Dynamic Portfolio Choice

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DOI:10.1093/acprof:oso/9780190241148.003.0009

Abstract and Keywords
The first-order condition for optimal portfolio choice is called the Euler equation. Optimal consumption can be computed by a static approach in a dynamic complete market and by orthogonal projection for a quadratic utility investor. Dynamic programming and the Bellman equation are explained. The envelope condition and hedging demands are explained. Investors with CRRA utility have CRRA value functions. Whether the marginal value of wealth is higher for a CRRA investor in good states or in bad states depends on whether risk aversion is less than or greater than 1. With IID returns, the optimal portfolio for a CRRA investor is the same as the optimal portfolio in a single-period model.

Keywords: Euler equation, static approach, orthogonal projection, quadratic utility, dynamic programming, envelope condition, hedging demands, constant relative risk aversion, transversality condition

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The first three sections of this chapter discuss topics in portfolio choice that are very much the same in dynamic markets as in single-period markets: the Euler equation, solving for optimal consumption in complete markets using the unique SDF process, and solving for optimal consumption for quadratic utility with orthogonal projections. The remainder of the chapter explains dynamic programming, which is a method for converting a dynamic optimization problem into a series of single-period problems. Section 9.4 explains dynamic programming in general, and Section 9.5 explains its application to portfolio choice. Key topics for portfolio choice are the envelope condition and hedging demands. Section 9.6 illustrates dynamic programming by solving for optimal portfolios with independent and identically distributed (IID) returns and CRRA utility.

9.1 Euler Equation
Consider a risk-averse investor with initial wealth $w_0$ and time-additive utility (8.2a) or (8.2b). The first-order condition for dynamic portfolio choice is called the Euler equation. It is:

\[(\forall i, t) \quad u(C_t) = \mathbb{E}[\delta u(C_{t+1})R_{t+1}] .\]  

(9.1)
This states that the investor is indifferent at the margin between consuming a bit more at $t$ and investing a bit more in asset $i$ to increase consumption at $t+1$. Assuming strictly monotone utility, the Euler equation is equivalent to the statement that the MRS

\[Z_{t+1} = \frac{\delta u(C_{t+1})}{u(C_t)} \]

is a single-period SDF. Compounding the single-period SDFs, we see that the Euler equation is equivalent to the statement that

\[M_t = \frac{\delta' u(C_t)}{u(C_t)} \]

(9.2)
is an SDF process.
As in a single-period model, the first-order condition must hold at any optimum from which it is feasible to make small variations in the consumption and portfolio decisions. For each asset $i$, each date $t$ and each event $A$ observable at $t$, assume there is some $\epsilon > 0$ such that each of the following variations on the optimum produces finite expected utility:

(i) It is feasible for the investor to reduce consumption by $\epsilon$ at $t$ when $A$ occurs, to invest $\epsilon$ in asset $i$, and to consume the value of this additional investment at $t+1$.

(ii) It is feasible for the investor to increase consumption by $\epsilon$ at $t$ when $A$ occurs, to finance this consumption by investing $\epsilon$ less in (or shorting) asset $i$, and to restore wealth to its optimal level by consuming less at $t+1$.

Under this assumption, the first-order condition is derived below (by the same logic as in the single-period model).

It is also shown below that the first-order condition is sufficient for optimality when the horizon is finite. In the infinite-horizon case, the first-order condition plus the transversality condition are jointly sufficient for optimality, if a constraint is imposed to preclude Ponzi schemes.

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**Necessity of the Euler Equation** To derive (9.1) at an optimum, first consider reducing consumption by $\epsilon$ at $t$ when $A$ occurs and investing $\epsilon$ in asset $i$, consuming the value of the investment at $t+1$. Consumption changes at $t$ by $-\epsilon 1_A$ and at $t+1$ by $\epsilon 1_A R_{t+1}$. The resulting change in expected utility is

\[
\mathbb{E}[1_A \delta' \mathbb{V}(C_t - \epsilon) - \mathbb{V}(C_t)] + \mathbb{E}[1_A \delta^{\epsilon} \mathbb{V}(C_{t+1} + \epsilon R_{t+1}) - \mathbb{V}(C_{t+1})] \leq 0.
\]

Letting $\epsilon \to 0$ (and using the monotone convergence theorem as in Section 2.1), we conclude that

\[
\mathbb{E}[-1_A \delta' \mathbb{V}(C_t)] + \mathbb{E}[1_A \delta^{\epsilon} \mathbb{V}(C_{t+1} R_{t+1})] \leq 0.
\]

(p.204) Now, consider increasing consumption by $\epsilon$ at $t$ when $A$ occurs by investing less in asset $i$. Reasoning in the same way, we obtain

\[
\mathbb{E}[1_A \delta' \mathbb{V}(C_t)] + \mathbb{E}[-1_A \delta^{\epsilon} \mathbb{V}(C_{t+1}) R_{t+1}] \leq 0.
\]

Therefore,
Because this is true for each event \( A \) observable at \( t \), the Euler equation (9.1) follows from the definition of a conditional expectation (Appendix A.8).

**Sufficiency of the Euler Equation** Suppose \( u \) is concave.

Suppose the Euler equation holds for a consumption process \( C \) that satisfies the intertemporal budget constraint (8.1) in conjunction with some wealth and portfolio processes. Let \((\hat{C}, \hat{W}, \hat{\pi})\) be any other solution of the intertemporal budget constraint (8.1). By concavity,

\[
\begin{align*}
    u(C_t) - u(\hat{C}_t) & \geq u(C_t)(C_t - \hat{C}_t), \\
    \text{(9.3)}
\end{align*}
\]

Suppose the horizon is finite. Because there is no bequest motive, we can assume \( C_T = W_T \) and \( \hat{C}_T = \hat{W}_T \). By (8.17) and (9.2),

\[
    W_0 + \frac{1}{u(C_0)} \sum_{t=1}^{T} \delta^t \mathbb{E}[u(C_t)Y_t] = \frac{1}{u(C_0)} \sum_{t=0}^{T} \delta^t \mathbb{E}[u(C_t)\hat{C}_t].
\]

and

\[
    W_0 + \frac{1}{u(C_0)} \sum_{t=1}^{T} \delta^t \mathbb{E}[u(C_t)Y_t] = -\frac{1}{u(C_0)} \sum_{t=0}^{T} \delta^t \mathbb{E}[u(C_t)C_t],
\]

Hence,

\[
    \sum_{t=0}^{T} \delta^t \mathbb{E}[u(C_t)(C_t - \hat{C}_t)] = 0.
\]

It follows from this and (9.3) that

\[
    \sum_{t=0}^{T} \delta^t \mathbb{E}[u(C_t) - u(\hat{C}_t)] \geq 0,
\]

which shows that the solution of the Euler equation is optimal.

**(p.205)** Consider the infinite-horizon case. Assume

\[
    \sum_{t=1}^{\infty} \mathbb{E}[u(C_t)Y_t]
\]

exists and is finite. Suppose the constraint (8.24) is imposed to preclude Ponzi schemes (with \( M_T = \delta^T u(C_T)u(C_0) \)). Suppose the wealth process corresponding to the solution of the Euler equation satisfies the transversality condition

\[
    \lim_{T \to \infty} \mathbb{E}[\delta^T u(C_T)W_T] = 0.
\]

(9.4)

Then, as explained in Section 8.5, we have

\[
    W_0 + \frac{1}{u(C_0)} \sum_{t=1}^{\infty} \delta^t \mathbb{E}[u(C_t)Y_t] = \frac{1}{u(C_0)} \sum_{t=0}^{\infty} \delta^t \mathbb{E}[u(C_t)C_t],
\]

and

\[
    W_0 + \frac{1}{u(C_0)} \sum_{t=1}^{\infty} \delta^t \mathbb{E}[u(C_t)Y_t] \geq \frac{1}{u(C_0)} \sum_{t=0}^{\infty} \delta^t \mathbb{E}[u(C_t)\hat{C}_t],
\]
Hence, (9.3) implies
\[ \sum_{t=0}^{\infty} \delta_t E[u(C_t) - u(\widehat{C}_t)] \geq 0. \]

9.2 Static Approach in Complete Markets

In a complete market, we can collapse the intertemporal budget constraints (one constraint for each date and state of the world) into a single budget constraint. We can then solve for optimal consumption in the same way that we can solve for it in a complete single-period market. This approach to solving for optimal consumption in a dynamic model is called the static approach.

Assume the market is complete and there are no arbitrage opportunities. Then, as discussed in Sections 8.3 and 8.4, there is a unique SDF process \( M \), and \( M \) is strictly positive.

Consider an investor with a finite horizon \( T \) (the analysis can be extended to an infinite horizon at the expense of dealing with Ponzi schemes and transversality conditions). The investor chooses consumption and portfolio processes, and her wealth process is then determined by the intertemporal budget constraint. As before, assume there is no bequest motive, so \( C_T = W_T \). From (8.17), the consumption process must satisfy
\[ \text{(p.206)} \]

\[ W_0 + \sum_{t=1}^{T} E[M_t Y_t] = \sum_{t=0}^{T} E[M_t C_t]. \]

(9.5)

Furthermore, by Exercise 8.3, for any consumption process satisfying (9.5), there exists a portfolio process such that, after consuming \( (C_0, \ldots, C_{T-1}) \), the investor has remaining wealth \( W_T = C_T \). Thus, any consumption process satisfying (9.5) is feasible.

Because the feasible consumption processes are precisely those that satisfy (9.5), the investor’s optimal consumption process must be the process that maximizes expected utility subject to (9.5). This optimization problem is called a static problem, because there is only a single budget constraint (as in a single-period model), the intertemporal budget constraints having been collapsed into (9.5).

As an example, consider CRRA utility with relative risk aversion \( \rho \). The first-order condition implies
and substituting this into (9.5) yields

\[ C_0 + C_0 \mathbb{E} \left[ \sum_{t=1}^{T} \delta^t \bar{S}_i M_t^{-1} \right] = W_0 + \sum_{t=1}^{T} \mathbb{E} [M,Y], \]

which can be solved as

\[ C_0 = \frac{W_0 - \mathbb{E} \left[ \sum_{t=1}^{T} \delta^t \mathbb{E} [M,Y] \right]}{1 + \mathbb{E} \left[ \sum_{t=1}^{T} \delta^t \bar{S}_i M_t^{-1} \right]}. \]

Thus,

\[ C_t = \frac{\left( W_0 + \mathbb{E} \left[ \sum_{t=1}^{T} \delta^t \mathbb{E} [M,Y] \right] \right) \delta^t \bar{S}_i M_t^{-1}}{1 + \mathbb{E} \left[ \sum_{t=1}^{T} \delta^t \bar{S}_i M_t^{-1} \right]}. \]

(9.7)

The optimal wealth process is, for \( t < T \),

\[ W_t = C_t + \mathbb{E} \left[ \sum_{t=1}^{T} \delta^t \mathbb{E} [M,Y] \right]. \]

9.3 Orthogonal Projections for Quadratic Utility

Section 3.7 characterizes the optimal portfolio for an investor with quadratic utility in a single-period model. The investor shorts the portfolio that best approximates her labor income, shorts the portfolio with return \( R_p \), and invests all of the proceeds from the short sales plus her initial wealth in the asset that best approximates a risk-free asset. In the presence of a risk-free asset, the optimum can be described thus: Short the best approximation to labor income and invest the proceeds of the short sale plus initial wealth in a portfolio on the mean-variance frontier. A similar characterization of the optimal portfolio can be made in a dynamic model. We will only consider the finite-horizon case. Denote the horizon by \( T \).

Consider an investor with quadratic utility function \( u(c) = -(c - \bar{c})^2/2 \) and no bequest motive. The Euler equation states that

\[ \delta u(C_t) = u(C_t) M_t \Rightarrow C_t = \xi_t - C_0 \delta^t M_t, \]

(9.8)

for an SDF process \( M \). Subtracting \( Y_t \) from both sides gives

\[ C_t - Y_t = \xi_t - Y_t - (C_0 \delta^t M_t). \]

(9.9)

At each date \( t \), \( C_t - Y_t \) is the amount the investor withdraws from her financial portfolio, so \( C - Y \) is a marketed consumption.
process in the sense of Section 8.4. Set $X_i = C_t - Y_t$ and $\hat{M}_t = \delta^t M_\infty$, so we can write (9.9) as

$$X = \zeta - Y - (\zeta - C_0) \hat{M}.$$  

(9.10)

Consider stochastic processes $A = (A_0, \ldots, A_T)$ with the property that $\mathbb{E}[A_t^2] < \infty$ for each $t$. Denote the set of such processes by $L^2$. Use the investor’s discount factor $\delta$ to define an inner product for $A, B \in L^2$:

$$(A, B) \overset{def}{=} \sum_{t=0}^{T} \delta^t \mathbb{E}[A_t B_t].$$  

(9.11)

The set $L^2$ equipped with this inner product is a Hilbert space (Section 3.8). Let $M_0$ denote the subset of marketed consumption processes in $L^2$. The set $M_0$ is a linear subspace of $L^2$. Let $M$ denote its closure. For any $A \in L^2$, the orthogonal projection of $A$ onto $M$ exists. It is $A_p \in M$ such that $A - A_p$ is orthogonal to $M$, in the sense that $(B, A - A_p) = 0$ for every $B \in M$. We will consistently use the subscript $p$ to denote projections onto $M$.

Assume the investor’s labor income process $Y$ and consumption process $C$ belong to $L^2$. Then, $X = C - Y \in L^2$. As already remarked, $X$ is marketed, so $X \in M$. Therefore, it equals its own projection onto $M$. Projections are linear, so the fact that $X = X_p$ combined with (9.10) gives

$$X = \zeta \ 1 - Y_p - (\zeta - C_0) \hat{M}_p.$$  

(9.12)

Here $1_p$ denotes the projection onto $M$ of the trivial stochastic process that equals 1 at every date and in every state of the world. Equation (9.10) shows that, as in a single-period model, the investor short sells the marketed consumption process that best approximates her labor income, short sells the consumption process that best approximates the SDF process, and invests her initial wealth plus the proceeds from the short sales into the consumption process that best approximates a constant.

The quantity $C_0$ in (9.12) is endogenous. We solve for it below using the budget constraint and show that
The consumption process \( \hat{M}_p / (\hat{M}_p \hat{M}_p) \) is a unit-cost version of the projection \( \hat{M}_p \), so it is analogous to \( \bar{R}_p \) in the single-period model. The formula (9.13) is a direct extension of the single-period formula (3.42).

\[
X = \zeta_1 p - Y_p - \left( \zeta \{ 1_p, \hat{M}_p \} - \{ Y_p, \hat{M}_p \} - W_0 \right) \frac{\hat{M}_p}{(\hat{M}_p \hat{M}_p)}.
\] (9.13)

Note that, for any \( A \in M \), we have

\[
\langle A, \hat{M}_p \rangle = \langle A, \hat{M} \rangle = \sum_{t=0}^{T} \mathbb{E}[A_M_t]
\]

Thus, the inner product with \( \hat{M}_p \) gives the fundamental value of a marketed consumption process. Using \( C_T = W_T \) (no bequest motive), \( Y_0 = 0 \), and (8.17), we therefore have

\[
\langle X, \hat{M}_p \rangle = W_0.
\]

This is a reformulation of the investor’s budget constraint. Substitute for \( \chi \) from (9.12) to obtain

\[
\zeta \{ 1_p, \hat{M}_p \} - \{ Y_p, \hat{M}_p \} - (\zeta - C_0) (\hat{M}_p \hat{M}_p) = W_0.
\]

This implies

\[
\zeta - C_0 = \frac{\{ Y_0, \hat{M}_p \} + \{ Y_0, \hat{M}_p \} - W_0}{(\hat{M}_p \hat{M}_p)}.
\]

Substitute this into (9.12) to obtain (9.13).

9.4 Introduction to Dynamic Programming

Many dynamic decision problems are easiest to solve when reduced to a series of single-period problems. This method is easiest to explain in a lattice (tree) model. The tree in Figure 9.1 represents a decision problem in which a person must decide at each of dates \( t = 0, 1, 2 \) whether to go up or down.

The reward earned at date \( t = 3 \) depends on the sequence of decisions made and is shown at the right side of the graph. Clearly, 24 is the maximum possible reward, and the optimal sequence of decisions is Down-Up-Up.
To see how dynamic programming works in this simple problem, consider each of the four nodes at date \( t = 2 \). If we were to reach the top node, the optimal decision from that point is obviously Up, leading to a reward of 14. From the second-highest node at date \( t = 2 \), the optimal decision is Down, leading to a reward of 18. These calculations lead to the value function at date 2, which lists the maximum terminal reward that can be reached from each of the date-2 nodes.

Having computed the value function at date 2, we can compute the value function at date 1 by considering at each node whether Up or Down produces the highest date-2 value. For example, at the top node at date 1, we can choose between the values 14 and 18. Obviously, we would choose 18, meaning Down. We do not have to look forward to date 3, because the information we need to make an optimal decision at date 1 is already encoded in the date-2 values. (p.210) Likewise, we can compute the value at date 0 by considering whether Up or Down produces the highest date-1 value. This process of computing the value function is called backward induction or backward recursion. The complete set of values is shown in Figure 9.2.

**Figure 9.1 A decision tree.** In this example, there are four dates 0, 1, 2, and 3, starting with date 0 on the left and ending with date 3 on the right. At dates 0, 1, and 2, the choice to be made is to go up or down. The rewards occur at date 3 and depend on the path taken. The rewards are shown in the date 3 nodes.
State Transition Equation

To represent these calculations as a mathematical formula, note that there are \( z \) nodes at each date \( t \). Index the nodes, starting from the bottom, as \( x = 1, \ldots, z \). Let \( \pi \) represent the decision variable, with \( \pi = 0 \) meaning Up and \( \pi = 1 \) meaning Down. If \( x \) is the node at date \( t \) and the decision \( \pi \) is taken at date \( t \), then \( 2x - \pi \) is the node at date \( t + 1 \), so we write

\[
x_{t+1} = 2x_t - \pi_t.
\]

(9.14)

**Bellman Equation**

Let \( V(x) \) denote the maximum attainable value starting from node \( x \) at date \( t \). The values at date 3 are the given rewards; for example, \( V(2) = 9 \). The values at dates \( t = 0, 1, 2 \) are the values we have computed by backward induction; for example, \( V_t(1) = 20 \).

The backward induction process is expressed mathematically as

\[
V(x) = \max_{\pi \in \{0,1\}} V_{t+1}(2x - \pi).
\]
This formula for $V_t$ in terms of $V_{t+1}$ is called the Bellman equation.

**Intermediate Rewards**

A variation of the decision problem is one in which there are rewards earned at each date and the objective is to maximize the sum of rewards over time. Consider a tree like that shown in Figure 9.1 but suppose there are rewards at each node. We can allow the reward earned at each node to depend on the decision (Up or Down) taken at that node. Denote the reward earned at date $t$ at node $x$ when decision $\pi$ is taken by $u(x, \pi)$. There is obviously no decision to be made at the terminal date $t = 3$, so we can write $u_f(x)$ for the terminal reward at node $x$.

Denote the decision taken at date $t$ at node $x$ by $\pi_t(x)$. We want to choose the decisions $\pi_t(x)$ so as to maximize

$$\sum_{t=0}^{2} u(x_t, \pi_t(x_t)) + u_f(x_3),$$

where the path through the tree is determined by the decisions $\pi_t(x_t)$ and the state transition equation (9.14). Let $V_f(x) = u_f(x)$ and for $t < 3$, define

$$V_t(x) = \max_{\pi \in \Pi(x)} \{ u(x_t, \pi) + V_{t+1}(2x - \pi) \}.$$  

(9.15b)

By this backward induction, we define the values at each node. Equation (9.15b) is the Bellman equation for this problem. To make the optimal decision at each date, it is again enough to look at the current reward and the values attainable at the next date, rather than looking forward to the end of the tree.

**Dynamic Programming under Uncertainty**

In dynamic programming under uncertainty, we use the maximum attainable expected utility as the value. This means that (9.15a) is replaced by

$$V_t(x) = \max_{\pi} \mathbb{E} V_{t+1}(X_{t+1}) \mid X_t = x,$$

(9.16a)

where $X_{t+1}$ denotes the random state (node) at date $t + 1$, the distribution of which may depend on the decision $\pi$ and the state $x$ at date $t$. Also, (9.15b) is replaced by

$$V_t(x) = \max_{\pi} \{ u(x_t, \pi) + \mathbb{E} V_{t+1}(X_{t+1}) \mid X_t = x \}.$$  

(9.16b)
In these equations, and in the statement of the Bellman equation throughout the book, the operator max means the least upper bound (supremum).

Obviously, in an infinite-horizon problem, we cannot calculate the value function by starting at T and working backward using the Bellman equation. Nevertheless, under certain conditions, we can find the value function by solving the Bellman equation, and the maximization in the Bellman equation produces the optimal decisions. This is discussed in the next section.

9.5 Dynamic Programming for Portfolio Choice
Dynamic programming is applicable only in a Markovian model, meaning that there are some variables the current values of which form a sufficient statistic for predicting future random variables (returns and labor income in our case). These variables play the role of the nodes in the decision tree shown earlier in the sense that they summarize the past in a manner that is sufficient for making optimal decisions. They are called state variables.

Assume the distribution of the vector of asset returns \( R_{t+1} \) depends on a vector of state variables \( X_t \). Likewise, allow the distribution of the labor income \( Y_{t+1} \) to depend on the vector \( X_t \). By “depend on,” it is meant that the distribution of \( R_{t+1} \) and \( Y_{t+1} \) conditional on all information at date \( t \) is the same as the distribution conditional on \( X_t \) only. For this to be useful, the vector \( X_t \) must have the same property, namely, that the distribution of \( X_{t+1} \) conditional on all information at date \( t \) is the same as the distribution conditional on \( X_t \) only. This defines \( p. 213 \) the sequence of random vectors \( X_t \) as a Markov process, and it implies that the sequence of random vectors \( (X_t, Y_t, R_t) \) is also a Markov process.\(^2\)

It follows from these assumptions that the distributions of \( X_u \), \( Y_u \), and \( R_u \) for all \( u > t \) conditional on information at date \( t \) are the same as the distributions conditional on \( X_t \) only. The properties assumed here are often expressed by saying that \( X_t \) is a sufficient statistic for predicting the future values of \( (X_t, Y_t, R_t) \).
because all information at date $t$ other than $X_t$ can be ignored for making those predictions.

The decision variable $\pi$ in the previous section is now the consumption-portfolio pair $(c, \pi)$, and the state variable (node) $x$ in the previous section is replaced by the state variables $X$ and wealth $w$. Recall that we are including the return of the money market account, if it exists, as one of the returns in the vectors $R_t$. Therefore, a portfolio is a vector $\pi$ such that $\pi_t = 1$.

*Every maximization over $\pi$ in the remainder of the chapter is to be understood as subject to the constraint $\pi_t = 1$."

Bellman Equation for Portfolio Choice

If an investor has a finite horizon and seeks only to maximize the expected utility of terminal wealth $\mathbb{E}[u(W_T)]$, then the Bellman equation is

$$V(x, w) = \max_{\pi} \mathbb{E}\left[V_{t+1}(X_{t+1}, Y_{t+1} + w\pi R_{t+1}) \mid X_t = x\right].$$

(9.17)

Here, the conditional expectation is over the distributions of $X_{t+1}$, $Y_{t+1}$, and $R_{t+1}$, given $X_t = x$. The future value $V_{t+1}$ depends on $X_{t+1}$ and $w_{t+1}$. We have used the intertemporal budget equation to substitute for $w_{t+1}$ so as to show how it depends on the decision $\pi$ and wealth $w$ at time $t$. The value function equals the utility function at the terminal date $T$, that is, $V_T(x, w) = u(w)$.

If there is consumption at each date as in (8.2a) and (8.2b), then the rewards $u_i$ in the previous section are $\delta u(C_0)$.

Therefore, the Bellman equation (9.16b) is

$$V(x, w) = \max_{\pi} \left(\delta u(c) + \mathbb{E}\left[V_{t+1}(X_{t+1}, Y_{t+1} + (w - c)\pi R_{t+1}) \mid X_t = x\right]\right).$$

(9.18)

When there is consumption at each date, it is convenient to define another value function $J_t(w, x) = \delta^t V_t(w, x)$. The relation of $J_t$ to $V_t$ is that $V_t$ is the maximum utility from date–$t$ onward, discounted to date 0, whereas $J_t$ is the same utility (p.214) discounted only to date $t$. Substitute $J_t$ for $V_t$ in (9.18) and cancel the factor $\delta^t$. This produces

$$J_t(x, w) = \max_{\pi} \left(u(c) + \delta \mathbb{E}\left[J_{t+1}(X_{t+1}, Y_{t+1} + (w - c)\pi R_{t+1}) \mid X_t = x\right]\right).$$

(9.19)
We will use the value function $J$ and the Bellman equation (9.19) whenever there is consumption at each date. If the horizon is finite and there is consumption at each date, then the value function $J_t$ equals the utility function at the terminal date $t = T$, that is, $J_T(x, w) = u(w)$.

The Bellman equation (9.19) can be simplified in the infinite-horizon case (8.2a), due to stationarity. When there is an infinite horizon, the maximum value that can be achieved from date $t$ onward discounted to date $t$ and starting from $X_t = x$ and $W_t = w$ is the same for every $t$, given $x$ and $w$. This is due to the fact there is always an infinite number of periods remaining at any date $t$ and to the fact that $X$ is Markov, so its distribution does not depend on the time index, given $x$. Thus, in the infinite-horizon case, $J$ does not depend on $t$, and the Bellman equation is

$$J(x, w) = \max_{\alpha_t} \left\{ u(c) + \delta E \left[ J(X_{t+1}, Y_{t+1} + (w - c)R_{t+1}) \mid X_t = x \right] \right\}.$$  

(9.20)
Optimal Portfolio and Hedging Demands

In principle, given knowledge of the value function $J$, we could perform the maximization in the infinite-horizon Bellman equation (9.20) in two steps. First, we can compute the optimal consumption $c^*$ and then find the optimal portfolio $\pi^*$. In the second step, we are maximizing

$$
E[J(x_{t+1}, y_{t+1} + (w - c^*) \pi_t R_{t+1}) | X_t = x],
$$

(9.21)

where $c^*$ denotes the consumption attaining the maximum in (9.20). Maximizing (9.21) is similar to a single-period portfolio choice problem, which of course is the point of dynamic programming, but there is an important difference between (9.21) and the objective function in a single-period problem. The difference is that the function $J$ in (9.21) depends on the state variables $X_{t+1}$ in addition to date $t+1$ wealth.

(p.215) The same two-step procedure could be used for the Bellman equations (9.17) and (9.19), given knowledge of the value functions. So, in general, the optimal portfolio at any date $t$ is the one that maximizes the expected value of the value function at $t+1$. In general, the value function at date $t+1$ is lower when the realized values of the state variables $X_{t+1}$ imply a less favorable distribution for future returns $R_{t+2}, R_{t+3}, \ldots$, and/or a less favorable distribution for future income $Y_{t+2}, Y_{t+3}, \ldots$.

Investors generally choose to hedge, to some extent, against such adverse changes in the state variables. Investors typically face a trade-off between hedging and achieving high returns, in addition to the trade-off between risk and return that appears in a single-period model. In continuous-time models, a formula for the optimal portfolio (and for the hedging demands) can be given in terms of the partial derivatives of the value function, the covariance matrix of the returns, and the covariances between returns and state variables (Section 14.5).

Envelope Condition

The envelope theorem states that the partial derivative of a value function with respect to a parameter is the same whether a choice variable is varied when the parameter is varied or held fixed at the previous optimum. In the Bellman equation (9.20), we can regard the expression being maximized as a function of $(x, w, \pi, c)$. The exogenous parameters
are \((x, w)\) and the choice variables are \((\pi, c)\). The partial
derivative of the maximum value with respect to \(w\) is the same
whether \(c\) is varied with \(w\) or held fixed at the optimum, so we
can suppose \(c\) is varied one-for-one with \(w\), leaving \(w - c\) fixed.
The partial derivative of the maximum with respect to \(w\)—the
left-hand side of (9.20)—is
\[
\frac{\partial}{\partial w} J(x, w).
\]
Taking \(dc/dw = 1\) and holding \(w - c\) fixed, the derivative with
respect to \(w\) of the expression being maximized on the right-
hand side of (9.20) is \(u(c)\). The envelope theorem implies the
equality of these expressions:
\[
\frac{\partial}{\partial w} J(x, w) = u(c)
\]
(9.22)
when \(c\) is the optimal choice. Naturally, equation (9.22) is
called the envelope condition. To repeat somewhat the
discussion above, the interpretation of (9.22) is that, because
the investor has optimized over consumption \(c\) and investment
\(w - c\), a small change in initial wealth can either be consumed
or invested, with (p.216) the value of either option being the
same. Thus, the value of a small amount of additional wealth is
the same as the value of a small amount of additional
consumption. The envelope condition also holds for the finite-
horizon Bellman equation (9.19).

CRRA Utility Implies a CRRA Value Function

Assume for the remainder of this section that there is no labor
income. Then, the value function of an investor with CRRA
utility has constant relative risk aversion for wealth with the
same risk aversion that the utility function has for
consumption.\(^5\) For both log and power utility, the value
function is separable in the state variables and in wealth. For
log utility, the value function is additively separable. For power
utility, the value function is multiplicatively separable.

If the horizon is finite, then the value function for log utility is
\(\gamma' \log w + f(x)\) for some numbers \(\gamma'\) and functions \(f\). This is true
when the objective is to maximize the expected utility of
terminal wealth, and it is also true when there is consumption
at each date. If the horizon is infinite, then \(J(x, w) = \gamma \log w + f(x)\) for
a constant \(\gamma\) and function \(f\).
If the horizon is finite, then the value function for power utility is \( f(x)w^{1-p} \) for some functions \( f \), where \( p \) is the relative risk aversion of the utility function. This is true when the objective is to maximize the expected utility of terminal wealth, and it is also true when there is consumption at each date. If the horizon is infinite, then \( J(x, w) = f(x)w^{1-p} \) for some function \( f \).

The fact that the value function has constant relative risk aversion for wealth with risk aversion \( p \) is often expressed, when \( p \neq 1 \), by saying that the value function (like the utility function) is homogeneous of degree \( 1 - p \). The utility and value functions are also said to be homothetic, a homothetic function being a monotone transform of a homogeneous function.

When the horizon is finite, the above description of the value function can be verified by induction using the Bellman equation. Consider power utility with consumption at each date. The value function \( J_t \) is equal to the utility function at date \( t = T \), so it is of the form \( f(x)w^{1-p} \) with \( f(x) = 1/(1 - p) \) for all \( x \). Suppose \( J_s(x, w) = f(x)w^{1-p} \) for all \( s = t + 1, \ldots, T \) for some functions \( f_s \).

Then,

\[
J_s(x, w) = \max_{c_t} \left\{ \mathbb{E} \left[ \int_{t+1}^{T} (w - c_t) \rho R_{t+1} | X_t = x \right] \right\} + \delta \mathbb{E} \left[ f_s(X_{t+1}) \left( (w - c_t) \rho R_{t+1} \right)^{1-p} | X_t = x \right]
\]

Let \( z \) (p.217) denote the consumption rate \( c/w \). We can maximize over \( z \) instead of \( c \). This produces

\[
J_s(x, w) = \max_{z_t} \left\{ \int_{t+1}^{T} z_t^{1-p} w^{1-p} + \delta \mathbb{E} \left[ f_s(X_{t+1}) \left( (1 - z_t) \rho R_{t+1} \right)^{1-p} | X_t = x \right] \right\}
\]

Setting

\[
f(x) = \max_{z_t} \left\{ \int_{t+1}^{T} z_t^{1-p} + \delta \mathbb{E} \left[ f_s(X_{t+1}) \left( (1 - z_t) \rho R_{t+1} \right)^{1-p} | X_t = x \right] \right\},
\]

we see that \( J_s(x, w) = f(x)w^{1-p} \) as claimed. By induction, the claim is true for all \( t \). The argument is similar for log utility and for maximizing the expected utility of terminal wealth.

With an infinite horizon, a similar argument shows that there is a CRRA solution of the Bellman equation. However, this does not prove that the value function has constant relative risk aversion, because there can in general be multiple...
solutions of the Bellman equation in infinite-horizon dynamic programming problems, only one of which is the value function. A simple example is presented in Exercise 9.4. Therefore, we give a different proof below that the value function has constant relative risk aversion in the infinite-horizon model with CRRA utility and no labor income.

Adopt the constraint \( c_t \leq w_t \) to rule out Ponzi schemes. This no-borrowing constraint is very sensible here, because we are assuming there is no future labor income \( y \) to borrow against. Consider any consumption and portfolio processes. Set \( z_t = c_t / w_t \). The constraint \( c_t \leq w_t \) is equivalent to \( z_t \leq 1 \). Let \( r_{t+1} \) denote the portfolio return from \( t \) to \( t+1 \). The intertemporal budget constraint specifies that, for all dates \( s \),

\[
w_s = w_{t+1} \prod_{s=t}^{u} (1 - z_s) r_{s+1}.
\]

By induction, for any dates \( t < u \),

\[
w_u = w_t \prod_{s=t}^{u} (1 - z_s) r_{s+1}.
\]

Therefore,

\[
C_u = w_t z_u \prod_{s=t}^{u} (1 - z_s) r_{s+1}.
\]

(9.23)

We can use the same formula for \( u = t \) by defining the product over an empty range to equal 1. It follows that, for any \( t \),

\[
\sum_{u=t}^{t} \delta^{u-t} \frac{1}{t-p} c_{u}^{t-p} = \frac{1}{t-p} w_t^{t-p} \sum_{u=t}^{t} \delta^{u-t} \left[ z_u \prod_{s=t}^{u} (1 - z_s) r_{s+1} \right]^{1-p}.
\]

Thus,

\[
J(X_t, w_t) = \max_{C_t} \left[ \sum_{u=t}^{t} \delta^{u-t} \frac{1}{t-p} c_{u}^{t-p} \right] = \max_{z_t} \left[ \frac{1}{t-p} w_t^{t-p} \sum_{u=t}^{t} \delta^{u-t} \left[ z_u \prod_{s=t}^{u} (1 - z_s) r_{s+1} \right]^{1-p} \right]
\]

where the optimization is over \( z_u \leq 1 \) for all \( u \) and over the portfolio process that affects the returns \( r_{s+1} \). The important fact is that

\[
f(X_t) \overset{\text{def}}{=} \max_{z_t} \left[ \frac{1}{t-p} \sum_{u=t}^{t} \delta^{u-t} \left[ z_u \prod_{s=t}^{u} (1 - z_s) r_{s+1} \right]^{1-p} \right]
\]

does not depend on \( w_t \), so the value function has the form claimed.
Chapter 10 presents the Intertemporal Capital Asset Pricing Model (ICAPM), which is a factor model for asset risk premia that uses market wealth and the state variables $x$ as the factors. The prices of risk for the state variables in the ICAPM are determined by how the state variables affect investors’ marginal values of wealth. Chapter 10 derives the ICAPM as an approximate relation in discrete time, and Chapter 14 derives it as an exact relation in continuous time. Chapter 14 links the prices of risk of the state variables to the hedging demands. Briefly, if investors desire to hold assets because they help to hedge changes in state variables, then this increased demand will produce higher prices and lower expected returns, so hedging demands are in inverse relation to prices of risk.

Here, we explain how state variables affect the marginal value of wealth for an investor with CRRA utility and no labor income. This example is useful for interpreting the prices of risk in the ICAPM. The results are different for the three cases $\rho < 1$, $\rho = 1$, and $\rho > 1$. Specifically, changes in any state variable cause the investor’s value and the investor’s marginal value of wealth to change in the same direction if $\rho < 1$ and to change in opposite directions if $\rho > 1$. Changes in state variables have no effect on the marginal value of wealth if $\rho = 1$ (log utility). The case $\rho > 1$ seems the most intuitive, as is discussed below.

The proof is very simple. We present it for the value function in an infinite-horizon model to economize on notation, but, as can easily be seen, the calculation is exactly the same for a finite horizon, both with and without consumption prior to the terminal date. First, consider $\rho = 1$. Then,

$$J(x, w) = y \log w + f(x) = \frac{\partial J(x, w)}{\partial w} = \frac{v}{w},$$

Thus, the marginal value of wealth is independent of all state variables. Now, consider $\rho \neq 1$. Then, $J(x, w) = f(x)w^{1-\rho}$, so for any $j = 1, \ldots, k$,

$$\frac{\partial J(x, w)}{\partial x_j} = w^{1-\rho}\frac{\partial f(x)}{\partial x_j}, \quad \frac{\partial^2 J(x, w)}{\partial x_i \partial x_j} = (1-\rho)w^{1-\rho}\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{v_{ij} + \rho v_j}{w},$$

Thus, the signs of the first partial derivative and the cross partial derivative are the same if $\rho < 1$ and are opposite if $\rho > 1$. 
To compare the different results for $\rho < 1$ and $\rho > 1$, consider a state variable $X_j$ having the property that increases in $X_j$ improve the investment opportunity set. Then, increases in $X_j$ will be beneficial for the investor, meaning that $\frac{\partial u(w, x)}{\partial x_j} > 0$. The result we have just derived shows that if $\rho < 1$, then increases in $X_j$ increase the marginal value of wealth, and if $\rho > 1$, then increases in $X_j$ lower the marginal value of wealth. Thus, for aggressive investors ($\rho < 1$), having an additional unit of wealth is prized more highly when investment opportunities are better. It seems plausible that additional wealth would be more valuable when investment opportunities are poorer rather than when they are better, because when they are poorer, it is more difficult to increase wealth through investments. This is the case if $\rho > 1$.

9.6 CRRA Utility with IID Returns
The simplest dynamic portfolio choice problems are with IID returns and LRT utility. In this section, we analyze power utility. Log utility and CARA utility are considered in the exercises, and portfolio choice with shifted log and shifted power utility is discussed in the end-of-chapter notes.

Assume the investor has no labor income, and assume that the return vectors $R_t, R_{t+1}, \ldots$ are IID. Thus, in particular, the conditional distribution of $R_{t+1}$ is independent of information at date $t$. The assumptions that returns are independent of conditioning information and there is no labor income imply that we do not need to keep track of conditioning information: The only state variable in the model is the investor’s wealth. Because there is no dependence on state variables, the description of the value function in Section 9.5 simplifies. The functions $f(x)$ do not depend on $X_t$; instead, they depend only on time. Likewise, $f(x)$ is just a constant when the horizon is infinite.

The main lesson from this section is that there is not much new in a dynamic portfolio choice model with IID returns compared to a single-period model. The optimal portfolio for CRRA utility is the same in a dynamic model as in a single-period model—it is the portfolio that achieves the maximum in (9.24) when $\rho \neq 1$ and the portfolio that maximizes the expected log return when $\rho = 1$. When there is consumption at each
date, optimal consumption is proportional to wealth. The constant of proportionality is time dependent when the horizon is finite and independent of time (a single number) when the horizon is infinite.

Assume there exists a finite number $B > 0$ satisfying

\[
\frac{1}{1-\rho} B^{1-\rho} = \max \left\{ \frac{1}{1-\rho} (\omega R_t)^{1-\rho} \right\}.
\]

(9.24)

Thus, $B$ is the certainty-equivalent end-of-period wealth for an investor with constant relative risk aversion $\rho$ in a single-period problem with initial wealth $W_0 = 1$ and consumption only at the end of the period. The existence of $B$ simply means that the maximum utility in this single-period problem is finite. In the infinite-horizon model, we will need the additional assumption

$$\delta B^{1-\rho} < 1,$$

(9.25)

where $\delta$ is the discount factor. This ensures that the maximum attainable lifetime utility is finite.

**Maximizing the Expected Utility of Terminal Wealth**

Suppose that the investor does not consume at dates $t = 0, \ldots, T - 1$ and seeks to maximize

$$E\left[ \frac{1}{1-\rho} W_T^{1-\rho} \right]$$

We will show that

$$V_t(w) = \frac{B^{1-\rho(t+1)}}{1-\rho} w^{1-\rho}$$

(9.26)

for each $t$. The argument is by induction. Clearly, (9.26) is true for $t = T$. Suppose it is true for $t + 1, \ldots, T$ for some $t$. Then, the Bellman equation gives us

$$V_t(w) = \max_{\pi} \mathbb{E}[V_{t+1}(\omega t R_{t+1})]$$

$$= \max_{\pi} \mathbb{E}\left[ \frac{1}{1-\rho} (\omega t R_{t+1})^{1-\rho} \right]$$

$$= B^{1-\rho(t+1)} w^{1-\rho} \max_{\pi} \mathbb{E}\left[ \frac{1}{1-\rho} (\omega R_{t+1})^{1-\rho} \right]$$

$$= B^{1-\rho(t+1)} w^{1-\rho} \cdot \frac{1}{1-\rho} B^{1-\rho}$$

$$= B^{1-\rho(t+1)} \cdot w^{1-\rho}.$$
Finite Horizon with Consumption at Each Date
Suppose now that the investor consumes at each date. Since we are assuming there is no bequest motive, the investor seeks to maximize

\[
\frac{1}{1-\rho} \mathbb{E}\left[ \sum_{t=0}^{T} \delta^{t} C_t^{-\rho} \right].
\]

Define \( A_t \) by

\[
\frac{1}{A_t} = \sum_{s=0}^{T-t} (\delta B^{t-s})^{-\rho}. \tag{9.27}
\]

(p.222) Note that the \( A_t \) increase with \( t \), culminating in \( A_T = 1 \).

We will show that the optimal consumption is

\[
C_t = A_t W_t, \tag{9.28}
\]

and the value function is

\[
J(x, w) = \frac{A_t^{-\rho}}{1-\rho} w^{1-\rho}. \tag{9.29}
\]

The proof is again by induction and again shows that the optimal portfolio is the one that maximizes the single-period objective function in (9.24).

First, note that (9.27) implies \( A_T = 1 \) and, for \( t < T \),

\[
\frac{1}{A_t} = 1 + \sum_{s=1}^{T-t} (\delta B^{t-s})^{-\rho} = 1 + (\delta B^{-\rho})^{1-\rho} \sum_{s=0}^{T-t} (\delta B^{-\rho})^{s} = 1 + \frac{(\delta B^{-\rho})^{1-\rho}}{A_{t+1}}. \tag{9.30}
\]

Thus,

\[
A_t = \frac{A_{t+1}}{A_{t+1} + (\delta B^{-\rho})^{1-\rho}}. \tag{9.31}
\]

Because the value function equals the utility at \( t = T \) and \( A_T = 1 \), we have \( J(x, w) = A_t w^{1-\rho}/(1-\rho) \) at \( t = T \). Suppose \( J_s(w) = A_s w^{1-\rho}/(1-\rho) \) for \( s = t+1, \ldots, T \) for some \( t \), where the \( A_s \) are defined in (9.27).

Define \( z = c/W \) and replace maximization over \( c \) in the Bellman equation with maximization over \( z \). We have
Because \( c_\tau \) is a positive constant, the maximum in \( \pi \) is achieved at the portfolio that maximizes the single-period objective function in (9.24), and (p.223) the maximum value of that objective is \( B^{1/p} / (1 - \rho) \) by definition. Therefore,

\[
J(\omega) = w^{1/p} \max \left\{ \frac{1}{1 - \rho} z^{1/p} + \delta A_{\omega} \right\}.
\]

(9.31)

The maximum here is achieved at

\[
z = \frac{y}{1 - \rho},
\]

where

\[
y = A_{\omega} (\delta B^{1/p})^{-1/p}.
\]

Note that

\[
\frac{y}{1 - \rho} = \frac{A_{\omega} (\delta B^{1/p})^{-1/p}}{1 - \rho y (\delta B^{1/p})^{-1/p} - A_{\omega}} = A_{\omega},
\]

using (9.30) for the last equality, so the optimal consumption is \( \omega_{\omega} \) as claimed in (9.28). Substitute the optimal \( z \) into (9.31) to obtain

\[
J(\omega) = w^{1/p} \left[ \left( \frac{y}{1 - \rho} \right)^{1/p} + y \left( \frac{1}{1 - \rho} \right)^{1/p} \right] = w^{1/p} \left( \frac{aw}{1 - \rho} \right)^{1/p}.
\]

Given that \( y / (1 + y) = A_{\omega} \), this verifies the formula (9.29) for the value function.

**Infinite Horizon**

Assume the horizon is infinite. Impose the no-borrowing constraint \( C_t \leq W_t \) to preclude Ponzi schemes. As remarked previously, this is a very natural constraint, because we are assuming there is no labor income to borrow against. Assume (9.25) holds. We will show that the optimal consumption is

\[
C_t = A W_t,
\]

(9.32)

where

\[
A = 1 - (\delta B^{1/p})^{1/p}.
\]

(9.33)

The value function is
(9.34)

\[ J(w) = \frac{A}{1 - \rho} w^{1 - \rho}. \]

Condition (9.25) is not very restrictive if \( \rho > 1 \). For example, it holds if \( \rho > 1 \) and there is a risk-free asset with return \( R_f > 1 \), because the maximum utility achievable in a single-period problem is at least as large as that achieved by investing everything in the risk-free asset. This implies \( B \geq R_f \), so \( B^{1 - \rho} \leq R_f^{1 - \rho} < 1 \). Thus, (9.25) is an issue primarily when \( \rho < 1 \). In that case, if \( \delta B^{1 - \rho} > 1 \), then (p.224) the maximum expected lifetime utility is infinite. This can be seen from the fact that the value \( J(w) \) is at least as large as the value of consuming nothing for \( n \) periods and then consuming all wealth, which is the value when maximizing the expected value of terminal wealth with \( n \) periods remaining and discounting by \( \delta \). Therefore, from (9.26),

\[ J(w) \geq \delta^{n} w^{1 - \rho} \rightarrow \infty \]

as \( n \rightarrow \infty \) when \( \delta B^{1 - \rho} > 1 \).

We write the Bellman equation here in terms of a generic function \( \hat{J} \) (not necessarily equal to the true value function \( J \)):

\[ \hat{J}(w) = \max_{c_{w}} \left\{ \frac{1}{1 - \rho} c^{1 - \rho} + \delta E[\hat{J}(w - c)(1 + R_f)] \right\}. \]

(9.35)

The value function \( J \) satisfies the Bellman equation (9.35), but there may in general be other functions \( \hat{J} \) that also satisfy (9.35). In a finite-horizon model, the value of \( J_f \) is given, and the Bellman equation can be used to compute \( J_f \) by backward induction. However, in an infinite-horizon model, this procedure is not possible, so we must use some other method to ensure that a solution of the Bellman equation is actually the value function. See the end-of-chapter notes for further discussion and also Exercise 9.4.

With CRRA utility, the value function is a constant times \( w^{1 - \rho} \), as shown in Section 9.5. Because the value function has the same sign as the utility function, it is of the form (9.34) for some constant \( A > 0 \). We show below that (9.33) is the unique constant \( A > 0 \) such that (9.34) satisfies the Bellman equation. Therefore, (9.34) with \( A \) defined by (9.33) is the value function.
In a finite-horizon model, the choices \((c, \eta)\) attaining the maximum in the Bellman equation, substituting the true value function \(j = j\) in the Bellman equation, are guaranteed to be optimal. This is also true in an infinite-horizon model if the transversality condition holds. The transversality condition is

\[
\lim_{t \to \infty} \delta^t E_t [W_t^*] = 0,
\]

(9.36)

where \(W^*\) denotes the wealth process produced by the choices \((c, \eta)\) attaining the maximum in the Bellman equation. This condition automatically holds if the utility function is bounded above, as is the case when \(\rho > 1\) (Exercise 9.5). It does not necessarily hold in general if the utility function is unbounded above (Exercise 9.4). We verify below that the transversality condition holds in this model and that the transversality condition and Bellman equation imply optimality.

As in the finite-horizon model, set \(z = c/w\). We can maximize over \(z \leq 1\) instead of over \(c \leq w\). We want to find \(A > 0\) satisfying

\[
A^\rho \left( z_{\frac{1}{1-\rho}} \right) = \max_{x} \left\{ \frac{1}{1-\rho} z^\rho - \delta A^\rho \left[ \frac{1}{1-\rho} (w(1-z)R_{\delta})^\rho \right] \right\} = w^\rho \max_{x} \left\{ \frac{1}{1-\rho} z^{1-\rho} + \delta A^\rho (1-z)^{1-\rho} \left[ \frac{1}{1-\rho} (\tau R_{\omega})^{1-\rho} \right] \right\},
\]

(9.37)

Because \(A^\rho (1-z)^{1-\rho}\) is positive, the maximum in \(\pi\) is achieved at the portfolio that maximizes the single-period objective function in (9.24), and the maximum value of that objective is \(B_{1-\rho}/(1-\rho)\) by definition. Making this substitution and canceling \(w^\rho\), we see that (9.37) is equivalent to

\[
\frac{1}{1-\rho} A^\rho = \max_{x} \left\{ \frac{1}{1-\rho} z^{1-\rho} + \frac{\delta A^\rho B_{1-\rho}}{1-\rho} (1-z)^{1-\rho} \right\}.
\]

The maximum here is achieved at

\[
z = \frac{\bar{y}}{1-\rho},
\]

where

\[
y = A(\delta B_{1-\rho})^{1-\rho}.
\]

Substituting the optimal \(z\), we see that (9.37) is equivalent to

\[
\frac{1}{1-\rho} A^\rho = \frac{1}{1-\rho} \left( \frac{\bar{y}}{1-\rho} \right)^{1-\rho} + y^{1-\rho} \left( \frac{\bar{y}}{1-\rho} \right)^{1-\rho} = \frac{1}{1-\rho} \left( \frac{\bar{y}}{1-\rho} \right)^{1-\rho}.
\]

Thus,
Let $\pi^*$ denote the portfolio that maximizes the single-period objective function in (9.24), and let $w^*$ denote the wealth process generated by this portfolio (p.226) and consumption $c_t^* = Aw_t^*$. We want to show that this portfolio and consumption are actually optimal. To this point, we have shown that they satisfy the Bellman equation, so

$$J(w_t^*) = u(c_t^*) + \delta E[J(w_{t+1})]$$

for all $t$. Start at $t = 0$ and substitute this recursively to obtain

$$J(w_0) = u(c_0^*) + \delta E[J(w_1)]$$

$$= u(c_0^*) + \delta E[u(c_0^*) + \delta E[J(w_1)]]$$

$$= E\left[\sum_{t=0}^{T-1} \delta^t u(c_t^*)\right] + \delta^T E[J(w_T^*)]$$

From the monotone convergence theorem (Appendix A.5), we have

$$E\left[\sum_{t=0}^{T-1} \delta^t u(c_t^*)\right] - E\left[\sum_{t=0}^{T-1} \delta^t u(c_t^*)\right]$$

as $T \to \infty$. Therefore,

$$J(w_0) = E\left[\sum_{t=0}^{T-1} \delta^t u(c_t^*)\right] + \lim_{T \to \infty} \delta^T E[J(w_T^*)].$$

(9.38)

Note that

$$w_T^* = w_0(1 - A)^T \Pi_{i=0}^{T-1} (\tau R_{i+1}).$$

Therefore,

$$\delta^T E[w_T^*] = \delta^T w_0^* (1 - A)^T \Pi_{i=0}^{T-1} (\tau R_{i+1})^T$$

$$= w_0^* (1 - A)^T \Pi_{i=0}^{T-1} (\delta B^T)^T.$$

Substitute

$$1 - A = (\delta B^T)^\frac{1}{\rho}$$

from (9.33) to obtain

$$\delta^T E[w_T^*] = w_0^* (\delta B^T)^\frac{1}{\rho},$$

which converges to zero as $T \to \infty$ by virtue of (9.25). The formula (9.34) for the value function therefore implies the transversality condition (9.36). From the transversality condition and (9.38), we conclude that $\pi^*$ and $c^*$ are optimal.
Notes and References

The portfolio choice method described for complete markets in Section 9.2 can also be applied, though less directly, when markets are incomplete or there are market frictions such as short sales constraints, margin requirements, or different borrowing and lending rates. Consider a finite-horizon finite-state model in which the market is incomplete. Add fictitious assets to complete the market. In the completed market, the investor can achieve expected utility at least as high as that achieved in the incomplete market, because not trading the new assets is always an option. If she can obtain higher expected utility in the completed market than in the incomplete market, then the consumption plan in the completed market must be infeasible in the incomplete market. The minimum expected utility attainable in any completed market is the expected utility attainable in the incomplete market, and the completed market solving the optimization problem “minimize the maximum attainable expected utility” is the market in which the new assets are priced by the SDF process \( M_t = \delta^t u(C_t) \), where \( C \) is the optimal consumption plan in the incomplete market. If we can solve the optimization problem to find what is called the least favorable fictitious completion, then we can find \( C \) and solve the incomplete market portfolio choice problem using the fictitious complete market as in Section 9.2. See He and Pearson (1991a).

The discussion of the optimal portfolio for a quadratic utility investor in Section 9.3 is based on Cochrane (2014). Mossin (1968) discusses maximizing the expected utility of terminal wealth with IID returns. He solves the problem with CRRA utility and shows that the optimal portfolio is the same in each period if and only if the investor has CRRA utility. Samuelson (1969) solves the finite-horizon problem with CRRA utility and consumption at each date. Hakansson (1970) solves the infinite-horizon problem with IID returns and CRRA utility (and also CARA utility).

Optimal portfolios for shifted log and shifted power utility can be deduced, in some circumstances, from the results for log and power utility. Consider shifted power utility with shift \( \zeta > 0 \) and with consumption only at the terminal date \( T \). Assume there is a zero-coupon bond maturing at \( T \). Recall that \( \zeta \) can be interpreted as a subsistence level of consumption. An
investor’s total consumption equals the subsistence level plus a surplus: \( c = \zeta + (c - \zeta) \). The investor can likewise separate her portfolio problem into two parts: She buys zero-coupon bonds maturing at \( t \) with a total face value of \( \zeta \) and she invests (p. 228) the remainder of her wealth in a portfolio to maximize power utility of the surplus consumption \( c - \zeta \). Thus, the optimal portfolio for an investor with shifted power utility involves a position in a zero-coupon bond, and the remaining wealth is invested in the portfolio that is optimal for an investor with power utility. The same is true for a shift \( \zeta < 0 \), if the utility function is regarded as defined for all \( c > \zeta \). In this case, the investor shorts zero-coupon bonds with a face value of \( \zeta \) and invests the proceeds from the short sale plus her wealth in the portfolio that is optimal for an investor with power utility. However, it is not very sensible to allow negative consumption, and it is more natural to restrict the domain of the utility function to \( c \geq 0 \) when \( \zeta < 0 \). When negative consumption is disallowed, the portfolio just described is infeasible. Back, Liu, and Teguia (2015) derive the solution to the problem when negative consumption is not allowed (in a continuous-time model).

The general properties of infinite-horizon stationary discounted dynamic programming problems are somewhat different for positive and negative utility functions, as mentioned earlier. If the utility function is bounded from below, we can add a constant and make it positive, or, if it is bounded from above, we can subtract a constant and make it negative, so the positive and negative cases include all utility functions that are bounded either from below or from above. Call these the positive (P) and negative (N) cases, respectively. If the utility function is bounded both from below and above, then it has the properties of both cases. Call this the bounded (B) case.

Here are important properties, with the cases in which they hold stated in parentheses. These properties are stated in terms of a portfolio choice problem, but they hold for general infinite-horizon stationary discounted dynamic programming problems (including those with state variables \( x_t \)). These results can be found in Hinderer (1970).
1. (P, N or B) The value function satisfies the Bellman equation.
2. (P, N, or B) Any optimal policy \((\alpha(\cdot), \pi(\cdot))\) must attain the maximum in the Bellman equation (employing the true value function in the Bellman equation) with probability 1.
3. (P) The value function is the smallest positive solution of the Bellman equation.
4. (B) The value function is the unique bounded solution of the Bellman equation.

**(p.229)** 5. (P or B) The value function can be computed by value iteration: Letting \(V^*_0\) denote the value function from a problem with intermediate consumption and horizon \(T\), we have \(V^*_0 \rightarrow V_0\) as \(T \rightarrow \infty\).
6. (N or B) If a policy \((\alpha(\cdot), \pi(\cdot))\) attains the maximum in the Bellman equation with probability 1, using the true value function \(J = J\), then the policy is optimal.

Exercise 9.4 is an example of the positive case in which there are multiple solutions of the Bellman equation. It is also an example in which attaining the maximum in the Bellman equation, using the true value function in the Bellman equation, is not a sufficient condition for optimality, due to a failure of the transversality condition. Exercise 9.5 asks for a proof of no. 6 in cases N and B. Log utility is unbounded above and below, so it fits none of the cases listed above. Exercise 9.1 asks for a proof of the transversality condition with log utility and IID returns.

In general, regardless of the boundedness of the utility function, the following is true: If \(J\) is any solution of the Bellman equation and if

\[
\lim_{T \rightarrow \infty} \mathbb{E} \left[ \sum_{t=1}^{T} \delta^t u(C_t) \right] = \mathbb{E} \left[ \sum_{t=1}^{\infty} \delta^t u(C_t) \right]
\]

(9.39a)

and

\[
\limsup_{T \rightarrow \infty} \delta^T \mathbb{E} \left[ \delta^T J(W_T) \right] \geq 0
\]

(9.39b)

for every sequence of decisions \((C_n, \pi_n)\), with \(W_t\) being the corresponding wealth process, and if

\[
\lim_{T \rightarrow \infty} \delta^T \mathbb{E} \left[ \delta^T J(W_T) \right] = 0,
\]

(9.39c)
where \( w^*_t \) is the wealth process corresponding to the decisions that attain the maximum in the Bellman equation relative to \( \hat{J} \), then (a) \( \hat{J} \) is the value function, and (b) the decisions attaining the maximum in the Bellman equation are optimal. See Exercise 9.6. This is also true of general infinite-horizon stationary discounted dynamic programming problems. This fact immediately implies no. 4 above, because (9.39) holds whenever \( u \) and \( \hat{J} \) are bounded. The \( \lim \sup \) equals the largest limit point of any subsequence, so (9.39b) holds if the limit is zero. Note that (9.39c) is the transversality condition (9.36), but relative to \( \hat{J} \).

(p.230) **Exercises**

**9.1.** Consider the infinite-horizon model with IID returns and no labor income. Assume

\[
\max_{\alpha} \mathbb{E}[\log R_{t+1}] < \infty.
\]

(a) Calculate the unique constant \( \gamma \) such that

\[
J(w) = \log \frac{w}{\gamma} + \gamma
\]

does the Bellman equation.

(b) Show that the transversality condition

\[
\lim_{t \to \infty} \delta^t \mathbb{E}[J(w_t^*)] = 0
\]

holds.

(c) Show that the optimal portfolio is the one that maximizes \( \mathbb{E}[\log R_{t+1}] \) and the optimal consumption is

\[
C_t = (1 - \delta)W_t.
\]

**9.2.** Consider the finite-horizon model with consumption at each date, state variables \( X_t \), log utility, and no labor income. Assume \( \max_{\alpha} \mathbb{E}[\log R_{t+1}] \) is finite for each \( t \) with probability 1. The value function at date \( T \) is

\[
J_T(x, w) = \log w.
\]

Define

\[
\gamma_t = \frac{1}{1 - \delta^{T-t}}.
\]

(a) Show that

\[
J_t(x, w) = \gamma_t \log w + f_t(x)
\]

for some functions \( f_t \).
(b) Show that the optimal portfolio at each date $t$ is the one that maximizes $E[\log R_{t+1}]$ and that the optimal consumption at each date $t$ is $C_t = W_t / Y_t$. Note: Similar results are true for power utility only when returns are IID. This exercise does not assume IID returns.

9.3. Consider the finite-horizon model with consumption at each date, IID returns, and no labor income. Suppose one of the assets is risk free with return $R_t$. Let $\mathbf{R}$ denote the vector of risky asset returns, let $\mu$ denote the expected value of $\mathbf{R}$, and let $\Sigma$ denote the covariance matrix of $\mathbf{R}$. Assume $\Sigma$ is nonsingular.

(p.231) (a) For constants $\alpha$, $\delta$, $\kappa_0$, and $Y_0$, define

$$J_0(w) = -\kappa_0 e^{\alpha w}$$

and

$$J(w) = \max_{\epsilon \in \mathbb{R}} \left( -e^{\alpha w} + \delta E[J_0((w - \epsilon)R_t + \phi[R - R_t])] \right).$$

Show that

$$J_0(w) = -\kappa_0 e^{\alpha w}$$

for constants $\kappa_1$ and $Y_1$.

(b) Using dynamic programming, deduce from the result of Part (a) that the value function $J_t$ of an investor with CARA utility and horizon $T < \infty$ has constant absolute risk aversion. How does the risk aversion depend on the remaining time $T - t$ until the horizon?

(c) If the investor has an infinite horizon, then the value function is independent of $t$. A good guess would therefore be $J(w) = -\kappa_0 e^{\alpha w}$ where $\kappa_0$ and $Y_0$ are such that $\kappa_1 = \kappa_0$ and $Y_1 = Y_0$ in Part (a); that is, $(\kappa_0, Y_0)$ is a fixed point of the map $(\kappa_0, Y_0) \mapsto (\kappa_1, Y_1)$ calculated in Part (a). Show that this implies

$$Y_0 = \frac{R_t - 1}{R_t} \alpha.$$

9.4. Suppose there is a single asset that is risk free with return $R_f > 1$. Consider an investor with an infinite horizon, utility function $u(c) = c$, and discount factor $\delta = 1/R_f$. Suppose she is constrained to consume $0 \leq C_t \leq W_t$. 
(a) Show that the value function for this problem is \( J(w) = w \).

(b) Show that the value function solves the Bellman equation.

(c) Show that \( \hat{J}(w) = 2w \) also solves the Bellman equation.

(d) Show that, using the true value function \( \hat{J}(w) = w \) in the Bellman equation, the suboptimal policy \( C_t = 0 \) for every \( t \) achieves the maximum for every value of \( w \).

9.5. Consider the infinite-horizon model with IID returns and no labor income. Denote the investor’s utility function by \( u(c) \).

(a) Case B: Assume there is a constant \( K \) such that \(-K \leq u(c) \leq K\) for each \( c \). Show that the transversality condition (9.36) holds.

(b) Case N: Assume \( u(c) \leq 0 \) for each \( c \) and \( J(w) > -\infty \) for each \( w \). Show that the transversality condition (9.36) holds. Hint: Use (9.38) and the definition of a value function to deduce that the limit in (9.36) is nonnegative.

9.6. Consider the infinite-horizon model with IID returns and no labor income. Denote the investor’s utility function by \( u(c) \).

Let \( \hat{J} \) be a function that solves the Bellman equation. Assume (9.3) holds. For arbitrary decisions \((C_t, \pi_t)\), assume \( E[u(C_t)] \) and \( E[\hat{J}(W_t)] \) are finite for each \( t \). Suppose \((C^*_t, \pi^*_t)\) attain the maximum in the Bellman equation. Show that \( \hat{J} \) is the value function and \((C^*_t, \pi^*_t)\) are optimal.

Notes:

(1.) Take \( Y_0 = 0 \) to be consistent with our usual convention that income at date 0 is already included in \( W_0 \).

(2.) A simple example of a Markov process is an AR(1) process:

\[ X_{t+1} = \sigma + AX_t + \varepsilon_{t+1}, \]

for a vector \( \sigma \), a square matrix \( A \), and a sequence of IID random vectors \( \varepsilon_t \).

(3.) In practice, we may want to solve for the consumption and portfolio simultaneously, or we may want to follow the
opposite procedure, solving for the portfolio first. Examples are given in Section 9.6.

(4.) An exception is when the investor has log utility (Exercise 9.2).

(5.) Likewise, CARA utility implies a CARA value function (Exercise 9.3).

(6.) The optimal portfolio is constant over time, meaning that the fraction of invested wealth that is invested in each asset remains constant over time. However, the investor will usually trade all assets each period. She trades because her wealth changes—buying assets when wealth rises and selling them to finance consumption when wealth falls. Furthermore, she trades to rebalance her portfolio weights to the optimum. For example, if the price of an asset held by the investor rises relative to others, then the investor’s portfolio weight on that asset will rise, and she will need to sell the asset to return the weight to the optimum.

(7.) Note that this is not the same as the transversality condition discussed in Sections 8.5 and 9.1. The transversality condition discussed in Chapter 8 states that the contribution of consumption at dates \( T, T+1, \ldots \) to the date-0 budget goes to zero as \( T \to \infty \). Condition (9.36) states that the contribution of consumption at dates \( T, T+1, \ldots \) to the date-0 expected lifetime utility goes to zero as \( T \to \infty \).

(8.) If there is consumption at each date, the investor buys zero-coupon bonds with face value of \( \zeta \) for each maturity date \( t = 1, \ldots, T \).

(9.) Recall that this is the LRT/DARA utility function that has increasing relative risk aversion. If \( \rho > 1 \), it is also bounded on the domain \([0, \infty)\) (Exercise 1.10).