Appendix Returns

1. Returns
There are several definitions of returns. This appendix shows how they are related to each other.

1.1. Gross Returns

A *gross return* over one period, \( t \) to \( t+1 \), is defined as

\[
\text{Gross return } R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t},
\]

where \( P_t \) is the price of the asset at the beginning of the period, \( P_{t+1} \) is the price at the end of the period, and \( D_{t+1} \) is any dividend paid at time \( t+1 \). I denote gross returns with an upper case \( R \). The definition in equation (A.1) assumes that we can measure prices—which is not true for some illiquid investments. For now, we assume assets are traded; I discuss illiquid asset returns in chapter 13.

As wealth cumulates over time, we compound gross returns multiplicatively. An asset earning gross returns of \( R_{t+1} = 1.10 \) three years in a row produces cumulated wealth at the end of three years of

\[
W_{t+3} = R_{t+1} R_{t+2} R_{t+3} = 1.10 \times 1.10 \times 1.10 = 1.3310,
\]

starting with $1 at time \( t \).
1.2. Arithmetic Returns

We can express a gross return as an arithmetic return, which I denote through the book as a lower case $r_{t+1}$:

$$\text{Arithmetic return} = R_{t+1} - 1.$$  

This is the most popular way of expressing rates of return. Arithmetic returns are also called “simple returns” or “net returns.” Often we just drop any modifier and say just “returns” (as I do throughout the book).

Arithmetic returns are not additive over time. Assume the gross return $R_{t+1} = 1.10$. If an asset earns arithmetic returns of $r_{t+1} = (1.10 - 1) = 10\%$ for three periods, then we earn

$$W_{t+3} = (1 + r_{t+1})(1 + r_{t+2})(1 + r_{t+3}) = 1.10 \times 1.10 \times 1.10 = 1.3310$$

at the end of three periods. We cannot add arithmetic returns. That is $1.3310 \neq 1 + 3 \times 0.10$, because of the compounding of wealth. Even though arithmetic returns are not additive, in some of the figures in this book, I simply cumulate arithmetic returns—to remove the effects of compounding for aesthetics.

Arithmetic returns are, however, additive across assets. Suppose that asset A has a return of 10\%, $r^A_{t+1} = 0.10$ and asset B has a return of 5\%, $r^B_{t+1} = 0.05$. Then, if we hold a portfolio of 50\% asset A and 50\% asset B, the arithmetic portfolio return is

$$r^P_{t+1} = 0.5 r^A_{t+1} + 0.5 r^B_{t+1} = 0.5 \times 0.10 + 0.5 \times 0.05 = 0.075$$

so arithmetic returns conveniently aggregate in a portfolio.

1.3. Log Returns

Log returns, or continuously compounded returns, are defined as the natural logarithm of gross returns and I denote them with an upper bar:

$$\text{Log return} = \ln (R_{t+1}) = \ln (1 + r_{t+1}).$$

Or we can equivalently state everything in terms of gross returns:

$$R_{t+1} = \exp (r_{t+1}) = 1 + r_{t+1}.$$  

(p.627) Log returns are also called growth rates, and growth occurs exponentially. That is, growth will eventually cause a
portfolio to increase to infinity or decrease to zero in the long run.

The log return corresponding to an arithmetic return of 10% is 
\[ \tilde{r}_{t+1} = \ln (1.10) = 9.53\%. \] The arithmetic return of \( r_{t+1} = 10\% \) and the log return of \( \tilde{r}_{t+1} = 9.53\% \) are both equivalent because they represent the same end-of-period wealth of $1.10, or gross return of \( R_{t+1} = 1.10 \), for every $1 invested at the beginning of the period. With the arithmetic return, the interest is returned only at the end of the period. In contrast, the log return assumes that interest is earned at every instant (hence returns are said to be "continuously compounded"). Since interest compounds on the interest at every moment in time, the continuously compounded interest rate does not need to be as high—it is only 9.53%—to reach the same end of period wealth of 1.10 at the end of the period.

Log returns aggregate over time. If we have an asset earning gross returns of \( R_{t+1} = 1.10 \) for three years, we can sum three years of log returns of \( \tilde{r}_{t+1} = 9.53\% \) to obtain the cumulated wealth at the end of three years:

\[
W_{t+3} = \exp (r_{t+1}) \exp (r_{t+2}) \exp (r_{t+3}) \\
= \exp (r_{t+1} + r_{t+2} + r_{t+3}) \\
= \exp (3 \times 0.0953) = 1.3310
\]

Log returns do not aggregate across assets. Suppose we take asset A with a gross return of \( R_{t+1}^A = 1.10 \) and asset B with a gross return of \( R_{t+1}^B = 1.05 \). These assets have log returns of 
\[ \tilde{r}_{t+1}^A = \ln (1.10) = 9.53\% \] and 
\[ \tilde{r}_{t+1}^B = \ln (1.05) = 4.88\% \], respectively. If we hold a portfolio of 50% asset A and 50% asset B, then the log return of this portfolio is 

\[
r_{t+1}^P = \ln (0.5 \exp (\tilde{r}_{t+1}^A) + 0.5 \exp (\tilde{r}_{t+1}^B)) \\
\geq \ln (0.5 \exp (0.0953) + 0.5 \exp (0.0448)) = \ln (1.075) = 7.23\% \\

\geq 0.5 \times \tilde{r}_{t+1}^A + 0.5 \times \tilde{r}_{t+1}^B
\]

That is, the log return of the portfolio is not the weighted average of each stock’s log return.

1.4. Arithmetic versus Log Returns

Many practitioners and academics have long argued about whether it is better to use arithmetic or log returns.\(^1\) Whether you use arithmetic or log returns is like (p. 628) arguing whether you should use pounds or kilograms in your recipe to make apple pie. They represent the same thing; it is a question...
of which is more useful. What is crucial is the recipe itself—you need the right model for the returns; otherwise it doesn’t matter whether you measure your apples in pounds or kilos.

In analyzing portfolios (e.g., as we do in chapter 7), it is easiest to use arithmetic returns because they aggregate when we form portfolios. When we analyze asset returns across time (e.g., as we do in chapters 8 and 11, which focus on long-run equity return predictability and how different assets hedge inflation in the long run, respectively), we use log returns because we can sum them across time. In both instances, we describe the same thing—returns.

This is not to say that whether you use pounds or kilos does not matter. As every dieter knows, you can’t switch between calories or kilojoules without the risk of severely wrecking your waistline. Consider the following return process for a gross return:

\[ R_{t+1} = 1.50 \]

\[ R_{t+1} = 0.50 \]

which is a 50%-50% sequence of winning or losing plus or minus 50% arithmetic returns and the probabilities are independent over time. The investor will eventually lose all her capital investing in this asset. (When you gain 50% and lose 50% you end up losing 25%!) The mean arithmetic return is zero, \( E[r_{t+1}] = \frac{1}{2} \times 0.5 + \frac{1}{2} \times -0.5 = 0 \) and may give a misleading picture of the long-run return of cumulated wealth. In the long run, your wealth shrinks to zero! The mean log return is

\[ E[r_{t+1}] = 0.5 \times \ln(1.5) + 0.5 \times \ln(0.5) = -14.38\% \]

so the negative sign at least indicates that this strategy loses wealth over time. The mean log return is also called the geometric mean. (When returns are risky, the arithmetic mean is always greater than the geometric mean.) But the better way is to look at the full distribution of returns, which is captured by the probability distribution function or density function (see chapter 2). The density function of this lottery is very simple—it provides you with risk (there is dispersion or variance) but zero return. You must have a good reason to
invest in the first place, (p.629) right? From this point of view, the mean and variance of arithmetic returns are useful statistical measures.

Here’s another example where using the arithmetic mean gets it right and using geometric means is wrong. Suppose that asset A and asset B have average gross returns of $E[R^A_t] = 1.5$ and $E[R^B_t] = 0.5$, respectively. An investor holding 50% of asset A and 50% of asset B breaks even, on average, with the arithmetic mean being zero. Taking the average of asset A and B’s geometric returns would actually result in a number less than zero. Log returns do not aggregate across portfolios. In both examples, however, we are better off looking at the full distribution of returns. (p.630)

Notes:

(1) The debate continues today. Advocates of the arithmetic mean typically like the fact that it is unbiased. Those who prefer log returns note that they measure the actual change in wealth, and they automatically assume that capital is reinvested. For some references see Cooper (1996), Jacquier, Kane, and Marcus (2003), and Missiakoulis, Dimitrios, and Etriotis (2010).

(2) An approximate relation is $E[r_{t+1}] = E[r_{t+1}] - \frac{1}{2} \sigma^2$, where $\sigma^2$ is the variance of arithmetic returns. This difference is a Jensen’s inequality term, and it drives the rebalancing premium. See chapter 4.