Approximation of Conformal Mappings Using Conformally Equivalent Triangular Lattices

Ulrike Bücking

Abstract Two triangle meshes are conformally equivalent if their edge lengths are related by scale factors associated to the vertices. Such a pair can be considered as preimage and image of a discrete conformal map. In this article we study the approximation of a given smooth conformal map f by such discrete conformal maps f^ε defined on triangular lattices. In particular, let T be an infinite triangulation of the plane with congruent strictly acute triangles. We scale this triangular lattice by $\varepsilon > 0$ and approximate a compact subset of the domain of f with a portion of it. For ε small enough we prove that there exists a conformally equivalent triangle mesh whose scale factors are given by $\log |f'|$ on the boundary. Furthermore we show that the corresponding discrete conformal (piecewise linear) maps f^ε converge to f uniformly in C^1 with error of order ε.

1 Introduction

Holomorphic functions build the basis and heart of the rich theory of complex analysis. Holomorphic functions with nowhere vanishing derivative, also called *conformal maps*, have the property to preserve angles. Thus they may be characterized by the fact that they are infinitesimal scale-rotations.

In the discrete theory, the idea of characterizing conformal maps as local scale-rotations may be translated into different concepts. Here we consider the discretization coming from a metric viewpoint: Infinitesimally, lengths are scaled by a factor, i.e. by $|f'(z)|$ for a conformal function f on $D \subset \mathbb{C}$. More generally, on a smooth manifold two Riemannian metrics g and $\tilde{g}$ are conformally equivalent if $\tilde{g} = \mathrm{e}^{2u} g$ for some smooth function u.

U. Bücking (✉)
Inst. für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136,
10623 Berlin, Germany
e-mail: buecking@math.tu-berlin.de

A.I. Bobenko (ed.), *Advances in Discrete Differential Geometry*,
DOI 10.1007/978-3-662-50447-5_3

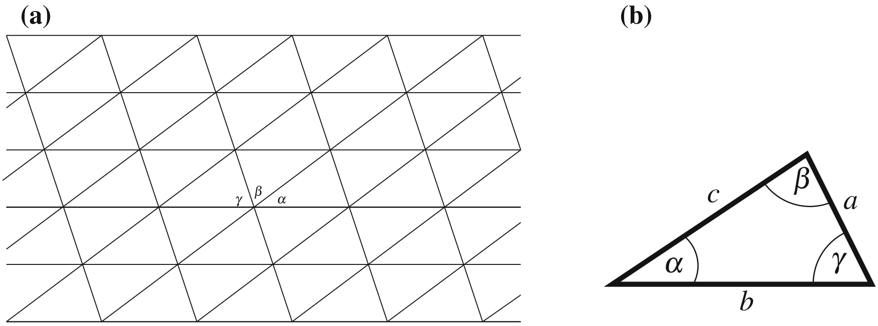

Fig. 1 Lattice triangulation of the plane with congruent triangles. **a** Example of a triangular lattice. **b** Acute angled triangle.

The smooth complex domain (or manifold) is replaced in this discrete setting by a triangulation of a connected subset of the plane $\mathbb{C}$ (or a triangulated piecewise Euclidean manifold).

1.1 Convergence for Discrete Conformal PL-Maps on Triangular Lattices

In this article we focus on the case where the triangulation is a (part of a) *triangular lattice*. In particular, let T be a lattice triangulation of the whole complex plane $\mathbb{C}$ with congruent triangles, see Fig. 1a. The sets of vertices and edges of T are denoted by V and E respectively. Edges will often be written as $e = [v_i, v_j] \in E$, where $v_i, v_j \in V$ are its incident vertices. For triangular faces we use the notation $\Delta[v_i, v_j, v_k]$ enumerating the incident vertices with respect to the orientation (counterclockwise) of $\mathbb{C}$.

On a subcomplex of T we now define a discrete conformal mapping. The main idea is to change the lengths of the edges of the triangulation according to scale factors at the vertices. The new triangles are then 'glued together' to result in a piecewise linear map, see Fig. 2 for an illustration. More precisely, we have

Definition 1.1 A *discrete conformal PL-mapping* g is a continuous and orientation preserving map of a subcomplex T_S of a triangular lattice T to $\mathbb{C}$ which is locally a homeomorphism in a neighborhood of each interior point and whose restriction to every triangle is a linear map onto the corresponding image triangle, that is the mapping is piecewise linear. Furthermore, there exists a function $u : V_S \to \mathbb{R}$ on the vertices, called *associated scale factors*, such that for all edges $e = [v, w] \in E_S$ there holds

$$|g(v) - g(w)| = |v - w|\mathrm{e}^{(u(v)+u(w))/2}, \tag{1}$$

where $|a|$ denotes the modulus of $a \in \mathbb{C}$.

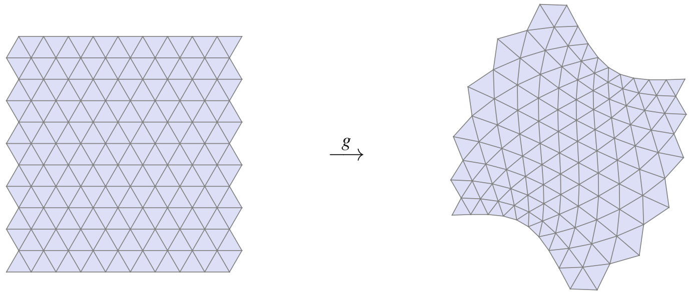

Fig. 2 Example of a discrete conformal PL-map g

Note that Eq. (1) expresses a linear relation for the logarithmic edge lengths, that is

$$2 \log |g(v) - g(w)| = 2 \log |v - w| + u(v) + u(w).$$

In fact, the definition of a discrete conformal PL-map relies on the notion of discrete conformal triangle meshes. These have been studied by Luo, Gu, Sun, Wu, Guo [8, 9, 14], Bobenko, Pinkall, and Springborn [1] and others.

As possible application, discrete conformal PL-maps can be used for discrete uniformization. The simplest case is a discrete Riemann mapping theorem, i.e. the problem of finding a discrete conformal mapping of a simply connected domain onto the unit disc. Similarly, we may consider a related Dirichlet problem. Given some function u_∂ on the boundary of a subcomplex T_S, find a discrete conformal PL-map whose associated scale factors agree on the boundary with u_∂. For such a Dirichlet problem (with assumptions on u_∂ and T_S) we will prove existence as part of our convergence theorem.

In this article we present a first answer to the following problem: *Given a smooth conformal map, find a sequence of discrete conformal PL-maps which approximate the given map.* We study this problem on triangular lattices T with acute angles and always assume for simplicity that the origin is a vertex. Denote by εT the lattice T scaled by $\varepsilon > 0$. Using the values of $\log |f'|$, we obtain a discrete conformal PL-map f^ε on a subcomplex of εT from a boundary value problem for the associated scale factors. More precisely, we prove the following approximation result.

Theorem 1.2 *Let $f : D \to \mathbb{C}$ be a conformal map (i.e. holomorphic with $f' \neq 0$). Let $K \subset D$ be a compact set which is the closure of its simply connected interior $int(K)$ and assume that $0 \in int(K)$. Let T be a triangular lattice with strictly acute angles. For each $\varepsilon > 0$ let T_K^ε be a subcomplex of εT whose support is contained in K and is homeomorphic to a closed disc. We further assume that 0 is an interior vertex of T_K^ε. Let $e_0 = [0, \hat{v}_0] \in E_K^\varepsilon$ be one of its incident edges.*

Then if $\varepsilon > 0$ is small enough (depending on K, f, and T) there exists a unique discrete conformal PL-map f^ε on T_K^ε which satisfies the following two conditions:

- *The associated scale factors $u^\varepsilon : V_K^\varepsilon \to \mathbb{R}$ satisfy*

$$u^\varepsilon(v) = \log|f'(v)| \quad \textit{for all boundary vertices } v \textit{ of } V_K^\varepsilon. \tag{2}$$

- *The discrete conformal PL-map is normalized according to $f^\varepsilon(0) = f(0)$ and $\arg(f^\varepsilon(\hat{v}_0) - f^\varepsilon(0)) = \arg(\hat{v}_0) + \arg(f'(\frac{\hat{v}_0}{2})) \pmod{2\pi}$.*

Furthermore, the following estimates for u^ε and f^ε hold for all vertices $v \in V_K^\varepsilon$ and points x in the support of T_K^ε respectively with constants C_1, C_2, C_3 depending only on K, f, and T, but not on v or x:

(i) The scale factors u^ε approximate $\log|f'|$ uniformly with error of order ε^2:

$$\left|u^\varepsilon(v) - \log|f'(v)|\right| \leqslant C_1\varepsilon^2. \tag{3}$$

(ii) The discrete conformal PL-mappings f^ε converge to f for $\varepsilon \to 0$ uniformly with error of order ε:

$$|f^\varepsilon(x) - f(x)| \leqslant C_2\varepsilon.$$

(iii) The derivatives of f^ε (in the interior of the triangles) converge to f' uniformly for $\varepsilon \to 0$ with error of order ε:

$$\left|\partial_z f^\varepsilon(x) - f'(x)\right| \leqslant C_3\varepsilon \quad \textit{and} \quad |\partial_{\bar{z}} f^\varepsilon(x)| \leqslant C_3\varepsilon$$

for all points x in the interior of a triangle Δ of T_K^ε. Here ∂_z and $\partial_{\bar{z}}$ denote the Wirtinger derivatives applied to the linear maps $f^\varepsilon|_\Delta$.

Note that the subcomplexes T_K^ε may be chosen such that they approximate the compact set K. Further notice that (3) implies that u^ε converges to $\log|f'|$ in C^1 with error of order ε, in the sense that also

$$\left|\frac{u^\varepsilon(v) - u^\varepsilon(w)}{\varepsilon} - \operatorname{Re}\left(\frac{f''((v+w)/2)}{f'((v+w)/2)}\right)\right| \leqslant \tilde{C}\varepsilon$$

on edges $[v, w]$ uniformly for some constant $\tilde{C}$.

The proof of Theorem 1.2 is given in Sect. 4. The arguments are based on estimates derived in Sect. 3.

The problem of actually computing the scale factors u for given boundary values u_∂ such that u gives rise to a discrete conformal PL-map (in case it exists) can be solved using a variational principle, see [1, 20]. Our proof relies on investigations using the corresponding convex functional, see Theorem 2.2 in Sect. 2.

Remark 1.3 The convergence result of Theorem 1.2 also remains true if linear interpolation is replaced with the piecewise projective interpolation schemes described in [1, 3], i.e., circumcircle preserving, angle bisector preserving and, generally, exponent-t-center preserving for all $t \in \mathbb{R}$. The proof is the same with only small adaptations. This is due to the fact that the image of the vertices is the same for all these interpolation schemes and these image points converge uniformly to the corresponding image points under f with error of order ε. The estimates for the derivatives similarly follow from Theorem 1.2(i).

1.2 Other Convergence Results for Discrete Conformal Maps

Smooth conformal maps can be characterized in various ways. This leads to different notions of discrete conformality. Convergence issues have already been studied for some of these discrete analogs. We only give a very short overview and cite some results of a growing literature.

In particular, linear definitions can be derived as discrete versions of the Cauchy-Riemann equations and have a long and still developing history. Connections of such discrete mappings to smooth conformal functions have been studied for example in [2, 6, 7, 13, 16, 19, 22].

The idea of characterizing conformal maps as local scale-rotations has lead to the consideration of circle packings, more precisely to investigations on circle packings with the same (given) combinatorics of the tangency graph. Thurston [21] first conjectured the convergence of circle packings to the Riemann map, which was then proven by [10, 11, 17].

The theory of circle patterns generalizes the case of circle packings. Also, there is a link to integrable structures via isoradial circle patterns. The approximation of conformal maps using circle patterns has been studied in [4, 5, 12, 15, 18].

The approach taken in this article constructs discrete conformal maps from given boundary values. Our approximation results and some ideas of the proof are therefore similar to those in [4, 5, 18] for circle patterns which also rely on boundary value problems.

2 Some Characterizations of Associated Scale Factors of Discrete Conformal PL-Maps

Consider a subcomplex T_S of a triangular lattice T and an arbitrary function $u : V_S \to \mathbb{R}$. Assign new lengths to the edges according to (1) by

$$\tilde{l}([v, w]) = |v - w| e^{(u(v)+u(w))/2} \tag{4}$$

In order to obtain new triangles with these lengths (and ultimately a discrete conformal PL-map) the triangle inequalities need to hold for the edge lengths $\tilde{l}$ on each triangle. If we assume this, we can embed the new triangles (respecting orientation) and immerse sequences of triangles with edge lengths given by $\tilde{l}$ as in (4). In order to obtain a discrete conformal PL-map, in particular a local homeomorphism, the interior angles of the triangles need to sum up to 2π at each interior vertex. The angle at a vertex of a triangle with given side lengths can be calculated. With the notation of Fig. 1b we have the half-angle formula

$$\tan\left(\frac{\alpha}{2}\right) = \sqrt{\frac{(-b+a+c)(-c+a+b)}{(b+c-a)(a+b+c)}} = \sqrt{\frac{1-(\frac{b}{a}-\frac{c}{a})^2}{(\frac{b}{a}+\frac{c}{a})^2-1}}. \tag{5}$$

The last expression emphasizes the fact that the angle does not depend on the scaling of the triangle. Careful considerations of this angle function depending on (scaled) side lengths of the triangle form the basis for our proof. In particular, we define the function

$$\theta(x,y) := 2\arctan\sqrt{\frac{1-(\mathrm{e}^{-x/2}-\mathrm{e}^{-y/2})^2}{(\mathrm{e}^{-x/2}+\mathrm{e}^{-y/2})^2-1}}, \tag{6}$$

so (5) can be written as

$$\alpha = \theta(x,y) \qquad \text{with} \quad \frac{b}{a} = \mathrm{e}^{-x/2} \ \text{ and } \ \frac{c}{a} = \mathrm{e}^{-y/2}.$$

Summing up, we have the following characterization of scale factors associated to discrete conformal PL-maps.

Proposition 2.1 *Let T_S be a subcomplex of a triangular lattice T and $u : V_S \to \mathbb{R}$ a function satisfying the following two conditions.*

(i) *For every triangle $\Delta[v_1, v_2, v_3]$ of T_S the* triangle inequalities *for $\tilde{l}$ defined by* (4) *hold, in particular*

$$|v_i - v_j| e^{(u(v_i)+u(v_j))/2} < |v_i - v_k| e^{(u(v_i)+u(v_k))/2} + |v_j - v_k| e^{(u(v_j)+u(v_k))/2} \tag{7}$$

for all permutations (ijk) of (123).

(ii) *For every interior vertex v_0 with neighbors $v_1, v_2, \ldots, v_k, v_{k+1} = v_1$ in cyclic order we have*

$$\sum_{j=1}^{k} \theta(\lambda(v_0, v_j, v_{j+1}) + u(v_{j+1}) - u(v_0), \lambda(v_0, v_{j+1}, v_j) + u(v_j) - u(v_0)) = 2\pi, \tag{8}$$

where $\lambda(v_a, v_b, v_c) = 2\log(|v_b - v_c|/|v_a - v_b|)$ for a triangle $\Delta[v_a, v_b, v_c]$.

Then there is a discrete conformal PL-map (unique up to post-composition with Euclidean motions) such that its associated scale factors are the given function $u : V_S \to \mathbb{R}$.

Conversely, given a discrete conformal PL-map on a subcomplex T_S *of a triangular lattice* T*, its associated scale factors* $u : V_S \to \mathbb{R}$ *satisfy conditions (i) and (ii).*

In order to obtain discrete conformal PL-maps from a given smooth conformal map we will consider a Dirichlet problem for the associated scale factors. Therefore we will apply a theorem from [1] which characterizes the scale factors u for given boundary values using a variational principle for a functional E defined in [1, Sect. 4]. Note that we will not need the exact expression for E but only the formula for its partial derivatives. In fact, the vanishing of these derivatives is equivalent to the necessary condition (8) for the scale factors to correspond to a discrete conformal PL-map.

Theorem 2.2 *([1]) Let* T_S *be a subcomplex of a triangular lattice and let* $u_\partial : V_\partial \to \mathbb{R}$ *be a function on the boundary vertices* V_∂ *of* T_S*. Then the solution* $\tilde{u}$ *(if it exists) of Eq. (8) at all interior vertices with* $\tilde{u}|_{V_\partial} = u_\partial$ *is the unique argmin of a locally strictly convex functional* $E(u) = E_{T_S}(u)$ *which is defined for functions* $u : V \to \mathbb{R}$ *satisfying the inequalities (7).*

The partial derivative of E *with respect to* $u_i = u(v_i)$ *at an interior vertex* $v_i \in V_{int}$ *with* k *neighbors* $v_{i_1}, v_{i_2}, \ldots, v_{i_k} v_{i_{k+1}} = v_{i_1}$ *in cyclic order is*

$$\frac{\partial E}{\partial u_i}(u) = 2\pi - \sum_{j=1}^{k} \theta(2\log\left(\frac{l_{i_{j+1},i_j}}{l_{i,i_{j+1}}}\right) + u_{i_j} - u_i, 2\log\left(\frac{l_{i_{j+1},i_j}}{l_{i,i_j}}\right) + u_{i_{j+1}} - u_i), \tag{9}$$

where $l_{j,k} = |v_j - v_k|$.

By Proposition 2.1 such a solution $\tilde{u}$ *are then scale factors associated to a discrete conformal PL-map.*

Remark 2.3 The functional E can be extended to a convex continuously differentiable function on $\mathbb{R}^V$, see [1] for details.

3 Taylor Expansions

We now examine the effect when we take $u = \log|f'|$ as 'scale factors', i.e. for each triangle we multiply the length $|v - w|$ of an edge $[v, w]$ by the geometric mean $\sqrt{|f'(v)f'(w)|}$ of $|f'|$ at the vertices. The proof of Theorem 1.2 is based on the idea that $u = \log|f'|$ almost satisfies the conditions for being the associated scale factors of an discrete conformal PL-map, that is conditions (i) and (ii) of Proposition 2.1, and therefore is close to the exact solution u^ε.

To be precise, suppose that εT is the equilateral triangulation of the plane. Assume without loss of generality that the edge lengths equal $\frac{\sqrt{3}}{2}\varepsilon > 0$ and edges are parallel to $e^{ij\pi/3}$ for $j = 0, 1, \ldots, 5$. Let the conformal function f, the compact set K, and the subcomplexes T_K^ε (with vertices V_K^ε and edges E_K^ε) be given as in Theorem 1.2. Let $v_0 \in V_{K,\text{int}}^\varepsilon$ be an interior vertex. Here and below $V_{K,\text{int}}^\varepsilon$ denotes the set of interior vertices having six neighbors in V_K^ε. Denote the neighbors of v_0 by $v_j = v_0 + \varepsilon\frac{\sqrt{3}e^{ij\frac{\pi}{3}}}{2}$ and consider the triangle $\Delta_j = \Delta[v_0, v_j, v_{j+1}]$ for some $j \in \{0, 1, \ldots, 5\}$. Taking $u = \log|f'|$, we obtain edge lengths of a new triangle $\tilde{\Delta}_j$, i.e. satisfying (7), if ε is small enough. Then the angle in $\tilde{\Delta}_j$ at the image vertex of v_0 is given by

$$\theta(\log|f'(v_0 + \varepsilon\frac{\sqrt{3}e^{ij\frac{\pi}{3}}}{2})| - \log|f'(v_0)|,\ \log|f'(v_0 + \varepsilon\frac{\sqrt{3}e^{i(j+1)\frac{\pi}{3}}}{2})| - \log|f'(v_0)|)$$

according to (6). Summing up these angles—that is inserting $\log|f'|$ into (8) instead of u at an interior vertex $v_0 \in V_{K,\text{int}}^\varepsilon$—we obtain the function

$$\mathscr{S}_{v_0}(\varepsilon) = \sum_{j=0}^{5}\theta(\log|f'(v_0 + \varepsilon\frac{\sqrt{3}e^{ij\frac{\pi}{3}}}{2})| - \log|f'(v_0)|,\ \log|f'(v_0 + \varepsilon\frac{\sqrt{3}e^{i(j+1)\frac{\pi}{3}}}{2})| - \log|f'(v_0)|)$$

We are interested in the Taylor expansion of $\mathscr{S}_{v_0}$ in ε. The symmetry of the lattice T implies that $\mathscr{S}_{v_0}$ is an even function, so the expansion contains only even powers of ε^n. Using a computer algebra program we arrive at

$$\mathscr{S}_{v_0}(\varepsilon) = 2\pi + C_{v_0}\varepsilon^4 + \mathscr{O}(\varepsilon^6). \tag{10}$$

Here and below, the notation $h(\varepsilon) = \mathscr{O}(\varepsilon^n)$ means that there is a constant $\mathscr{C}$, such that $|h(\varepsilon)| \leqslant \mathscr{C}\varepsilon^n$ holds for all small enough $\varepsilon > 0$. The constant of the ε^4-term is

$$C_{v_0} = -\frac{3\sqrt{3}}{32}\text{Re}\left(S(f)(v_0)\overline{\left(\frac{f''}{f'}\right)'(v_0)}\right),$$

where $S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$ is the Schwarzian derivative of f. We will not need the exact form of this constant, but only the fact that it is bounded on K.

Analogous results to (10) hold for all triangular lattices εT with edge lengths $a^\varepsilon = \varepsilon\sin\alpha$, $b^\varepsilon = \varepsilon\sin\beta$, $c^\varepsilon = \varepsilon\sin\gamma$, also if the angles are larger than $\pi/2$. We assume without loss of generality the edge directions being parallel to 1, $e^{i\alpha}$ and $e^{i(\alpha+\beta)}$. Arguing as above, we consider the function

$$
\begin{aligned}
\mathscr{S}_{v_0}(\varepsilon) = \ &\theta(2\log\frac{\sin\alpha}{\sin\gamma} + \log|\frac{f'(v_0+\varepsilon\sin\beta)}{f'(v_0)}|,\ 2\log\frac{\sin\alpha}{\sin\beta} + \log|\frac{f'(v_0+\varepsilon\sin\gamma\mathrm{e}^{i\alpha})}{f'(v_0)}|)\\
&+\theta(2\log\frac{\sin\beta}{\sin\alpha} + \log|\frac{f'(v_0+\varepsilon\sin\gamma\,\mathrm{e}^{i\alpha})}{f'(v_0)}|,\ 2\log\frac{\sin\beta}{\sin\gamma} + \log|\frac{f'(v_0+\varepsilon\sin\alpha\,\mathrm{e}^{i(\alpha+\beta)})}{f'(v_0)}|)\\
&+\theta(2\log\frac{\sin\gamma}{\sin\beta} + \log|\frac{f'(v_0+\varepsilon\sin\alpha\,\mathrm{e}^{i(\alpha+\beta)})}{f'(v_0)}|,\ 2\log\frac{\sin\gamma}{\sin\alpha} + \log|\frac{f'(v_0-\varepsilon\sin\beta)}{f'(v_0)}|)\\
&+\theta(2\log\frac{\sin\alpha}{\sin\gamma} + \log|\frac{f'(v_0-\varepsilon\sin\beta)}{f'(v_0)}|,\ 2\log\frac{\sin\alpha}{\sin\beta} + \log|\frac{f'(v_0-\varepsilon\sin\gamma\,\mathrm{e}^{i\alpha})}{f'(v_0)|})\\
&+\theta(2\log\frac{\sin\beta}{\sin\alpha} + \log|\frac{f'(v_0-\varepsilon\sin\gamma\,\mathrm{e}^{i\alpha})}{f'(v_0)}|,\ 2\log\frac{\sin\beta}{\sin\gamma} + \log|\frac{f'(v_0-\varepsilon\sin\alpha\,\mathrm{e}^{i(\alpha+\beta)})}{f'(v_0)}|)\\
&+\theta(2\log\frac{\sin\gamma}{\sin\beta} + \log|\frac{f'(v_0-\varepsilon\sin\alpha\,\mathrm{e}^{i(\alpha+\beta)})}{f'(v_0)}|,\ 2\log\frac{\sin\gamma}{\sin\alpha} + \log|\frac{f'(v_0+\varepsilon\sin\beta)}{f'(v_0)}|).
\end{aligned}
$$

Again, $\mathscr{S}_{v_0}$ is an even function. Using a computer algebra program we arrive at

$$\mathscr{S}_{v_0}(\varepsilon) = 2\pi + C_{v_0}\varepsilon^4 + \mathscr{O}(\varepsilon^6), \tag{11}$$

with corresponding constant

$$
\begin{aligned}
C_{v_0} = -\frac{\sin\alpha\sin\beta\sin\gamma}{4}\ \mathrm{Re}\Bigg(&S(f)(v_0)\overline{\left(\frac{f''}{f'}\right)'}(v_0)\\
&+c(\alpha,\beta,\gamma)\left(\frac{1}{2}\left(\frac{f''}{f'}\right)^2\left(\frac{f''}{f'}\right)' - \frac{1}{3}\left(\frac{f''}{f'}\right)'''\right)\Bigg),
\end{aligned}
$$

where $c(\alpha,\beta,\gamma) = \cos\beta\sin^3\beta + \cos\gamma\sin^3\gamma\mathrm{e}^{2i\alpha} + \cos\alpha\sin^3\alpha\mathrm{e}^{2i(\alpha+\beta)}$.

Our key observation is that we can control the sign of the $\mathscr{O}(\varepsilon^4)$-term in (10) if we replace $\log|f'(x)|$ by $\log|f'(x)| + a\varepsilon^2|x|^2$, where $a\in\mathbb{R}$ is some suitable constant. In particular, for positive constants $M^\pm, C^\pm$ consider the functions

$$w^\pm = \log|f'| + q^\pm \qquad \text{with } q^\pm(v) = \begin{cases} \pm\varepsilon^2(M^\pm - C^\pm|v|^2) & \text{for } v\in V^\varepsilon_{K,\mathrm{int}},\\ 0 & \text{for } v\in\partial V^\varepsilon_K.\end{cases}$$

Here and below ∂V^ε_K denotes the set of boundary vertices of V^ε_K.

Then we obtain for equilateral triangulations with edge length $\frac{\sqrt{3}}{2}\varepsilon$ the following Taylor expansion for all interior vertices $v_0\in V^\varepsilon_{K,\mathrm{int}}$ whose neighbors are also in $V^\varepsilon_{K,\mathrm{int}}$:

$$
\begin{aligned}
\sum_{j=0}^{5}\theta(w^\pm(v_0+\varepsilon\frac{\sqrt{3}}{2}\mathrm{e}^{ij\frac{\pi}{3}}) - w^\pm(v_0),\, w^\pm(v_0+\varepsilon\frac{\sqrt{3}}{2}\mathrm{e}^{i(j+1)\frac{\pi}{3}}) - w^\pm(v_0))&\\
= 2\pi + (C_{v_0}\mp\frac{3\sqrt{3}}{2}C^\pm)\varepsilon^4 + \mathscr{O}(\varepsilon^5).& \qquad (12)
\end{aligned}
$$