Forwards, Futures, and More Option Pricing

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Abstract and Keywords
Forward measures are defined. Forward and futures contracts are explained. The spot-forward parity formula is derived. A forward price is a martingale under the forward measure. A futures price is a martingale under a risk neutral probability. Forward prices equal futures prices when interest rates are nonrandom. The expectations hypothesis is explained. The option pricing formulas of Margrabe (exchange options), Black (options on forwards), and Merton (random interest rates) are derived. Implied volatilities and local volatility models are explained. Heston’s stochastic volatility model is derived.

Keywords: forward measure, forward contract, futures contract, spot-forward parity, spot-futures parity, expectations hypothesis, Margrabe’s formula, Black’s formula, Merton’s formula, Heston model

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17.1 Forward Measures

A probability created by using a discount bond as the numeraire is called a forward probability or, more commonly, a forward measure. We create the forward measure using the same steps used to create the risk-neutral probability and the probability $\mathbb{P}$ in Chapter 16. Let $P$ denote the price of a discount bond maturing at $T$, so $P_T = 1$. Let $M$ be an SDF process. Assume $MP$ is a martingale. The Radon-Nikodym derivative for the forward measure is $M_T P_T / P_0 = M_T / P_0$. In other words, we define

$$\mathbb{P}^F(A) = \frac{1}{P_0} E[M_T 1_A]$$

for each event $A$ observable at date $T$. Denote expectation with respect to $\mathbb{P}^F$ by $E^F$. By a standard result cited several times before (Appendix A.12), if $Y$ is a stochastic process such that $MY$ is a martingale under the physical probability, then $MY/(MP) = Y/P$ is a martingale under the forward measure $\mathbb{P}^F$.

The forward measure equals the risk-neutral probability when the short rate is nonrandom. They are equal because $1/P_0$ in (17.1) equals $1/R_T$ in the definition (15.5) of the risk-neutral probability when $r$ is nonrandom. Thus, the two probabilities

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are the same. When interest rates are random, the forward measure is sometimes more useful than the risk-neutral probability.

17.2 Forwards and Futures
A forward contract is a contract to make an exchange at a future date. One party (the “long”) agrees at date \( t \) to purchase an asset at date \( u > t \) from another party (the “short”) at a price \( F_t \). The price \( F_t \) is called the forward price, and it is fixed at date \( t \) as part of the contract. When the additional specificity seems convenient, we write \( F_u(t) \) for the forward price at \( t \) for a contract maturing at \( u \). We assume neither party to a forward contract makes a payment to the other at date \( t \) or at any date prior to \( u \) (though in reality, some collateral may be required). Let \( S \) denote the price of the underlying asset. Then, the value to the long party at the maturity \( u \) of a forward contract is \( S_u - F_u(t) \), and the value to the short party at maturity is \( F_u(t) - S_u \).

It is useful to compute how the value of a forward contract evolves over time. Suppose we enter into a forward as the long at date \( s \) at some forward price \( K \) (for example, \( K = F_s \)). The value at \( t > s \) can be seen as follows. We could unwind the long position at \( t \) by selling a forward at the market forward price \( F_t(u) \). The delivery/receipt obligations for the underlying asset of the short and long forwards would cancel. On the long forward, we are obligated to pay \( K \) at \( u \). On the short forward, we receive \( F_t(u) \) at \( u \). Thus, we would receive a net cash flow of \( F_t(u) - K \) at \( u \). A cash flow of \( F_t(u) - K \) at date \( u \) is worth \( (F_t(u) - K)P_t(u) \) at date \( t \), where, as before, \( P_t(u) \) denotes the price at \( t \) of a discount bond maturing at \( u \). Therefore, the value of the long forward at \( t \) must be \( (F_t(u) - K)P_t(u) \).

\[ \text{(p.435)} \] Another useful observation is that \( F_t(u)P_t(u) \) is the value of a non-dividend-paying portfolio for \( t \leq u \). Specifically, we can go long the forward contract at date 0 at price \( F_0(u) \) and simultaneously buy \( F_0(u) \) units of the discount bond that matures at \( u \). The previous paragraph shows that the value of the forward at each date \( t \in [0, u] \) is \( (F_t(u) - K)P_t(u) \) with \( K = F_0(u) \). The value of the bonds at date \( t \) is \( KP_t(u) \). So, the value of the
portfolio consisting of the long forward and the bonds is $F_d(u)P_d(u)$. This portfolio is a way to own the underlying asset at date $u$ by making a payment at date 0 and having no further cash flows between date 0 and date $u$. The cost of buying the asset without interim cash flows is the portfolio value at date 0, namely, $F_d(u)P_d(u)$. Date 0 is arbitrary in this discussion. The forward and discount bonds can be used in this way to buy the underlying asset without interim cash flows at any date $t < u$ at cost $F_t(u)P_d(u)$.

Because $FP$ is the value of a non-dividend-paying portfolio, $MFP$ is a local martingale for any SDF process $M$. Assume $MFP$ is actually a martingale. Then, as discussed in Section 17.1, $MFP/(MP)=F$ is a martingale under the forward measure. To summarize, given regularity conditions, the forward price $F_t(u)$ is a martingale under the forward measure defined by the discount bond with the same maturity $u$.

**Spot-Forward Parity and Convergence**

It is easy to replicate a forward contract if the underlying asset has a constant dividend yield. Replicating a forward is called creating a synthetic forward. To do so, we simply buy the asset with borrowed funds. Dividends paid by the asset before the forward matures are reinvested in the asset. This produces a portfolio that has cash flows only at the forward maturity. Let $q$ denote the dividend yield, and let $u$ denote the date at which the forward matures. At any date $t < u$, we can buy $e^{-q(t-r)}$ units of the underlying asset and finance the purchase by shorting $e^{-q(t-r)}S_t/P_d(u)$ units of the discount bond maturing at $u$. Through reinvestment of dividends, we will have a full share of the underlying asset at date $u$. We will have to pay $e^{-q(t-r)}S_t/P_d(u)$ at $u$ to cover the short of the discount bond. Note that this is a value known at date $t$. Because this strategy replicates the forward, the market forward price at $t$ must be

$$F_t(u) = \frac{e^{-q(t-r)}S_t}{P_d(u)}.$$  

(17.2)

Equation (17.2) is called spot-forward parity (the price $S_t$ is the “spot price” of the asset at date $t$). It is not strictly necessary to assume that the asset pays continuous dividends. Exercise
17.1 presents the spot-forward parity formula for an underlying asset that pays discrete known dividends at a finite number of known dates.

A special case of spot-forward parity is that \( F_{u}(u) = S_{u} \). This is called spot-forward convergence, because it reflects the fact that spot and forward prices must converge as the forward approaches maturity. Of course, at maturity, the forward contract is actually a spot contract, whence the convergence. Spot-forward convergence must hold even for assets that do not have constant dividend yields and even for assets for which dividends are not known at the time the forward is initiated. Hence, spot-forward convergence is more general than spot-forward parity.

Futures Contracts

A futures contract is an exchange-traded forward. The exchange requires collateral (margin) and marking of the contract to market. Marking to market means that when the futures price increases, the long party receives the change in the futures price (as a deposit to her margin account) and the short party pays the change in the futures price (as a debit to her margin account). The delivery price on the contract is simultaneously adjusted to the market futures price. The reverse occurs when the futures price falls. Marking to market causes a futures contract to always have a zero value. It could be canceled with no further cash flows by making an offsetting trade (selling if long and buying if short). The marking-to-market cash flows occur at the end of each trading day on an actual exchange, but it is simpler to model them as occurring continuously.

Let \( S \) denote the price of the underlying asset, and let \( \hat{F}(u) \) denote the futures price at \( r \) for a contract maturing at \( u \). We sometimes drop the \( u \) argument when the maturity date is understood. We must have \( \hat{F}(u) = S_{u} \). In other words, there must be spot-futures convergence, just as there is spot-forward convergence. Assume \( \hat{F} \) is an Itô process. In this model, the Itô differential \( d\hat{F} \) is the cash flow that is received by the long party and paid by the short party. Assume the cash flows are paid to or debited from the money market account; that is, they earn interest at the instantaneous risk-free rate. Thus, if
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we purchase at date $s$ a futures contract with maturity $u$ and hold it until date $t \in [s, u]$, we will have

$$V_t \equiv R_t \int_u^t \frac{dR_t}{R_s}$$

(17.3)

at date $t$. The process $V$ is a self-financing wealth process.

**Expectations Hypotheses**

The expectations hypothesis for forward prices states that the forward price is the best predictor of the future price of the underlying asset in the sense that $F_t(u) = E[S_u]$. There is also an expectations hypothesis for futures prices which states that $\hat{F}_t(u) = E[S_u]$. To see the motivation for the hypothesis, consider the forward price and suppose for simplicity that there is a constant risk-free rate $r$ and the underlying asset has a constant dividend yield $q$. As is shown below, forward prices equal futures prices when there is a constant risk-free rate. Under these hypotheses, the spot-forward parity formula (17.2) is

$$\hat{F}_t(u) = F_t(u) = e^{r(u-t)q} S_t$$

(17.4)

Given spot-forward parity, the expectations hypothesis is equivalent to

$$S_t = e^{r(u-t)q} E[S_u]$$

The right-hand side is how the underlying asset would be valued at $t$ if investors were risk neutral. Thus, the expectations hypothesis is a hypothesis about risk-neutral valuation.

The expectations hypothesis has been shown to be invalid empirically. This is not surprising, because risk neutrality is an unreasonable assumption. With risk-averse investors, we can derive the expectations hypothesis only by changing probabilities. The expectations hypothesis for the forward price should be true under the forward measure. The expectations hypothesis for the futures price should be true under the risk-neutral probability.

The expectations hypothesis for the forward price should be true under the forward measure, because, as explained above, the forward price is a martingale under the forward measure,
given regularity conditions. When we combine the martingale property with spot-forward convergence, we obtain

$$F_t(u) = E_t^u[S_t].$$

(17.5)
The expectations hypothesis for the futures price should be true under the risk-neutral probability, because $V$ defined in (17.3) is a self-financing wealth process, so $MV$ is a local martingale. Under regularity conditions, it is a martingale. In this case,

$$\frac{V}{K} = \int_0^t \frac{1}{K_s} d\hat{F}_t,$$

is a martingale under the risk-neutral probability. For this to be true, it must be that $\hat{F}$ is a martingale or at least a local martingale under the risk-neutral probability. Assume it is a martingale and apply spot-futures convergence to obtain

$$\hat{F}_t = E_t^u[S_t].$$

(17.6)

Equality of Forwards and Futures Prices

As noted in Section 17.1, if the short rate is nonrandom, then the risk-neutral probability and the forward measure are the same. Thus, the two expectations hypotheses (17.5) and (17.6) imply that forward prices equal futures prices when the short rate is nonrandom.

17.3 Margrabe, Black, and Merton Formulas

This section derives three extensions of the Black-Scholes formula. Each extension builds on the previous one, and each builds on and generalizes the Black-Scholes formula. We price exchange options (Margrabe, 1978), options on forwards and futures (Black, 1976), and options on assets that can pay either continuous or discrete dividends, allowing for random interest rates (Merton, 1973b).

Exchange Options

An exchange option is an option to exchange two assets. The party who is long an exchange option has the choice whether to exercise it and will do so when the value of the asset she receives is greater than the value of the asset she delivers. Call the asset she receives asset 1 and the asset she delivers asset 2. Let $S_1$ and $S_2$ denote the prices of the assets. The value of the option at the date $T$ that it matures is

$$\max(0, S_{1T} - S_{2T}).$$
The value to the seller of the option is the negative of this (as with puts and calls, the values for the short and long sum to zero). Calls and puts are special cases of exchange options. A call option is an exchange option in which cash is exchanged for the underlying asset. A put option is an exchange option in which the underlying asset is exchanged for cash. Here, cash means a position in the money market account or in the discount bond that matures at the option maturity date.

Assume the two assets have constant dividend yields \( q_1 \) and \( q_2 \), constant volatilities \( \sigma_1 \) and \( \sigma_2 \), and a constant correlation \( \rho \).

Then, the volatility of \( S_1/S_2 \) \((p.439)\) and also the volatility of \( S_2/S_1 \) is equal to \( \sigma \), defined as \( \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \). Assume there is an SDF process \( M \) such that \( MS_1 \) and \( MS_2 \) are martingales (sufficient conditions, including Novikov's condition, can be adapted from Section 16.2). We do not need to assume that there is a money market account, nor do we need to assume that there are traded discount bonds. The existence of those assets is neither necessary nor useful for valuing or hedging an exchange option.

We want to show that there is a self-financing wealth process \( W \) for which \( W_T = \max(0, S_{1T} - S_{2T}) \). We then want to calculate \( \mathbb{E}[M_T \max(0, S_{1T} - S_{2T})] \). We can do these two things directly (Exercises 17.3–17.4). However, we can also derive them from the Black-Scholes formula. The argument is based on a change of numeraire. Assume for simplicity that the assets do not pay dividends (\( q_1 = q_2 = 0 \)). The more general case is obtained from this special case by using the fact that \( e^{-q(T-t)S_1} \) is the price of a non-dividend-paying asset, as in Section 16.8. Use asset 2 as the numeraire. The price of asset 2 in this numeraire is \( S_2/S_1 = 1 \). Thus, asset 2 is risk free in this numeraire with rate of return equal to 0. Define \( R^* = 1 \). The price of asset 1 in this numeraire is \( S_1/S_2 \). For valuing assets in the new numeraire, \( M^* = MS_2/S_2 \) is an SDF process. This is analogous to the relation between nominal and real SDF processes discussed in Section 13.7. To see that \( M^* \) is an SDF process in the new numeraire, it suffices to observe that (i) \( M^*_0 = M_0 = 1 \) and (ii) for any non-dividend-paying asset with price \( s \) in the original numeraire and price \( S/S_1 \) in the new numeraire, we have \( M^*S/S_2 = MS/S_2 \), which is a local martingale. We are assuming \( MS_1 \) and \( MS_2 \) are martingales, so \( M^*S^* = MS_1/S_2 \) is a martingale, and \( M^*R^* = MS_2/S_2 \).
is a martingale. Hence, with an appropriate constraint on portfolio processes, the assumptions of Section 16.2 are satisfied for the underlying asset price $S^*$, money market price $R^*$, and SDF process $M^*$.

The payoff of the exchange option is

$$S_T \max\left(0, \frac{S_T^* - 1}{S_T^*} \right) = S_T \max\left(0, S_T^* - 1 \right).$$

(17.7)

(p.440) In the new numeraire, the payoff of the exchange option is $\max(0, S_T^* - 1)$. The market completeness result of Section 16.4 shows that there is a trading strategy in the underlying asset with price $S^*$ and money market account with price $R^*$ that has payoff $\max(0, S_T^* - 1)$. When we convert to the original numeraire, the payoff is the payoff of the exchange option. Thus, the payoff of the exchange option can be replicated using the assets with prices $S_1$ and $S_2$ (Exercise 17.2).

Furthermore,

$$\mathbb{E}[M_T \max(0, S_{1T} - S_{2T})] = S_{20} \mathbb{E}[M_T \max(0, S_T^* - 1)],$$

and the expectation on the right-hand side is given by the Black-Scholes call option formula. Multiplying the Black-Scholes call formula (for the underlying with price $S^*$ and strike equal to 1) by $S_{20}$ shows that the value of the exchange option is the Black-Scholes call option formula (16.5), where we input

- underlying asset price $S_0 = S_{10}$
- dividend yield $q = q_1$
- interest rate $r = q_2$
- strike price $K = S_{20}$
- volatility $\sigma = \sqrt{q_1^2 - 2q_1\sigma_1 + \sigma_2^2}$.

This is called Margrabe’s formula.

**Options on Forwards**

Options on forwards work slightly differently from standard options. When an option on a forward is exercised, a forward contract is created with forward price equal to the option strike. The strike is paid and the underlying asset is delivered when the forward matures. Exercise of a call on a forward creates a long forward for the party that exercises the option.
Exercise of a put on a forward creates a short forward for the party that exercises the option. The option counterparty becomes the counterparty to the forward contract.

Consider European options on a forward contract. Let $T$ denote the date at which the options mature and $u \geq T$ the date at which the forward matures. Let $K$ denote the strike. Assume there is a traded discount bond maturing at $u$ with price $P_u(u)$. Let $F(u)$ denote the forward price. Section 17.2 shows that $FP$ is a self-financing wealth process (the portfolio consists of a long forward and discount bonds maturing at the forward maturity date). Hence, $MP$ and $MFP$ are local martingales for any SDF process $M$. Assume there is an SDF process $M$ such that $MP$ and $MFP$ are martingales, and assume the volatility $\sigma$ of the forward price is constant.

If an option is exercised at $T$, then the value of the long forward that is created is shown in Section 17.2 to equal $F_T(u)P_T(u) - KP_T(u)$. Therefore, the value of the call option at its maturity is

$$\max(0, F_T(u)P_T(u) - KP_T(u)).$$

(17.8a)

Likewise, the value of a European put at its maturity is

$$\max(0, KP_T(u) - F_T(u)P_T(u)).$$

(17.8b)

Calls and puts on forwards are equivalent to exchange options. A call on a forward is equivalent to an exchange option in which the asset with price $KP$ (which is $K$ units of the discount bond maturing at $u$) is exchanged for the asset with price $FP$. A put on a forward is equivalent to the reverse exchange option. Thus, the values of calls and puts on forwards are given by Margrabe’s formula. Specifically, they are given by the Black-Scholes formula (16.5), where we input

- underlying asset price $S_0 = F(u)P_T(u)$,
- dividend yield $q = 0$,
- interest rate $r = \text{yield}$ of the discount bond maturing when the forward matures,
- strike price $K$. 


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• volatility \( \sigma \) of the forward price.

By the definition of the discount bond yield, the underlying asset price can be written as \( S_0 = e^{-\gamma T} \hat{F}_d(u) \).

Options on Futures

Futures options are similar to options on forwards except for marking to market. When a futures option is exercised, a futures contract is created with futures price equal to the option strike. The futures contract is immediately marked to market, so the futures price is reset to the market futures price and a cash flow equal to the difference between the option strike and the market futures price is transferred. The remainder of this subsection assumes that the short rate is nonrandom (we must acknowledge that this is an unreasonable assumption for valuing options on bond futures).

Then, the futures price equals the forward price. Hence, the values of futures options differ from the values of forward options only because of the marking to market of the futures contract created by exercise of a futures option.

Consider European options on a futures contract. Let \( T \) denote the date at which the options mature, \( u \geq T \) the date at which the futures matures, \( \hat{F}_d(u) \) the futures price, and \( K \) the option strike. The value of a call on a futures at the call’s maturity date \( T \) is \( \max(0, \hat{F}_d(T) - K) \), and the value of a put on a futures is \( \max(0, K - \hat{F}_d(T)) \). These equal the values of forward options given in (17.1) divided by \( P_f(u) \). Therefore, the values at date 0 of futures options equal (p.442) the values at date 0 of forward options divided by \( P_f(u) \). Consequently, with a nonrandom short rate, the values of calls and puts on futures are given by the Black-Scholes formula (16.5), where we input

• underlying asset price \( S_0 = \hat{F}_d(u)P_d(T) \).
• dividend yield \( q = 0 \),
• interest rate \( r = \) yield \( \gamma \) of the discount bond maturing when the option (not the futures) matures,
• strike price \( K \),
• volatility \( \sigma \) of the futures price.

By the definition of the discount bond yield, the underlying asset price can be written as \( S_0 = e^{-\gamma T} \hat{F}_d(u) \). 

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Arbitrary Dividends and Random Interest Rates

Now, consider European options on an underlying asset that may pay either continuous or discrete dividends. We do not assume that the interest rate is constant. We do not even need to assume the existence of a money market account. However, we do assume that there is a traded discount bond that matures when the options mature. Also, we assume there is a traded forward contract that matures when the options mature. Assume the forward price has a constant volatility, and assume there is an SDF process $M$ such that $MP$ and $MFP$ are martingales.

We use the discount bond and the forward to create the portfolio with price $FP$ discussed in Section 17.2. As explained there, the portfolio is a way to own the asset at the date the bond and forward mature without receiving dividends in the interim. The process $FP$ is a self-financing wealth process with value $FP(T)P(T) = F(T) = S_T$ at date $T$, using successively the facts that the discount bond pays 1 at its maturity and spot-forward convergence. Thus, an option on the portfolio with price $FP$ that matures at $T$ is equivalent to an option on the underlying asset. To avoid assuming a constant risk-free rate, consider the option as an exchange option (as in the discussion of forward options) where $K$ units of the discount bond maturing at $T$ are exchanged for the portfolio with price $FP$. Then, option values are given by the formula for options on forwards, (p.443) substituting $u = T$ for the maturity of the forward and discount bond. Specifically, option values are given by the Black-Scholes formula (16.5), where we input

- underlying asset price $S_0 = F(T)P(T)$,
- dividend yield $q = 0$,
- interest rate $r = \text{yield of the discount bond maturing when the option matures}$,
- strike price $K$,
- volatility $\sigma$ of the forward price.

By the definition of the discount bond yield, the underlying asset price can be written as $S_0 = e^{rT}F(T)$. 


An obvious difficulty with applying this formula is that there may not be a traded forward contract. This is not a problem if the underlying asset has a constant dividend yield or pays known dividends at known dates, because in those cases the forward can be created synthetically (see Exercise 17.1 for an example with known discrete dividends paid at known dates).

Suppose for the remainder of this section that the underlying asset has a constant dividend yield \( q \). Then, the spot-forward parity formula (17.2) implies that the underlying asset price above is

\[
S_0 = F_c(T)P_c(T) = e^{-qT}S_0.
\]

Inspection of the Black-Scholes formula shows that inputting this initial asset price and a dividend yield of 0 is equivalent to inputting an initial asset price of \( S_0 \) and a dividend yield of \( q \) (this is how the formula (16.5) is derived from (16.4)). Thus, with a constant dividend yield but a possibly random short rate (or a nonexistent money market account), the values of European puts and calls are given by the Black-Scholes formula (16.5) with two substitutions: We should use the yield of the discount bond maturing when the option matures as the interest rate \( r \), and we should use the volatility of the forward price as the volatility \( \sigma \). When the short rate is nonrandom, the spot-forward parity formula implies that the volatility of the forward price equals the volatility of the underlying asset price. In general, the two volatilities differ, with the difference depending on the volatility of the discount bond price. The difference is small for options with short maturities, because discount bonds have small volatilities when they are near maturity. Exercise 17.5 presents an example.

The market completeness result for exchange options (Exercises 17.2 and 17.3) shows that the value \( V \) of an exchange option satisfies

\[
\frac{dV}{V} = \frac{dS_1}{S_1} + (1-n) \frac{dS_2}{S_2}
\]

for some \( n \). In our current context, \( S_1 = FP \) and \( S_2 = KP \). Because \( K \) is a constant, \( dS_2/S_2 = dP/P \). With a constant dividend yield, spot-forward parity implies that \( dS_1/S_1 = dS/S \), where \( S \) is the underlying asset price as before. Thus,

\[
\frac{dV}{V} = \frac{dS}{S} + (1-n) \frac{dP}{P}.
\]

This shows that, with random interest rates, the value of a European option on an asset with a constant dividend yield
can be replicated using the underlying asset and the discount bond that matures when the option matures.

17.4 Implied and Local Volatilities
Volatilities must be estimated before option pricing formulas can be applied. A common way to estimate a volatility for use in an option pricing formula (and for other purposes) is to imply it from the price of a liquid (that is, frequently and easily traded) option. The implied volatility is the volatility that, when input into the option pricing formula, produces the market price. It is the volatility that matches the model to the market. In practice, implied volatilities will be different for options with different strikes and for options with different maturities. There is an easy extension to the Black-Scholes formula that allows us to interpret different implied volatilities for different maturities as the result of the actual volatility changing nonrandomly over time. There are further extensions that accommodate different implied volatilities at both different strikes and different maturities. We first consider matching implied volatilities at different maturities.
Term Structure of Volatilities

Let $T$ denote the maturity date of an option. Suppose

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(t) dB_t,$$

where $\sigma(\cdot)$ is a nonrandom function of time. The Black-Scholes formula is valid in this circumstance if we replace the volatility $\sigma$ in the formula by

$$\sigma_{av}(T) \overset{\text{def}}{=} \sqrt{\frac{1}{T} \int_0^T \sigma(t)^2 dt}.$$

(17.9)

The proof of the formula with this extension is virtually unchanged. For example, we obtain

$$\log S_T = \log S_0 + \left( \frac{1}{2} \sigma_{av}(T)^2 \right) T + \int_0^T \sigma(t) dB^*_t,$$

where $B^*$ is a Brownian motion under the risk-neutral probability. The random variable

$$\int_0^T \sigma(t) dB^*_t$$

is normally distributed with mean zero and variance $T \sigma_{av}(T)^2$ under the risk-neutral probability. Therefore, the derivations of (A) and (B) in Section 16.4 go through unchanged. Using this extension, we can usually fit different implied volatilities at different maturities (for example, for the at-the-money option at each maturity) by constructing a nonrandom function $\sigma(t)$ such that, for each $T$, $\sigma_{av}(T)$ equals the implied volatility for the option with time $T$ to maturity. The mapping $t \mapsto \sigma(t)$ is called the term structure of implied volatilities.
Smiles and Smirks

When we fix the maturity and consider options with different strikes, we usually find that implied volatilities are higher for low strikes and for high strikes than they are for strikes near the current value of the underlying asset. In other words, implied volatilities are typically higher for deep out-of-the-money and for deep in-the-money options than they are for options that are near the money. When we graph implied volatilities against strikes, plotting strikes on the horizontal axis and implied volatilities on the vertical axis, the plot looks like a smile, and the pattern is commonly called an option smile. Implied volatilities for equity index options are commonly higher for low strikes than for high strikes, so the smile appears twisted. It is called a smirk. The smirk means that there are especially high prices for deep out-of-the-money puts compared to the Black-Scholes formula. This may reflect high demand for insurance against crashes, which would correspond to a risk-neutral distribution for $\log S_T$ that is negatively skewed instead of normal.

Local Volatility

A local volatility model is a model in which volatility changes both with time and with the underlying asset price. In other words, volatility is a function $\sigma(t, s)$ of time $t$ and the underlying asset price $s$. A function of this form is called a volatility surface. Subject to $MR$ and $MS$ being martingales, the market completeness result of Section 16.4 is still valid in this context, and the pricing formula

$$e^{-\tau S_t \text{prob}^T(A)} - e^{-\tau T \text{prob}^Y(A)}$$

(p.446) for a call and corresponding formula for a put are still valid ($A$ is the event $S_T > K$ for a call, as before). There are different proposals in the literature for functional forms for $\sigma$ to match the model to market prices. In all such models, the probabilities $\text{prob}^T(A)$ and $\text{prob}^Y(A)$ are computed numerically.

Practitioners typically construct different volatility surfaces each day (or more frequently) to match market option prices. There is an inconsistency in doing this, because if the function $\sigma(t, s)$ changes each day, then it must depend on more than time and the underlying asset price. There must be an omitted factor. Stochastic volatility models described in the next section do not suffer from this inconsistency and are at least
somewhat useful for producing smiles and smirks of the sort seen in market data, though they generally have too few parameters to match a given smile or smirk exactly.

17.5 Stochastic Volatility
If the volatility $\sigma$ is a stochastic process not locally perfectly correlated with the asset price $S$, then options cannot be replicated using the underlying asset and the risk-free asset. In this circumstance, we cannot price an option unless we make some assumption about the market price of risk for the volatility process. This is called equilibrium asset pricing—as opposed to arbitrage pricing—because it involves an assumption about which SDF process is appropriate for pricing the option. The assumption might be justified, for example, by assuming there is a representative investor with a specific utility function and specific aggregate consumption process.

The model of Heston (1993) is quite tractable (because it is an affine model—see Chapter 18). Set $V_t = \sigma^2_t$. Heston assumes

$$dV_t = \kappa(\theta - V_t)dt + \sqrt{V_t}dB_{1,t}^V$$

(17.10a)

$$\frac{dS}{S} = \mu dt + \sqrt{V_t} \left[ \rho dB_{1,t} + \sqrt{1 - \rho^2} dB_{2,t} \right]$$

(17.10b)

where $\mu$, $\kappa > 0$, $\theta > 0$, $\gamma$ and $\rho$ are constants, and $B_1$ and $B_2$ are independent Brownian motions under the physical probability. This implies that $\rho$ is the instantaneous correlation of $dS/S$ and $dV$. It is known that the solution of (17.10a) starting from $V_0 > 0$ has the property that, with probability 1, $V_t \geq 0$ for all $t$, as a variance must be. Intuitively, the reason for the nonnegativity of $V$ is that the instantaneous standard deviation $\sqrt{V_t}$ in (17.10a) vanishes as $V_t \to 0$; hence, the drift dominates as $V_t \to 0$, pulling $V$ up toward $\theta$.

The market is not complete, because there are two Brownian motions in (17.2) and only a single underlying asset. There are many SDF processes. The condition $\lambda = \mu - r$ means that the two-dimensional vector $\lambda$ of prices of risk satisfies a single equation. Thus, there is one degree of indeterminacy. Assume the risk-free rate is constant, and assume the asset does not pay dividends. Any SDF process $M$ satisfies
\[ \frac{dM_t}{M_t} = -r\, dt - \lambda_1 \, dB_{1t} - \lambda_2 \, dB_{2t} + \frac{\epsilon_t}{\langle \epsilon \rangle}, \]

(17.11)

where \( \epsilon \) is a local martingale uncorrelated with \( B_1 \) and \( B_2 \) (it is spanned by Brownian motions independent of \( B_1 \) and \( B_2 \)). From

\[ \sigma \lambda = \mu - r \iff \left( \frac{dM_t}{M_t} \right) \left( \frac{dS}{S} \right) = (\mu - r) dt, \]

we obtain the restriction

\[ \left( \rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2 \right) \langle V \rangle = \mu - r. \]

(17.12)

This is a single equation in the two prices of risk \( \lambda_1 \) and \( \lambda_2 \), so, as remarked before, there is one degree of indeterminacy.

In the remainder of this section, we choose to use the prices of risk that satisfy

\[ \lambda_1 = \phi \sqrt{V_t} \]

(17.13a)

for some constant \( \phi \). This is the “equilibrium assumption” mentioned above. We now regard \( \phi \) as another parameter of the model. Conditions (17.12) and (17.13a) imply

\[ \lambda_2 = \frac{\mu - r - \rho \phi \sqrt{V_t}}{\sqrt{1 - \rho^2}} V_t, \]

(17.13b)

Thus, the equilibrium assumption identifies a unique SDF process \( M \) (ignoring the orthogonal component \( d\epsilon/\epsilon \)), given a value for the parameter \( \phi \). This SDF process has the property that \( M_R \) and \( MS \) are martingales (see the end-of-chapter notes).

The value of a European call option is

\[ \mathbb{E}[M_t \max(0, S_T - K)]. \]

**The derivation of (16.14) is still valid in this context,** so we have

\[ \mathbb{E}[M_t \max(0, S_T - K)] = S_0 \text{prob}^\gamma(A) - e^{-rT} \text{prob}^\delta(A), \]

(17.14)

where \( A \) is the event \( S_T > K \), and \( \text{prob}^\gamma \) and \( \text{prob}^\delta \) are the probabilities defined in terms of \( M \). To calculate these probabilities of the event \( A \), we need to know the dynamics of \( S \) and \( V \) under the two probabilities. Changing probabilities does not change volatilities, so what we need to know are the drifts. As in Section 16.3, the expected rate of return of \( S \) is
\[\frac{dS}{S} = (r + \delta^V)dt + \sqrt{V} \left[ \rho dB'_S + \sqrt{1 - \rho^2} dB'_I \right]\]

(17.15a)

for \(i \in \{R, S\}\), where \(B'_I\) and \(B'_S\) are independent Brownian motions under \(\text{prob}^R\) and where \(\delta^R = 0\) and \(\delta^S = 1\). We verify this below and also show that

\[dV = \kappa(\delta^I - V)dt + \gamma \sqrt{V} \, dB'_I,\]

(17.15b)

Where

\[
\begin{align*}
\kappa^R &= \kappa + \gamma \phi, \\
\kappa^S &= \kappa + \gamma(\phi - \rho), \\
\theta^R &= \frac{\sigma^2}{2 \kappa^R}, \\
\theta^S &= \frac{\sigma^2}{2 \kappa^S}.
\end{align*}
\]

The result that \(\kappa^R \theta^R = \kappa^S \theta^S = \kappa \theta\) is important. Under the assumption that \(\kappa \theta \geq V^I/2\) (footnote 5), we have \(V_i > 0\) for all \(i\) with probability 1 under the physical probability, under \(\text{prob}^I\), and under \(\text{prob}^S\). This is discussed further in the end-of-chapter notes.

To compute the probabilities in the option pricing formula (17.14), it may be helpful to represent them as solutions of PDEs. Denote the conditional probabilities given date–information by \(\text{prob}^I(t, S, V)\) and \(\text{prob}^S(t, S, V)\). These conditional probabilities are martingales under the respective probabilities (because they are conditional expectations of \(I_s\)), so their drifts are zero. Calculate the drifts from Itô’s formula using (17.4) and equate the drifts to zero to obtain

\[Q'_i + (r + \delta^V)SQ'_i + \kappa(\delta^I - V)Q'_i + \frac{1}{2} \sigma^2 Q'^2_{2i} + \frac{1}{2} \sigma^2 V Q'_v + \gamma \rho \sigma V Q'_w = 0\]

(17.16)

(p.449) for \(i \in \{R, S\}\). These equations should be solved subject to the condition that the solutions lie between 0 and 1 and subject to the terminal condition

\[Q^K(T, a, b) = 0\]

if \(a > K\),

\[Q^K(T, a, b) = 1\]

otherwise.

Heston (1993) presents the solutions of the PDEs as integrals. This is a closed-form solution for the option price in the same sense that the Black-Scholes formula is a closed-form solution (the cumulative normal distribution function in the Black-
Scholes formula is computed by integrating the normal density function).

The Radon-Nikodym derivative of the risk-neutral probability relative to the physical probability is \( \xi_t = \frac{M_t R_T}{R_0} \). Set \( \xi_t = \mathbb{E}[\xi_T] \). Because \( MR \) is a martingale, we have \( \xi_t = \frac{M_t R_0}{R_0} \). Therefore,

\[
\frac{d\xi}{\xi} = -\lambda_1 dB_1 - \lambda_2 dB_2 + \frac{d\xi}{\xi}.
\]

Girsanov’s theorem implies that

\[
 dB_j^R = dB_j - \left( \frac{\xi}{T} \right) dB_j = dB_j + \lambda_j dt
\]

is the differential of a Brownian motion under \( \text{prob}^R \) for \( j = 1, 2 \).

Substitute the formulas for the \( \lambda \) in (17.3) and substitute the \( dB_j^R \) for the \( dB_j \) in (17.2) to obtain (17.4) for \( i = R \).

The argument for \( i = S \) is very similar. The Radon-Nikodym derivative of \( \text{prob}^S \) relative to the physical probability is \( \xi_T = \frac{M_T S_T}{S_0} \). Set \( \xi_t = \mathbb{E}[\xi_T] \). Because \( MS \) is a martingale, we have \( \xi_t = \frac{M_t S_t}{S_0} \). Use Itô’s formula and the fact that

\[
(dS/S)(dM/M) = (r - \rho) dt
\]

to obtain

\[
\frac{d\xi}{\xi} = \left( \rho \sqrt{V} - \lambda_1 \right) dB_1 + \left( \sqrt{1 - \rho^2} V - \lambda_2 \right) dB_2 + \frac{d\xi}{\xi}.
\]

Girsanov’s theorem implies that

\[
 dB_j^S = dB_j - \left( \frac{\xi}{T} \right) dB_j = dB_j + \left( \lambda_1 - \rho \sqrt{V} \right) dt
\]

are differentials of independent Brownian motions under \( \text{prob}^S \).

Substitute the formulas for the \( \lambda \) in (17.3) and substitute the \( dB_j^S \) for the \( dB_j \) in (17.2) to obtain (17.4) for \( i = S \).
Notes and References

The forward measure concept first appears in Jamshidian (1989). The derivation of the exchange option formula from the Black-Scholes formula in Section 17.3 is attributed by Margrabe (1978) to Steve Ross. Cox and Ross (1976b) discuss local volatility models in which the volatility is \( \sigma \) for a constant \( y \). These are called constant elasticity of variance (CEV) models. Seminal local volatility models designed to match market option prices are the binomial tree of Rubinstein (1994) and the trinomial tree of Derman and Kani (1998). Jackwerth and Rubinstein (1996) use nonparametric methods to construct a risk-neutral distribution for the underlying asset that produces option prices matching market prices. Hobson and Rogers (1998) propose a class of models in which the volatility is adapted to \( S \) but not a function of the contemporaneous value of \( S \). These models are related to discrete-time GARCH option pricing models (Duan, 1995; Heston and Nandi, 2000).

If the market is incomplete, as in the case of stochastic volatility, then the introduction of a zero-net-supply nonspanned asset will generally (unless, for example, all investors have LRT utility functions with the same cautiousness parameter) change equilibrium prices, so attempting to compute what the price of a nonspanned option would be if it were traded is somewhat problematic. Detemple and Selden (1991) provide a general equilibrium analysis of the effect of introducing a nontraded option on the price of the underlying asset.

The process (17.10a) for the variance in the Heston model is called a square-root process. In the special case \( \kappa \theta = \gamma^2 / 4 \), the solution of (17.10a) is the square of an Ornstein-Uhlenbeck process (Exercise 12.6). In general, the solution of (17.10a) can be represented as a time-changed squared Bessel process. If \( \kappa \theta \geq \gamma^2 / 2 \), then \( V_t \) is not just nonnegative but in fact strictly positive for all \( t \) (with probability 1). This condition is equivalent to the “dimension” of the Bessel process being at least 2. See, for example, Back (2005, Appendix B3). If \( \kappa \theta \geq \gamma^2 / 2 \), we say that the boundary \( (V = 0) \) is inaccessible, whereas the boundary is accessible if the inequality does not hold.
The volatility process in the Heston model can be normalized by defining $Y = V / \sqrt{\theta}$. The dynamics (17.10a) for $V$ imply

$$dY = \left( \frac{\theta}{\sqrt{\theta}} - \kappa Y \right) dt + \sqrt{Y} \, dB_1.$$

The condition for the boundary ($V = 0 \Rightarrow Y = 0$) being inaccessible is that the constant $\kappa \theta / \sqrt{\theta}$ in the drift of $Y$ be at least $1/2$. This constant is the same under $\text{prob}^p$ and $\text{prob}^\delta$ as under the physical probability in the Heston model. Thus, the boundary being inaccessible under the physical probability implies that it is also inaccessible under $\text{prob}^p$ and $\text{prob}^\delta$. This equivalence under the various (p.451) probabilities of the boundary being accessible is a necessary condition for them to be equivalent as probability measures.

The assumption in Section 17.4 that $\kappa \theta \geq \sqrt{\theta} / 2$ implies that the Heston model is a member of the extended affine class of models defined by Cheridito, Filipović, and Kimmel (2007). This ensures that $MR$ and $MS$ are strictly positive martingales if we assume $B_1$ and $B_2$ are the only sources of uncertainty (or more generally that $\varepsilon$ is a martingale independent of $B_1$ and $B_2$).

The univariate process $V$ with the price of risk specification $\lambda_1 = \sqrt{\theta} V$ is a member of the completely affine class, but the form (17.13b) for $\lambda_2$ implies that the joint process $(V, \log S)$ is only extended affine. Exercise 17.9 illustrates that the price of risk assumption (17.3) can be generalized while retaining the extended affine property. See Section 18.7 for further discussion. Exercise 17.7 is adapted from Hull and White (1987).

Exercises

**17.1.** Consider a forward contract on an asset that pays a single known discrete dividend $x$ at a known date $T < u$, where $u$ is the date the forward matures. Suppose there are traded discount bonds maturing at $T$ and $u$. Let $S$ denote the price of the asset. Prove the following spot-forward parity formula for $t < T$:

$$F_t(x) = \frac{S_t e^{T x}}{P_t(x)}.$$

**17.2.** Suppose $V^*$ is a self-financing wealth process with risky asset price $S^* = S_1 / S_2$ and money market price $R^* = 1$, meaning

$$\frac{dV^*}{V^*} = \pi^* \frac{dS^*}{S^*} + (1 - \pi^*) \frac{dR^*}{R^*}.$$
for some \( \pi^* \). Define \( V = S_V V^* \). Show that \( V \) is a self-financing wealth process with risky asset prices \( S_1 \) and \( S_2 \) and no money market account.

**17.3.** Suppose the prices of two non-dividend-paying assets are given by

\[
\frac{dS_i}{S_i} = \mu_i dt + \sigma_i dB_i
\]

where the \( B_i \) are Brownian motions with correlation \( \rho \). The \( \mu_i, \sigma_i \), and \( \rho \) can be stochastic processes. However, assume the volatility of \( S_i/S_2 \) is a constant \( \sigma \). Assume there is an SDF process such that \( MS_1 \) and \( MS_2 \) are martingales. Let \( X_T = S_T f(S_{1T}/S_{2T}) \) for some nonnegative function \( f \), for example, \( f(\sigma) = \max(0, \sigma - 1) \) as in (17.7). Assume \( \mathbb{E}[M_T X_T] < \infty \). Define \( Z = S_i/S_2 \).

**p.452** (a) Show that

\[
\frac{dZ}{Z^*} = \sigma dB^*
\]

where \( B^* \) is a Brownian motion under the probability \( \mathbb{P}^{S_i} \) and satisfies

\[
\sigma dB^* = (\mu_1 - \mu_2 - \rho \sigma_1 \sigma_2 + \sigma_2^2) dt + \sigma_1 dB_1 - \sigma_2 dB_2.
\]

(b) Show that

\[
\mathbb{E}^S_t f(Z_t) = \mathbb{E}^{S^*}_t f(Z_t) + \int_0^t \psi_s dB^*_s
\]

for some stochastic process \( \psi \) and all \( 0 \leq t \leq T \).

(c) Define \( W = S_2 \mathbb{E}^{S^*}_t f(Z_t) \). Note that \( W_T = X_T \). Show that \( W \) is a self-financing wealth process generated by the portfolio process in which the fraction

\[
\pi = \frac{S^*}{W}
\]

of wealth is invested in asset 1 and \( 1 - \pi \) is invested in asset 2.

**17.4.** Adopt the assumptions of Exercise 17.3. Let \( A \) denote the event \( S_{1T} > S_{2T} \).

(a) Show that, for \( i = 1, 2 \),

\[
\mathbb{E}[M_T S_{1T} 1_A] = S_{1^0} \mathbb{P}^{S_i}(A)
\]

Conclude that the value at date 0 of an option to exchange asset 2 for asset 1 at date \( T \) is

\[
S_{1^0} \mathbb{P}^{S_i}(A) - S_{2^0} \mathbb{P}^{S_i}(A).
\]
(b) Define $Y = S_2/S_1$. Show that

$$d\log Y = -\frac{1}{2} \sigma^2 dt + \sigma dB^*$,$$

where $B^*$ is a Brownian motion under the probability measure $\operatorname{prob}^Y$. Use this fact and the fact that $A$ is the event $\log Y_T < 0$ to show that $\operatorname{prob}^Y(A) = N(d_1)$, where

$$d_1 = \frac{\log(S_0/S_1) + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}}.$$ 

(c) Define $Z = S_1/S_2$. Show that

$$d\log Z = -\frac{1}{2} \sigma^2 dt + \sigma dB^*,$$

where $B^*$ is a Brownian motion under the probability measure $\operatorname{prob}^Z$. Use this fact and the fact that $A$ is the event $\log Z_T > 0$ to show that $\operatorname{prob}^Z(A) = N(d_2)$, where

$$d_2 = d_1 - \sigma \sqrt{T}.$$

17.5. Suppose the price $S$ of a non-dividend-paying asset has a constant volatility $\sigma$. Assume the volatility at date $t$ of a discount bond maturing at $T > t$ is

$$\frac{\phi}{T}(1 - e^{-\kappa(T-t)})$$

(17.17)

for constants $\kappa > 0$ and $\phi > 0$. Assume the discount bond and stock have a constant correlation $\rho$. Let $P$ denote the price of the discount bond.

(a) Calculate the volatility of $S/P$ as a function $\hat{\sigma}(t)$.

(b) Define

$$\sigma_{\text{avg}} = \sqrt{T} \int_0^T \hat{\sigma}(t)^2 dt.$$ 

Show that

$$\sigma_{\text{avg}} = \sigma^2 + \frac{1}{T} \left( \sigma^2 - 2 \kappa \rho \phi - (2\phi^2 - 2\kappa \rho \phi \left( \frac{1+\kappa T}{\kappa T} \right) \right)$$

(c) Apply results from Section 17.3 to derive a formula for the value of a European call option on the asset with price $S$ that matures at $T$.

(d) Use l’Hôpital’s rule to show that $\sigma_{\text{avg}} \approx \sigma$ for small $T$.

(e) Show that $\sigma_{\text{avg}} > \sigma$ for large $T$ if $\rho$ is sufficiently small.

Note: The bond volatility (17.17) arises in the Vasicek term structure model, with $\kappa$ being the rate of mean reversion of the
short rate process and \( \phi \) being the (absolute) volatility of the short rate process (Section 18.3).

17.6. Consider a European call option on an asset that pays a single known discrete dividend \( x \) at a known date \( T < u \), where \( u \) is the date the option expires. Assume the asset price \( S \) drops by \( x \) when it goes ex-dividend at date \( T \) (i.e., \( S_T = \lim_{t \to T} S_t - x \)) and otherwise is an Itô process. Suppose there are traded discount bonds maturing at \( T \) and \( u \). Assume the volatility of the process \( Z_t \) is a constant \( \sigma \) during \([0, u]\).

\[
Z_t = \begin{cases} \frac{[S_t - xP_T(T)]}{S_t} & \text{if } t < T \\ \frac{P_t(u)}{P_t(u)} & \text{if } T \leq t \leq u \end{cases}
\]

(a) Show that the value at date 0 of the call option is

\[
[S_0 - xP_T(T)]N(d_1) - e^{\gamma u}KN(d_2),
\]

where \( \gamma \) is the yield at date 0 of the discount bond maturing at \( u \) and

\[
d_1 = \frac{\log(S_0 - xP_T(T)) + (\gamma - \frac{1}{2} \sigma^2)u}{\sigma \sqrt{u}},
\]

\[
d_2 = d_1 - \sigma \sqrt{u}.
\]

(b) Interpret the process \( Z \).

17.7. Let \( S \) denote the price of a non-dividend-paying asset. Assume

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,
\]

\[
d\sigma_t = \phi(\sigma_t)dt + \gamma(\sigma_t)dB_t,
\]

for some functions \( \phi(\cdot) \) and \( \gamma(\cdot) \), where \( B_t \) and \( B_t^* \) are independent Brownian motions and \( \mu \) may be a stochastic process. Assume the price of risk for \( B_t^* \) equals \( \lambda(\sigma_t) \) for some function \( \lambda(\cdot) \). Assume there is a constant risk-free rate \( r \).

(a) Show that

\[
\frac{dS_t}{S_t} = r dt + \sigma_t dB_t^*,
\]

\[
d\sigma_t = \phi(\sigma_t)dt + \gamma(\sigma_t) dB_t^*,
\]

for some function \( \phi^*(\cdot) \), where \( B_t^* \) and \( B_t^* \) are independent Brownian motions under the risk-neutral probability.

(b) Use iterated expectations to show that the date-0 value of a call option equals

\[
\mathbb{E}_T^* [S_N(d_1) - e^{rT}KN(d_2)],
\]
(17.18)
where
\begin{align*}
d_1 &= \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2_T)T}{\sigma_{\text{avg}} \sqrt{T}}, \\
d_2 &= d_1 - \sigma_{\text{avg}} \sqrt{T},
\end{align*}
and the risk-neutral expectation in (17.18) is taken over the random “average” volatility
\[\sigma_{\text{avg}} = \sqrt{T} \int_0^T \sigma^2_t \, dt\]
on which \(d_1\) and \(d_2\) depend.

(c) Implement the Black-Scholes formula numerically. Plot the value of an at-the-money (\(S_0 = K\) call option as a function of the volatility \(\sigma\). Observe that the option value is approximately an affine (linear plus constant) function of \(\sigma\).

(d) Explain why the value of an at-the-money call option on an asset with random volatility is approximately given by the Black-Scholes formula with
\[E^{\mathbb{Q}}\left[\exp\left(\int_0^T \sigma_t^2 \, dt\right)\right]\]
in input as the volatility.

(e) Part (c) should indicate that the Black-Scholes value is not exactly linear in the volatility. Neither is it uniformly concave nor uniformly convex; instead, it has different shapes in different regions. Explain why if it were concave (convex) over the relevant region, then the Black-Scholes formula with
\[E^{\mathbb{Q}}\left[\exp\left(\int_0^T \sigma_t^2 \, dt\right)\right]\]
in input as the volatility would overstate (understate) the value of the option.

17.8. Set \(V_t = \log S_t\), where \(\sigma_t\) is the volatility of a non-dividend-paying asset with price \(S\). Assume
\[\frac{\sigma_t}{S_t} = \mu_t \, dt + \sigma_t \, dB_1,\]
\[dV_t = \kappa(\theta - V_t) dt + \left(\rho \, dB_2 + \sqrt{1 - \rho^2} \, dB_1\right),\]
where \(\mu, \kappa, \theta, \rho,\) and \(\gamma\) are constants and \(B_1\) and \(B_2\) are independent Brownian motions under the physical probability measure. Assume there is a constant risk-free rate.
(a) Show that any SDF process must satisfy
\[ \frac{dM_t}{M_t} = -r \, dt - \frac{\mu - r}{\sigma} \, dB_t + \lambda_t \, dB_\lambda + \epsilon_t \, \, d\epsilon_t \]
(17.19)
for some stochastic process \( \lambda_t \), where \( \epsilon_t \) is a local martingale uncorrelated with \( B_1 \) and \( B_2 \).

(b) Assume that \( \lambda_t \) in the previous part is a constant. Show that
\[ \frac{dS}{S} = r \, dt + \sigma_t \, dB_t \]
\[ dV_t = \kappa_t(\theta_t - V_t) \, dt - \frac{\kappa_t \sigma_t}{\sigma_t} \, \frac{d\sigma_t}{\sigma_t} \, dt + \sqrt{1 - \rho^2} \, dB_t^* + \sqrt{\rho^2} \, dB_t^\lambda \]
for some constants \( \kappa_t \) and \( \theta_t \), where \( B_t^* \) and \( B_t^\lambda \) are independent Brownian motions under the risk-neutral probability corresponding to \( M \).

(c) Let \( W(t, S, V) \) denote the conditional probability \( \text{prob}_t(S_t > K) \) for a constant \( K \). Show that \( W \) must satisfy the PDE
\[ W_t = r S W_t + \left[ \kappa_t \theta_t - \kappa_t V_t - \rho_t \mu + r \right] S W_t + \frac{1}{2} \kappa_t \theta_t^2 W_t + \rho_t \mu S W_t + \frac{1}{2} \rho_t^2 S^2 W_t = 0. \]

17.9. In the Heston model (17.2), define \( Y_1 = V / \gamma^2 \) and \( Y_2 = \log S - \rho V / \gamma \).

(a) Derive the constants \( a_{ij}, b_{ij}, \) and \( \beta \) such that
\[ \frac{dY_1}{dY_2} = \begin{pmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \, dt + \begin{pmatrix} \sqrt{Y_1} \\ \sqrt{Y_2} \end{pmatrix} \, dB_1 + \begin{pmatrix} 0 \\ \beta \sqrt{Y_1} \end{pmatrix} \, dB_2.

(b) Consider a price of risk specification
\[ \lambda_u = c_{u0} c_{u1} Y_u + c_{u2} \, \psi_u, \]
\[ \lambda_\alpha = c_{\alpha0} c_{\alpha1} Y_\alpha + c_{\alpha2} \, \psi_\alpha, \]
for constants \( c_{ij} \). Derive \( c_{20} \) and \( c_{21} \) as functions of \( c_{10} \) and \( c_{11} \) from the fact that (17.12) must hold for all \( V \). Note: The specification in Section 17.4 is the special case \( c_{10} = 0 \).

(p.457) (c) Assume that \( M \) defined in terms of \( c_{10} \) and \( c_{11} \) is such that \( M_R \) is a martingale. Derive constants \( a_t^* \) and \( b_t^* \) in terms of \( c_{10} \) and \( c_{11} \) such that
\[ \frac{dY_1}{dY_2} = \begin{pmatrix} a^*_1 & b^*_1 \\ a^*_2 & b^*_2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \, dt + \begin{pmatrix} \sqrt{Y_1} \\ \sqrt{Y_2} \end{pmatrix} \, dB_1^* + \begin{pmatrix} 0 \\ \beta \sqrt{Y_1} \end{pmatrix} \, dB_2^* \]
where the $B^*_t$ are independent Brownian motions under the risk-neutral probability.

(d) Assume $\kappa \theta / \gamma^2 \geq 1/2$. Under what condition on $c_{10}$ is $\sigma_t^2 \geq 1/2$?

17.10. Consider an American call option with strike $K$ on an asset that pays a single known discrete dividend $x$ at a known date $T < u$, where $u$ is the date the option expires. Assume the asset price $S$ drops by $x$ when it goes ex-dividend at date $T$ (i.e., $S_T = \lim_{t \to T} S_t - x$) and otherwise is an Itô process. Assume there is a constant risk-free rate $r$.

(a) Show that if $x < (1 - e^{\rho u - T} - x)$, then the call should not be exercised early.

For the remainder of the exercise, assume $x > (1 - e^{\rho u - T} - x)$.

Assume the volatility of the process

$$Z_t = \begin{cases} S_t - e^{rt - \frac{\sigma^2}{2} t} & \text{if } t < T \\ S_T & \text{if } T \leq t \leq u \end{cases}$$

is constant over $[0, u]$. Let $V(t, S_t)$ denote the value of a European call on the asset with strike $K$ maturing at $u$. Let $S^*$ denote the value of the stock price just before $T$ such that the holder of the American option would be indifferent about exercising just before the stock goes ex-dividend. This value is given by $S^* - K = V(T, S^* - x)$. Exercise is optimal just before $T$ if $\lim_{t \to T} S_t > S^*$, and equivalently, if $S_T > S^* - x$. Let $A$ denote the event $S_T > S^* - x$ and let $C$ denote the set of states of the world such that $S_T \leq S^* - x$ and $S_u > K$. The cash flows to a holder of the option who exercises optimally are $(S_T + x - K)1_A$ at (or, rather, “just before”) date $T$ and $(S_u - K)1_C$ at date $u$.

(b) Show that the value at date 0 of receiving $(S_T + x - K)1_A$

at date $T$ is

$$(S_0 - e^{-rT}x)N(d_1) - e^{-r(T - x)}N(d_2),$$

where

$$d_1 = \frac{\log S_0 - rT + \log S^* - x + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}},$$

$$d_2 = d_1 - \sigma \sqrt{T}.$$
(p.458) (c) Show that the value at date 0 of receiving \((S_u - K)_{1c}\) at date \(u\) is

\[
(S_0 - e^{-r_0}K)M(-d_1, d_2 - \sqrt{U}/u) - e^{-r_0}KM(-d_1, d_2 - \sqrt{U}/u),
\]

where \(M(a, b, \rho)\) denotes the probability that \(\xi_1 < a\) and \(\xi_2 < b\) when \(\xi_1\) and \(\xi_2\) are standard normal random variables with correlation \(\rho\), and where

\[
d_1 = \frac{\log S_0 - r_0 T + (r_0 + \sigma^2/2)T}{\sigma \sqrt{T}}, \\
d_2 = d_1 - \sigma \sqrt{u}.
\]

Notes:

(1.) There are related empirical results that are surprising. This is discussed in the end-of-chapter notes.

(2.) By Itô’s formula,

\[
\frac{d(S_t/S_0)}{S_t/S_0} = \frac{dS_t}{S_t} - \frac{dS_t}{S_0} - \frac{dS_t}{S_t},
\]

Therefore,

\[
\left(\frac{d(S_t/S_0)}{S_t/S_0}\right)^2 = \left(\frac{dS_t}{S_0}\right)^2 - 2\sigma^2 \frac{dS_t}{S_t} + \frac{dS_t}{S_0} = (\sigma^2 - 2\rho \sigma^2 + \sigma^2)dt = \sigma^2 dt.
\]

(3.) We do not need to assume the discount bond is traded directly if there is a money market with a nonrandom interest rate, because then the discount bond can be created from the money market account. Also, we do not need to assume that the forward is traded directly if it can be created synthetically using the underlying asset as discussed in connection with the spot-forward parity formula.

(4.) Here, we use the notation of Chapter 13, and \(\sigma\) is the row vector \((\sqrt{\rho}, \sqrt{1-\rho})\) that premultiplies the column vector of Brownian motion differentials in (17.10b).

(5.) If \(\mu \neq r\), then \(\nu\) must be always strictly positive in order for (17.12) to hold—that is, the asset must always be risky \((\nu_t > 0)\) in order to earn a risk premium. A necessary and sufficient condition to have, with probability 1, \(\nu_t > 0\) for all \(t\), is that \(\kappa \theta \geq \nu^2/2\). For the remainder of this section, consider only parameters satisfying this restriction.