dynamic markets—with state-independent utility—is not possible for CARA utility or for other LRT utility functions.

Consider a discrete-time model with horizon $T$. Suppose the market is complete, all investors have the same discount factor $\delta$, and investors' beliefs are mutually absolutely continuous. Let $C$ denote aggregate consumption. Scale the weights $\lambda_h$ in the social planner's objective function so that they sum to 1.
Log Utility

The social planning problem is

$$\max \sum_{h=1}^{H} \sum_{t=0}^{T} \lambda_h \delta^t \mathbb{E}[z_h \log C_h] \text{ subject to } \sum_{h=1}^{H} C_h = C_r.$$ 

We cannot solve this pointwise (in each date and state) because doing so would produce $C_{ht}$ that depend on $z_h$ and hence are not measurable with respect to information at date $t$. However, it is easy to modify the problem so it can be solved pointwise. For any $C_{ht}$ that is measurable with respect to date–$t$ information, iterated expectations implies $\mathbb{E}[z_h \log C_h] = \mathbb{E}[Z_h \log C_h]$, where $Z_h = \mathbb{E}[z_h]$. The social planning problem can be restated as

$$\max \sum_{h=1}^{H} \sum_{t=0}^{T} \lambda_h \delta^t \mathbb{E}[Z_h \log C_h] \text{ subject to } \sum_{h=1}^{H} C_h = C_r.$$ (21.8)

The solution is

$$C_{ht} = \left( \frac{\lambda_h Z_h}{Z_t} \right) C_r,$$ (21.9)

where

$$Z_t = \sum_{h=1}^{H} \lambda_h Z_{ht}.$$ (21.10)

Substitute this into the objective function in (21.8) to see that the social planner’s utility is

$$\sum_{t=0}^{T} \delta^t \mathbb{E}[Z_t \log C_t] = \sum_{t=0}^{T} \delta^t \mathbb{E}[Z_t \log C_t] = \sum_{t=0}^{T} \delta^t \left[ \sum_{h=1}^{H} \lambda_h Z_{ht} \log \left( \frac{\lambda_h z_h}{Z_t} \right) \right].$$ (21.11)

As in Section 21.2, define the weighted-average beliefs

$$P_m(A) = \sum_{h=1}^{H} \lambda_h P_h(A),$$

and let $E_m$ denote the expectation operator. Use iterated expectations and the fact that the Radon-Nikodym derivative of $P_m$ with respect to $P$ is $\sum_{h=1}^{H} \lambda_h z_h$ to (p.560) obtain

$$\mathbb{E}[Z_t \log C_t] = \mathbb{E} \left[ \sum_{h=1}^{H} \lambda_h z_h \log C_t \right] = \mathbb{E} \left[ \log C_t \right].$$

Therefore, the social planner’s utility (21.11) is a constant plus
Thus, the social planner is a representative investor with log utility and beliefs $P_m$.

**Linear Risk Tolerance**

Assume first that all investors have CARA utility. As for log utility, substitute $Z_{P} = E[\tilde{z}]$ for $\tilde{z}$ in the social planning problem and solve the optimization problem pointwise to calculate the social planner’s utility. The result is that the social planner’s utility is

$$
-\left( \prod_{h=1}^{H} (\rho_h \sigma_h)^{\gamma/h} \right) \sum_{t=0}^{T} \delta^t E[Z_t e^{\gamma C_t}]
$$

(21.12a)

where

$$
Z_t = \prod_{h=1}^{H} Z_t^{\gamma/h}.
$$

(21.12b)

This is a straightforward extension of (21.7). This would imply the existence of a representative investor with state-independent CARA utility and beliefs $P_m$ if it were true that

$$
E[Z_t e^{\gamma C_t}] = \gamma E[d e^{\gamma C_t}] 
$$

(21.13)

for some probability $P_m$ and a positive constant $\gamma$ (which we could drop). By iterated expectations,

$$
\gamma E[d e^{\gamma C_t}] = \gamma E[dp_m][e^{\gamma C_t}] = \gamma E[e^{\gamma C_t}E[dp_m] \mid dp]
$$

Therefore, (21.13) holds for all consumption processes $C$ if and only if

$$
Z_t = \gamma E[dp_m] \mid dp
$$

(21.14) *(p.561)* for each $t$. Taking $t = T$ here yields $dP_m/dP = Z_T / \gamma$, so (21.14) implies $Z_t = E[dZ_T]$. Thus, a necessary and sufficient condition for (21.13) is that $Z$ be a martingale relative to $P$.

In the log case, $Z$ defined in (21.10) is a martingale, so we obtain a representative investor with state-independent utility. However, $Z$ defined in (21.12b) is a supermartingale but not a martingale (see below). Thus, the most we can say in the CARA case is that there is a representative investor with state-dependent utility.
An analogous result is true for general LRT utility functions in complete dynamic markets. For general LRT utility, whether $Z$ is a supermartingale or a submartingale depends on the cautiousness parameter (Exercise 21.2).

The supermartingale/submartingale property of $Z$ has economic implications. We can interpret $Z$ as a random discounting factor. A supermartingale decreases on average, so when $Z$ is a supermartingale, the future is discounted more on average, producing a higher risk-free rate in equilibrium.

The fact that $Z$ is a supermartingale in the case of CARA utility follows from the fact that a geometric average is smaller than an arithmetic average. To see the relation between geometric and arithmetic averages, apply Jensen’s inequality to the logarithm function to obtain

\[
E\left(\prod_{h=1}^{H} \frac{Z_{h+1}}{Z_{h}}\right) = \exp\left(\sum_{h=1}^{H} \frac{1}{H} \log \frac{Z_{h+1}}{Z_{h}}\right)
\]

\[
\leq \exp\left(\log \sum_{h=1}^{H} \frac{1}{H} \frac{Z_{h+1}}{Z_{h}}\right)
\]

\[
= \sum_{h=1}^{H} \frac{1}{H} \frac{Z_{h+1}}{Z_{h}}
\]

This implies

\[
E\left[\frac{Z_{H}}{Z_{1}}\right] = E\left[\prod_{h=1}^{H} \frac{Z_{h+1}}{Z_{h}}\right] \leq E\left[\sum_{h=1}^{H} \frac{1}{H} \frac{Z_{h+1}}{Z_{h}}\right] = 1.
\]

The supermartingale property also follows directly from Hölder’s inequality:

\[
E\left[\prod_{h=1}^{H} \frac{Z_{h+1}}{Z_{h}}\right] \leq \prod_{h=1}^{H} E\left[\frac{Z_{h+1}}{Z_{h}}\right] = 1.
\]
21.4 Short Sales Constraints and Overpricing

When investors have heterogeneous beliefs, investors who are optimistic about an asset should be long the asset in equilibrium, and investors who are pessimistic should be short. There are many investors who cannot short sell and many others who find it costly to short (because they do not obtain use of the proceeds and must post additional margin on which they earn no interest). In the presence of short sales constraints, optimistic investors will hold the asset in equilibrium, and pessimistic investors may be on the sideline. Short selling increases the supply of an asset available to those who want to buy it. Curtailing short selling limits the available supply, and, of course, any limitation of supply good should increase the price. Thus, in the presence of short sales constraints, prices may be too high (relative to average beliefs).

To illustrate this, suppose there is a single risky asset (the market portfolio) and all investors have CARA utility. The payoff of the risky asset is market wealth \( \bar{w}_m \). Suppose the risky asset cannot be sold short. Normalize the shares of the risky asset so that the total supply is 1 share, and assume each investor is endowed with \( 1/H \) shares. Assume there is a risk-free asset in zero net supply. Assume investors have CARA utility with the same absolute risk aversion \( \sigma \), agree that \( \bar{w}_m \) is normally distributed, and agree on the variance \( \sigma^2 \) of \( \bar{w}_m \). Let \( \mu_h \) denote the mean of \( \bar{w}_m \) perceived by investor \( h \).

Given a price \( p \) for the risky asset, it follows from (2.19) that the optimal number of shares of the risky asset for investor \( h \) to hold, if she faced no short sales constraints, is

\[
\frac{\mu_h - PR_t}{\sigma^2}.
\]

When short sales are not allowed, investor \( h \)'s optimal demand is

\[
\theta_h = \begin{cases} 
\frac{\mu_h - PR_t}{\sigma^2} & \text{if } \mu_h \geq PR_t, \\
0 & \text{otherwise}.
\end{cases}
\]

For a given price \( p \), aggregate demand is

\[
\theta_m = \sum_{h=1}^{H} \frac{\mu_h - PR_t}{\sigma^2}.
\]

(21.16)

Market clearing requires \( \theta_m = 1 \).
As usual, since we have not introduced a date–0 consumption good and normalized prices by taking the price of date–0 consumption to be 1, there is one degree of indeterminacy in equilibrium prices. It is convenient to normalize prices by taking }R\text{ = 1 (that is, by taking a quantity of the risk-free asset that } (p.563)\text{ pays 1 unit of the consumption good at date 1 to be the numeraire at date 0). The equation }\theta_m = 1 \text{ can then be solved for the equilibrium price } p \text{ of the risky asset.}

To simplify the solution of the equation }\theta_m = 1 \text{, it is convenient to modify the model by assuming there is a continuum of investors, of total mass equal to 1, and that }\mu_h \text{ is uniformly distributed across investors on some interval } (\mu^* - \Delta, \mu^* + \Delta). \text{ The parameter }\mu^* \text{ is the average belief. In the following formulas, we should interpret }\alpha \text{ as aggregate absolute risk aversion: It equals individual risk aversion because the mass of investors is normalized to equal 1.}

In the modified model, in the absence of short sales constraints, aggregate demand would be

\[ \theta_m = \frac{1}{2\Delta} \int_{\mu^*-\Delta}^{\mu^*+\Delta} \frac{\mu - P}{\alpha \sigma^2} \, d\mu = \frac{\mu^* - P}{\alpha \sigma^2}, \]

just as if all investors agreed that }\mu^* \text{ is the mean of }\bar{\omega}_m. \text{ The market clearing condition }\theta_m = 1 \text{ would imply }P = \mu^* - \alpha \sigma^2. \text{ This is the average expectation minus a discount for risk.}

If the market clearing price in the unconstrained case is below the expectation of the most pessimistic investor—that is, if }\alpha \sigma^2 > \Delta\text{—then all investors are long the asset in the unconstrained case. Hence, the imposition of a short sale constraint has no effect. However, if }\Delta > \alpha \sigma^2\text{, then some investors short sell the asset in the unconstrained case, and constraining short sales affects their demands and hence affects the equilibrium price.}

Assume }\Delta > \alpha \sigma^2\text{, so the short sales constraint is binding on some investors. Consider any price }P > \mu^* - \Delta\text{. Generalizing (21.16) and using the condition }R_f = 1\text{, aggregate demand is}

\[ \theta_m = \frac{1}{2\Delta} \int_{\mu^*+\Delta}^{\mu^*} \frac{e^{\mu^* - P}}{\alpha \sigma^2} \, d\mu = \frac{e^{\mu^* - P} - \frac{\mu^* - P}{45\alpha \sigma^2}}{45\alpha \sigma^2}. \]

Hence, the market clearing condition }\theta_m = 1 \text{ implies
The difference between the price with and without the short sales constraint is

\[ \Delta - 2\sqrt{\Delta \sigma^2} + \alpha \sigma^2 = (\sqrt{\Delta} - \sqrt{\sigma^2})^2 > 0. \]

This confirms that constraining short selling increases the asset price when \( \Delta > \alpha \sigma^2 \). Furthermore, the price is increasing in \( \Delta \):

\[ \frac{\partial P}{\partial \Delta} = \frac{2}{\Delta} > 0 \]

when \( \Delta > \alpha \sigma^2 \). Therefore, greater dispersion of beliefs (greater \( \Delta \)) leads to higher prices.

(p.564) 21.5 Speculative Trade and Bubbles

The previous section presents an example in which, in the presence of short sales constraints, only relatively optimistic investors hold the risky asset, and the asset price is higher than it would be if all investors possessed the average beliefs. Even more interesting phenomena arise in dynamic models. For example, it need not be that optimistic investors always hold the asset. Instead, at any given point in time, pessimistic investors may value the asset more, because of the right to resell it later to the optimistic investors. Buying an asset when we regard its fundamental value as low in order to resell it later to others with higher valuations is speculative trading. Due to speculative trading, asset prices can be even higher than they would be if all investors possessed optimistic beliefs. This is a type of bubble in the asset price.

To illustrate this, consider the following discrete-time example. Suppose there are two investors (or two classes of investors) \( h = 1, 2 \) who are risk neutral and have the same discount factor \( \delta \). Suppose the horizon \( T \) is finite. Assume there is a risk-free asset in each period. Consider a risky asset (not necessarily the market portfolio) that pays a dividend \( D_t \) in period \( t \).

Assume there are no margin requirements for purchasing the risky asset, but short sales of the risky asset are prohibited.

Let \( E_{ht} \) denote conditional expectation at date \( t \), given the beliefs of investor \( h \). The equilibrium price in the penultimate period must be

\[ P_{T-1} = \delta_{\text{max}} E_{ht} [D_T] \]

and in other periods it must satisfy
\[ P_t = \delta \max_h \mathbb{E}_h [D_{t+1} + P_{t+1}] \]  
(21.17)

If \( P_t \) (or \( P_{T-1} \)) were more than this, then neither investor would be willing to hold the asset, preferring to consume more in period \( t \) (or \( T - 1 \)). If it were less, then one of the investors would want to buy an infinite amount on margin.

The fundamental value of the asset at date \( t \) for investor \( h \) is

\[ V_h \text{def} \mathbb{E}_h \left[ \sum_{t'=t}^{T} \delta^{t'-t} D_t \right] \]

We must have \( P_t \geq V_{ht} \), because otherwise investor \( h \) would want to buy an infinite amount of the asset, planning to buy and hold. Set \( V_t = \max(V_{1t}, V_{2t}) \). We might expect, based on the single-period model, that the price at \( t \) is set by the most optimistic investor, meaning \( P_t = V_r \). However, as remarked before, the equilibrium price can exceed the fundamental value of even the optimistic investors in a dynamic model, due to the value inherent in the opportunity to resell the asset. To see this, suppose that, in some state of the world, investor 2 is the most optimistic about the fundamental value at date \( t \), that is, \( V_t = V_{2t} > V_{1t} \), but investor 1 is the most optimistic about investor 2’s future valuation in the sense that

\[ \mathbb{E}_1[D_{t+1} + V_{2t+1}] \geq \mathbb{E}_2[D_{t+1} + V_{2t+1}] \]

Because \( P_{t+1} \geq V_{2t+1} \), this implies

\[ P_t \geq \delta \mathbb{E}_1[D_{t+1} + P_{t+1}] \geq \delta \mathbb{E}_2[D_{t+1} + V_{2t+1}] \]

\[ > \delta \mathbb{E}_2[D_{t+1} + V_{2t+1}] = V_{2t} = V_r \]

A specific (infinite-horizon) numerical example from Harrison and Kreps (1978) is presented in Exercise 21.3.
21.6 Notes and References
Rubinstein (1974) proves the existence of a representative investor in a single-period model with log or CARA utility. The discussion in Section 21.3 of representative investors in dynamic models with heterogeneous beliefs is based on Jouini and Napp (2006). The existence of a representative investor depends on the market being frictionless. Detemple and Murthy (1997) and Basak and Croitoru (2000) study dynamic models with log utility and heterogeneous beliefs in the presence of margin requirements and other portfolio constraints. Cao and Ou-Yang (2009) is the source for Part (c) of Exercise 21.1, which establishes that the existence of an asset with a payoff that is quadratic in market wealth is sufficient to implement Pareto optima when investors have CARA utility and agree that market wealth is normally distributed.

The idea that short sales constraints increase the prices of assets when investors have heterogeneous beliefs is due to Lintner (1969) and Miller (1977). It is commonly called Miller’s model. The model in Section 21.4 is due to Chen, Hong, and Stein (2002), who emphasize that overpricing is increasing in the dispersion of beliefs. Gallmeyer and Hollifield (2008) study the effect of a market-wide short sales constraint and show that it may either raise or lower asset prices, depending on investors’ elasticities of intertemporal substitution. They also show that the imposition of a short sales constraint increases the equilibrium interest rate. Hong and Stein (2003) develop a theory of market crashes based on heterogeneous beliefs and short-sale constraints.

The idea that speculative trade can cause prices to be above the fundamental values of even optimistic investors is due to Harrison and Kreps (1978), who analyze an infinite-horizon version of the model presented in Section 21.5. Exercise 21.3 presents a numerical example given by Harrison and Kreps (1978). (p.566) The role of the short sales constraint in the Harrison-Kreps model is to ensure the existence of equilibrium: In its absence, risk-neutral investors with heterogeneous beliefs would want to go infinitely short and long. Cao and Ou-Yang (2005) show that the price can be above the fundamental value of optimistic investors and can in other
times be below the fundamental value of pessimistic investors when risk-averse investors have heterogeneous beliefs.

Scheinkman and Xiong (2003) analyze a continuous-time version of the Harrison-Kreps model. In the Scheinkman-Xiong model, investors observe processes that forecast future dividends. They disagree on the precisions with which the various signal processes forecast dividends, which Scheinkman and Xiong interpret as reflecting overconfidence of investors. There are many other papers that model heterogeneous beliefs with learning about fundamentals. Those papers are surveyed in Section 23.6.

It is generally regarded as a puzzle that the volume of trading in financial markets is as high as it is. Speculative trading is one possible explanation for the magnitude of observed volume. Harris and Raviv (1993), Kandel and Pearson (1995), and Cao and Ou-Yang (2009) present models of volume with heterogeneous beliefs.

Anderson, Ghysels, and Juergens (2005) and David (2008a) ask whether heterogeneity in beliefs can explain the equity premium puzzle. With different models, they reach different conclusions. Anderson, Ghysels, and Juergens (2005) also test whether heterogeneity in beliefs is a priced risk factor. Banerjee, Kaniel, and Kremer (2009) show that stock returns can exhibit momentum when investors have “higher order” differences in beliefs. For a survey of the implications of heterogeneous beliefs for asset prices and trading volume, see Hong and Stein (2007).

Exercises 21.1. Suppose each investor $h$ has CARA utility with absolute risk aversion $\alpha_h$. Assume the information in the economy is generated by $\tilde{w}_m$. Assume investor $h$ believes $\tilde{w}_m$ is normally distributed with mean $\mu_h$ and variance $\sigma^2$, where $\sigma$ is the same for all investors.

(a) Show that the Radon-Nikodym derivative of investor $h$'s probability $P_h$ with respect to the average probability $P$ is
Show that the sharing rule (21.6) is equivalent to

\[
\tilde{w}_h = \frac{1}{H} \sum_{t=1}^{H} \left[ \log \left( \frac{w_{t+1}}{w_t} \right) + \frac{\nu_t^2 / \alpha_t^2}{2} \right] + \frac{1}{T} \tilde{w}_m + \frac{1}{T} \sum_{t=1}^{H} \left[ \log \left( \frac{\tilde{w}_{t+1}}{\tilde{w}_t} \right) + \frac{\nu_t^2 / \alpha_t^2}{2} \right] \cdot \tilde{w}_m.
\]

(c) Show that if investors also disagree about the variance of \( \tilde{w}_m \) then the sharing rule (21.6) is quadratic in \( \tilde{w}_m \).

21.2. Assume all investors have constant relative risk aversion \( \rho \) and the same discount factor \( \delta \). Solve the social planning problem in a finite-horizon discrete-time model to show that the social planner’s utility is

\[
\delta^T \mathbb{E} \left[ \sum_{t=0}^{T} \delta^t \frac{C^1 - \rho}{1 - \rho} \right]
\]

for some stochastic process \( Z \). Show that \( Z \) is a supermartingale relative to the average beliefs if \( \rho > 1 \). Hint: For the last statement, use a conditional version of the Minkowski inequality. The Minkowski inequality states that for random variables \( \tilde{x}_n \) and any \( \rho > 1 \),

\[
\mathbb{E} \left[ \left( \sum_{n=1}^{H} \tilde{x}_n^\rho \right)^{1/\rho} \right] \leq \sum_{n=1}^{H} \mathbb{E} \left[ \tilde{x}_n \right]^{1/\rho}.
\]

21.3. Consider an infinite-horizon version of the model in Section 21.5 in which both investors agree the dividend process is a two-state Markov chain, with states \( D = 0 \) and \( D = 1 \). Suppose the investors’ beliefs \( P_h \) satisfy, for all \( t \geq 0 \),

\[
P(D_{t+1} = 0| D_t = 0) = 1/2, \quad P(D_{t+1} = 1| D_t = 0) = 1/2, \\
P(D_{t+1} = 0| D_t = 1) = 2/3, \quad P(D_{t+1} = 1| D_t = 1) = 1/3, \\
P(D_{t+1} = 0| D_t = 0) = 2/3, \quad P(D_{t+1} = 1| D_t = 0) = 1/3, \\
P(D_{t+1} = 0| D_t = 1) = 1/4, \quad P(D_{t+1} = 1| D_t = 1) = 3/4.
\]

Assume the discount factor of each investor is \( \delta = 3/4 \). For \( s = 0 \) and \( s = 1 \), set

\[
V_h(s) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t D_t = s \right].
\]

For each \( h \), use the pair of equations
\[
\frac{V_f(0)}{\delta} = P_f(D_{t+1} = 0|D_t = 0)V_f(0) + P_f(D_{t+1} = 1|D_t = 0)[1 + V_f(1)]
\]

(p.568) to calculate \(V_f(0)\) and \(V_f(1)\). Show that investor 2 has the highest fundamental value in both states \([V_f(0) > V_f(0)\) and \(V_f(1) > V_f(1)\)] but investor 1 is the most optimistic in state \(D = 0\) about investor 2’s future valuation, in the sense that

\[
P_f(D_{t+1} = 0|D_t = 0)V_f(0) + P_f(D_{t+1} = 1|D_t = 0)[1 + V_f(1)] > P_f(D_{t+1} = 0|D_t = 0)V_f(0) + P_f(D_{t+1} = 1|D_t = 0)[1 + V_f(1)]
\]