On the Variational Interpretation of the Discrete KP Equation

Raphael Boll, Matteo Petrera and Yuri B. Suris

Abstract We study the variational structure of the discrete Kadomtsev-Petviashvili (dKP) equation by means of its pluri-Lagrangian formulation. We consider the dKP equation and its variational formulation on the cubic lattice $\mathbb{Z}^N$ as well as on the root lattice $Q(A_N)$. We prove that, on a lattice of dimension at least four, the corresponding Euler-Lagrange equations are equivalent to the dKP equation.

1 Introduction

We developed the theory of pluri-Lagrangian problems (integrable systems of variational origin) in recent papers [2–6, 15, 16], influenced by the fundamental insight of [11–13, 17]. In the present paper, we consider the pluri-Lagrangian formulation of the *discrete bilinear Kadomtsev-Petviashvili* (dKP) equation on three-dimensional lattices and its consistent extension to higher dimensional lattices. This equation belongs to integrable octahedron-type equations which were classified in [1]. A Lagrangian formulation of this equation was given in [13]. There, the authors consider a discrete 3-form on the lattice $\mathbb{Z}^3$ together with the corresponding Euler-Lagrange equations which are shown to be satisfied on solutions of the dKP equation. They also show that this 3-form is closed on solutions of the dKP equation, namely, the so-called 4D closure relation is satisfied. The main goal of the present paper is to provide a more precise understanding of the findings in that paper. More concretely:

- In the framework of the pluri-Lagrangian formulation, we construct the elementary building blocks of Euler-Lagrange equations, which, in the present situation, are the so-called 4D corner equations.

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- In the two-dimensional case, as noticed in [4], the corresponding 3D corner equations build a consistent system. Its solutions are more general then the solutions of the underlying hyperbolic system of quad-equations. On the contrary, in the present three-dimensional situation, the system of 4D corner equations is not consistent in the usual sense (i.e., it does not allow to determine general solutions with the maximal number of initial data). However, this system turns out to be equivalent, in a sense which we are going to explain later, to the corresponding hyperbolic system, namely the dKP equation.
- We provide a rigorous consideration of the branches of the logarithm functions involved in the Euler-Lagrange equations. This leads to the following more precise result: the system of 4D corner equations is equivalent, and thus provides a variational formulation, to two different hyperbolic equations, namely the dKP equation itself and its version obtained under inversion $x \mapsto x^{-1}$ of all fields which will be denoted by dKP^{-}.

One can consider the dKP equation on the cubic lattice $\mathbb{Z}^3$ and its higher dimensional analogues $\mathbb{Z}^N$, but, as discussed in [1, 8, 9] another natural setting the dKP equation (and related octahedron-type equations) is the three-dimensional root lattice

$$Q(A_3) := \{(n_i, n_j, n_k, n_\ell) : n_i + n_j + n_k + n_\ell = 0\}.$$

Also in this setting, the dKP equation can be extended in a consistent way to the higher dimensional lattices $Q(A_N)$ with $N > 3$.

Both lattices have their advantages and disadvantages. The cubic lattice $\mathbb{Z}^N$, on the one hand, is more manageable and easier to visualize. Its cell structure is very simple: for every dimension N, all N-dimensional elementary cells are N-dimensional cubes. On the other hand, it is less natural to consider dKP on the lattice $\mathbb{Z}^3$, because this equation depends on the variables assigned to six out of eight vertices of a (three-dimensional) cube.

The root lattice $Q(A_N)$, in contrast, has a more complicated cell structure, because the number of different N-dimensional elementary cells increases with the dimension N. For instance, for $N = 3$ there are two types of elementary cells octahedra and tetrahedra. Moreover, especially in higher dimensions, a visualization of the elementary cells is difficult, if not impossible. However, this lattice is more natural for the consideration of dKP from the combinatorial point of view, because this equation depends on variables which can be assigned to the six vertices of an octahedron, one of the elementary cells of the lattice. Furthermore, the four-dimensional elementary cells are combinatorially smaller (they contain only 10 vertices, as compared with 16 vertices of a four-dimensional cube) and possess higher symmetry than the cubic ones. Since they support the equations which serve as variational analogue of the dKP equation, this leads to a simpler situation.

We will see that a four-dimensional cube is combinatorially equivalent to the sum of four elementary cells of the root lattice $Q(A_4)$. Therefore, several results in the cubic case can be seen as direct consequences of results of the more fundamental $Q(A_N)$-case.

Let us start with some concrete definitions valid for an arbitrary N-dimensional lattice $\mathscr{X}$.

Definition 1.1 (*Discrete 3-form*) A *discrete 3-form* on $\mathscr{X}$ is a real-valued function $\mathscr{L}$ of oriented 3-cells σ depending on some field $x : \mathscr{X} \to \mathbb{R}$, such that $\mathscr{L}$ changes the sign by changing the orientation of σ.

For instance, in $Q(A_N)$, the 3-cells are tetrahedra and octahedra, and, in $\mathbb{Z}^N$, the 3-cells are 3D cubes.

Definition 1.2 (*3-dimensional pluri-Lagrangian problem*) Let $\mathscr{L}$ be a discrete 3-form on $\mathscr{X}$ depending on $x : \mathscr{X} \to \mathbb{R}$.

- To an arbitrary 3-manifold $\Sigma \subset \mathscr{X}$, i.e., a union of oriented 3-cells which forms an oriented three-dimensional topological manifold, there corresponds the *action functional*, which assigns to $x|_{V(\Sigma)}$, i.e., to the fields in the set of the vertices $V(\Sigma)$ of Σ, the number

$$S_\Sigma := \sum_{\sigma \in \Sigma} \mathscr{L}(\sigma).$$

- We say that the field $x : V(\Sigma) \to \mathbb{R}$ is a critical point of S_Σ, if at any interior point $n \in V(\Sigma)$, we have

$$\frac{\partial S_\Sigma}{\partial x(n)} = 0. \tag{1}$$

  Equation (1) are called *discrete Euler-Lagrange equations* for the action S_Σ.
- We say that the field $x : \mathscr{X} \to \mathbb{R}$ solves the *pluri-Lagrangian problem* for the Lagrangian 3-form $\mathscr{L}$ if, *for any 3-manifold* $\Sigma \subset \mathscr{X}$, the restriction $x|_{V(\Sigma)}$ is a critical point of the corresponding action S_Σ.

In the present paper, we focus on the variational formulation of the dKP equation on $Q(A_N)$ and $\mathbb{Z}^N$. Let us formulate the main results of the paper.

On the lattice $Q(A_N)$, we consider discrete 3-forms vanishing on all tetrahedra. One can show (see Corollary 2.5) that, for an arbitrary interior vertex of any 3-manifold in $Q(A_N)$, the Euler-Lagrange equations follow from certain elementary building blocks. These so-called 4D corner equations are the Euler-Lagrange equations for elementary 4-cells of $Q(A_N)$ different from 4-simplices, so-called 4-ambo-simplices. Such a 4-ambo-simplex has ten vertices. Therefore, the crucial issue is the study of the system consisting of the corresponding ten corner equations. In our case, each corner equation depends on all ten fields at the vertices of the 4-ambo-simplex. Therefore, one could call this system *consistent* if any two equations are functionally dependent. It turns out that this is *not* the case. We will prove the following statement:

Theorem 1.3 *Every solution of the system of ten corner equations for a 4-ambo-simplex in $Q(A_N)$ satisfies either the system of five dKP equations or the system of five dKP$^-$ equations on the five octahedral facets of the 4-ambo-simplex.*

Thus, one can prescribe arbitrary initial values at seven vertices of a 4-ambo-simplex. We will also prove the following theorem:

Theorem 1.4 *The discrete 3-form* $\mathscr{L}$ *is closed on any solution of the system of corner equations.*

In [4, 15], it was shown that in dimensions 1 and 2 the analogues of the property formulated in Theorem 1.4 are related to more traditional integrability attributes.

For the case of the cubic lattice $\mathbb{Z}^N$, the situation is similar: one can show (see Corollary 4.2) that, for an arbitrary interior vertex of any 3-manifold in $\mathbb{Z}^3$, the Euler-Lagrange equations follow from certain elementary building blocks. These so-called 4D corner equations are the Euler-Lagrange equations for elementary 4D cubes in $\mathbb{Z}^N$. A 4D cube has sixteen vertices, but in our case the action on a 4D cube turns out to be independent of the fields on two of the vertices. Therefore, the crucial issue is the study of the system consisting of the corresponding fourteen corner equations. Six of the fourteen corner equations depend each on thirteen of the fourteen fields. There do not exist pairs of such equations which are independent of one and the same field. All other equations depend each on ten of the fourteen fields. Therefore, one could call this system *consistent* if it would have the minimal possible rank 2 (assign twelve fields arbitrarily and use two of the six corner equations—depending on thirteen fields—to determine the remaining two fields, then all twelve remaining equations should be satisfied automatically). It turns out that the system of the fourteen corner equations is *not* consistent in this sense. We will prove the following analogue of Theorem 1.3:

Theorem 1.5 *Every solution of the system of fourteen corner equations for a 4D cube in* $\mathbb{Z}^N$ *satisfies either the system of eight dKP equations or the system of eight dKP*$^-$ *equations on the eight cubic facets of the 4D cube.*

Thus, one can prescribe arbitrary initial values at nine vertices of a 4D cube. Correspondingly, we will also prove the following statement:

Theorem 1.6 *The discrete 3-form* $\mathscr{L}$ *is closed on any solution of the system of corner equations.*

The paper is organized as follows: we start with the root lattice $Q(A_N)$, thus considering the combinatorial issues and some general properties of pluri-Lagrangian systems. Then we introduce the dKP equation and its pluri-Lagrangian structure. In the second part of the paper the present similar considerations for the cubic lattice $\mathbb{Z}^N$.

2 The Root Lattice $Q(A_N)$

We consider the root lattice

$$Q(A_N) := \{n := (n_0, n_1, \dots, n_N) \in \mathbb{Z}^{N+1} : n_0 + n_1 + \dots + n_N = 0\},$$

where $N \geq 3$. The three-dimensional sub-lattices $Q(A_3)$ are given by

$$Q(A_3) := \{(n_i, n_j, n_k, n_\ell) : n_i + n_j + n_k + n_\ell = \text{const}\}.$$

We consider fields $x : Q(A_N) \to \mathbb{R}$, and use the shorthand notations

$$x_{\bar{\imath}} = x(n - e_i), \qquad x = x(n), \qquad \text{and} \qquad x_i = x(n + e_i),$$

where e_i is the unit vector in the ith coordinate direction. Furthermore, the shift functions T_i and $T_{\bar{\imath}}$ are defined by

$$T_i x_\alpha := x_{i\alpha} \quad \text{and} \quad T_{\bar{\imath}} x_\alpha := x_{\bar{\imath}\alpha}$$

for a multiindex α. For simplicity, we sometimes abuse notations by identifying lattice points n with the corresponding fields $x(n)$.

We now give a very brief introduction to the Delaunay cell structure of the n-dimensional root lattice $Q(A_N)$ [7, 14]. Here, we restrict ourselves to a very elementary description which is appropriate to our purposes and follow the considerations in [1]. For each N there are N sorts of N-cells of $Q(A_N)$ denoted by $P(k, N)$ with $k = 1, \ldots, N$:

- Two sorts of 2-cells:

  $P(1, 2)$: black triangles$\lfloor ijk \rfloor := \{x_i, x_j, x_k\}$;
  $P(2, 2)$: white triangles$\lceil ijk \rceil := \{x_{ij}, x_{ik}, x_{jk}\}$;
- Three sorts of 3-cells:

  $P(1, 3)$: black tetrahedra$\lfloor ijk\ell \rfloor := \{x_i, x_j, x_k, x_\ell\}$;
  $P(2, 3)$: octahedra$[ijk\ell] := \{x_{ij}, x_{ik}, x_{i\ell}, x_{jk}, x_{j\ell}, x_{k\ell}\}$;
  $P(3, 3)$: white tetrahedra$\lceil ijk\ell \rceil := \{x_{ijk}, x_{ij\ell}, x_{ik\ell}, x_{jk\ell}\}$;
- Four sorts of 4-cells:

  $P(1, 4)$: black 4-simplices$\llbracket ijk\ell m \rrbracket := \{x_i, x_j, x_k, x_\ell, x_m\}$;
  $P(2, 4)$: black 4-ambo-simplices$\lfloor ijk\ell m \rfloor := \{x_{\alpha\beta} : \alpha, \beta \in \{i, j, k, \ell, m\}, \alpha \neq \beta\}$;
  $P(3, 4)$: white 4-ambo-simplices$\lceil ijk\ell m \rceil := \{x_{\alpha\beta\gamma} : \alpha, \beta, \gamma \in \{i, j, k, \ell, m\},$
  $\alpha \neq \beta \neq \gamma \neq \alpha\}$;
  $P(4, 4)$: white 4-simplices$\llbracket ijk\ell m \rrbracket := \{x_{ijk\ell}, x_{ijkm}, x_{ij\ell m}, x_{ik\ell m}, x_{jk\ell m}\}$.

The facets of 3-cells and 4-cells can be found in Appendix 1.

In the present paper we will consider objects on *oriented* manifolds. We say that a black triangle $\lfloor ijk \rfloor$ and white triangle $\lceil ijk \rceil$ are positively oriented if $i < j < k$ (see Fig. 1). Any permutation of two indices changes the orientation to the opposite one.

When we use the bracket notation, we always write the letters in brackets in increasing order, so, e.g., in writing $\lfloor ijk \rfloor$ we assume that $i < j < k$ and avoid the notation $\lfloor jik \rfloor$ or $\lfloor ikj \rfloor$ for the negatively oriented triangle $-\lfloor ijk \rfloor$.

There is a simple recipe to derive the orientation of facets of an N-cell: On every index in the brackets we put alternately a "+" or a "−" starting with a "+" on the

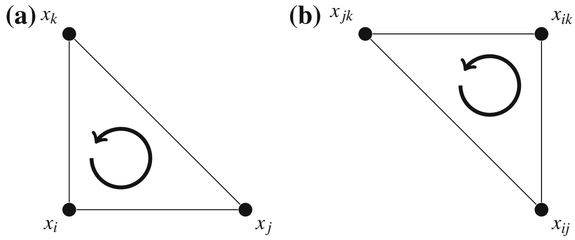

Fig. 1 Orientation of triangles: **a** the black triangle $\lfloor ijk \rfloor$; **b** the white triangle $\lceil ijk \rceil$

last index. Then we get each of its facets by deleting one index and putting the corresponding sign in front of the bracket. For instance, the black 4-ambo-simplex

$$\begin{matrix} & + & - & + & - & + & \\ \lfloor & i & j & k & \ell & m & \rfloor \end{matrix}$$

has the five octahedral facets $[ijk\ell]$, $-[ijkm]$, $[ij\ell m]$, $-[ik\ell m]$, and $[jk\ell m]$.

The following two definitions are valid for arbitrary N-dimensional lattices $\mathscr{X}$.

Definition 2.1 (*Adjacent N-cell*) Given an N-cell σ, another N-cell $\bar{\sigma}$ is called *adjacent* to σ if σ and $\bar{\sigma}$ share a common $(N-1)$-cell. The orientation of this $(N-1)$-cell in σ must be opposite to its orientation in $\bar{\sigma}$.

The latter property guarantees that the orientations of the adjacent N-cells agree.

Definition 2.2 (*Flower*) A 3-manifold in $\mathscr{X}$ with exactly one interior vertex x is called a *flower* with center x. *The flower* at an interior vertex x of a given 3-manifold is the flower with center x which lies completely in the 3-manifold.

As a consequence, in $Q(A_N)$, in each flower every tetrahedron has exactly three adjacent 3-cells and every octahedron has exactly four adjacent 3-cells.

Examples for open 3-manifolds in $Q(A_N)$ are the three-dimensional sub-lattices $Q(A_3)$. Here, the flower at an interior vertex consists of eight tetrahedra (four black and four white ones) and six octahedra.

Examples of closed 3-manifolds in $Q(A_N)$ are the set of facets of a 4-ambo-simplex (consisting of five tetrahedra) and the set of facets of a 4-ambo-simplex (consisting of five tetrahedra and five octahedra).

The elementary building blocks of 3-manifolds are so-called 4D corners:

Definition 2.3 (*4D corner*) A *4D corner* with center x is a 3-manifold consisting of all facets of a 4-cell adjacent to x.

In $Q(A_N)$, there are two different types of 4D corners: a corner on a 4-simplex (consisting of a four tetrahedra) and a corner on a 4-ambo-simplex (consisting of two tetrahedra and three octahedra), see Appendix 2 for details.

The following combinatorial statement will be proven in Appendix 3:

Theorem 2.4 *The flower at any interior vertex of any 3-manifold in $Q(A_N)$ can be represented as a sum of 4D corners in $Q(A_{N+2})$.*

Let $\mathscr{L}$ be a discrete 3-form on $Q(A_N)$. The *exterior derivative* $d\mathscr{L}$ is a discrete 4-form whose value at any 4-cell in $Q(A_N)$ is the action functional of $\mathscr{L}$ on the 3-manifold consisting of the facets of the 4-cell. For our purposes, we consider discrete 3-forms $\mathscr{L}$ vanishing on all tetrahedra. In particular, we have

$$d\mathscr{L}(\llbracket ijk\ell m\rrbracket) \equiv 0 \quad \text{and} \quad d\mathscr{L}(\llbracket ijk\ell m\rrbracket) \equiv 0$$

since a 4-simplices only contain tetrahedra. The exterior derivative on a black 4-ambo-simplex $\lfloor ijk\ell m\rfloor$ is given by

$$\begin{aligned} \underline{S}^{ijk\ell m} &:= d\mathscr{L}(\lfloor ijk\ell m\rfloor) \\ &= \mathscr{L}([ijk\ell]) + \mathscr{L}(-[ijkm]) + \mathscr{L}([ij\ell m]) + \mathscr{L}(-[ik\ell m]) + \mathscr{L}([jk\ell m]). \end{aligned} \tag{2}$$

The exterior derivative on a white 4-ambo-simplex $\lceil ijk\ell m\rceil$ is given by

$$\begin{aligned} \bar{S}^{ijk\ell m} &:= d\mathscr{L}(\lceil ijk\ell m\rceil) \\ &= \mathscr{L}(T_m[ijk\ell]) + \mathscr{L}(-T_\ell[ijkm]) + \mathscr{L}(T_k[ij\ell m]) + \mathscr{L}(-T_j[ik\ell m]) \\ &\quad + \mathscr{L}(T_i[jk\ell m]). \end{aligned} \tag{3}$$

Accordingly, the Euler-Lagrange equations on black 4-ambo-simplices $\lfloor ijk\ell m\rfloor$ are

$$\begin{aligned} &\frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{ij}} = 0, \quad \frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{ik}} = 0, \quad \frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{i\ell}} = 0, \quad \frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{im}} = 0, \quad \frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{jk}} = 0, \\ &\frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{j\ell}} = 0, \quad \frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{jm}} = 0, \quad \frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{k\ell}} = 0, \quad \frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{km}} = 0, \quad \frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{\ell m}} = 0. \end{aligned} \tag{4}$$

and the Euler-Lagrange equations on white 4-ambo-simplices $\lceil ijk\ell m\rceil$ are

$$\begin{aligned} &\frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{ijk}} = 0, \quad \frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{ij\ell}} = 0, \quad \frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{ijm}} = 0, \quad \frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{ik\ell}} = 0, \quad \frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{ikm}} = 0, \\ &\frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{i\ell m}} = 0, \quad \frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{jk\ell}} = 0, \quad \frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{jkm}} = 0, \quad \frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{j\ell m}} = 0, \quad \frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{k\ell m}} = 0. \end{aligned} \tag{5}$$

The last two systems are called *corner equations*.

The following statement is an immediate consequence of Theorem 2.4:

Theorem 2.5 *For discrete every 3-form on $Q(A_N)$ and every 3-manifold in $Q(A_N)$ all corresponding Euler-Lagrange equations can be written as a sum of corner equations.*

3 The dKP Equation on $Q(A_N)$

We will now introduce the dKP equation on the root lattice $Q(A_3)$. Every oriented octahedron $[ijk\ell]$ $(i < j < k < \ell)$ in $Q(A_3)$ supports the equation

$$x_{ij}x_{k\ell} - x_{ik}x_{j\ell} + x_{i\ell}x_{jk} = 0. \tag{6}$$

We can extend this system in a consistent way (see [1]) to the four-dimensional root lattice $Q(A_4)$ and higher-dimensional analogues, such that the five octahedral facets $[ijk\ell]$, $[jk\ell m]$, $-[ik\ell m]$, $[ijm\ell]$, and $-[ijkm]$ of the black 4-ambo-simplex $\lfloor ijk\ell m \rfloor$ support the equations

$$\begin{aligned} &x_{ij}x_{k\ell} - x_{ik}x_{j\ell} + x_{i\ell}x_{jk} = 0, \\ &x_{jk}x_{\ell m} - x_{j\ell}x_{km} + x_{jm}x_{k\ell} = 0, \\ &x_{k\ell}x_{im} - x_{km}x_{i\ell} + x_{ik}x_{\ell m} = 0, \\ &x_{\ell m}x_{ij} - x_{i\ell}x_{jm} + x_{j\ell}x_{im} = 0, \\ &x_{im}x_{jk} - x_{jm}x_{ik} + x_{km}x_{ij} = 0 \end{aligned} \tag{7}$$

and the five octahedral facets $T_m[ijk\ell]$, $T_i[jk\ell m]$, $-T_j[ik\ell m]$, $T_k[ij\ell m]$, and $-T_\ell[ijkm]$ of the white 4-ambo-simplex $\lceil ijk\ell m \rceil$ support the equations

$$\begin{aligned} &x_{ijm}x_{k\ell m} - x_{ikm}x_{j\ell m} + x_{i\ell m}x_{jkm} = 0, \\ &x_{ijk}x_{i\ell m} - x_{ij\ell}x_{ikm} + x_{ijm}x_{ik\ell} = 0, \\ &x_{jk\ell}x_{ijm} - x_{jkm}x_{ij\ell} + x_{ijk}x_{j\ell m} = 0, \\ &x_{k\ell m}x_{ijk} - x_{ik\ell}x_{jkm} + x_{jk\ell}x_{ikm} = 0, \\ &x_{i\ell m}x_{jk\ell} - x_{j\ell m}x_{ik\ell} + x_{k\ell m}x_{ij\ell} = 0. \end{aligned} \tag{8}$$

In both systems one can derive one equation from another by cyclic permutations of indices $(ijk\ell m)$.

We propose the following discrete 3-form $\mathscr{L}$ defined on oriented octahedra $[ijk\ell]$:

$$\mathscr{L}([ijk\ell]) := \frac{1}{2}\left(\Lambda\left(\frac{x_{ij}x_{k\ell}}{x_{ik}x_{j\ell}}\right) + \Lambda\left(\frac{x_{ik}x_{j\ell}}{x_{i\ell}x_{jk}}\right) + \Lambda\left(-\frac{x_{i\ell}x_{jk}}{x_{ij}x_{k\ell}}\right)\right), \tag{9}$$

where

$$\Lambda(z) := \lambda(z) - \lambda\left(\frac{1}{z}\right) \quad \text{and} \quad \lambda(z) := -\int_0^z \frac{\log|1-x|}{x}dx. \tag{10}$$

The discrete 3-form (9) has its motivation in [13]. Indeed, in [13], the authors consider a similar discrete 3-form on the cubic lattice $\mathbb{Z}^N$. One can also consider our 3-form on the cubic lattice $\mathbb{Z}^N$. Then one would assign to each 3D cube the 3-form at its inscribed octahedron. This 3-form differs from their one by an additive constant and a slightly different definition of the function $\lambda(z)$: they use the function

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-x)}{x}dx \tag{11}$$

instead of $\lambda(z)$. Our choice of $\lambda(z)$ allows us for a more precise consideration of the branches of the occurring logarithm.

Observe that the expression (9) only changes its sign under the cyclic permutation of indices $(ijk\ell m)$. This follows from $\Lambda(z) = -\Lambda(z^{-1})$. As a consequence, the exterior derivatives $\underline{S}^{ijk\ell m}$ and $\bar{S}^{ijk\ell m}$ defined in (2) and (3), respectively, are invariant under the cyclic permutation of indices $(ijk\ell m)$. Therefore, one can obtain all corner equations in (4) and (5) by (iterated) cyclic permutation $(ijk\ell m)$ from

$$\frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{ij}} = 0, \quad \frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{ik}} = 0, \quad \text{and} \quad \frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{ijk}} = 0, \quad \frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{ij\ell}} = 0.$$

Let us study separately the corner equations on black and white 4-ambo-simplices. The corner equations which live on the black 4-ambo-simplex $\lfloor ijk\ell m\rfloor$ are given by

$$\frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{ij}} = \frac{\partial \mathscr{L}([ijk\ell])}{\partial x_{ij}} + \frac{\partial \mathscr{L}(-[ijkm])}{\partial x_{ij}} + \frac{\partial \mathscr{L}([ij\ell m])}{\partial x_{ij}} = 0$$

and

$$\frac{\partial \underline{S}^{ijk\ell m}}{\partial x_{ik}} = \frac{\partial \mathscr{L}([ijk\ell])}{\partial x_{ik}} + \frac{\partial \mathscr{L}(-[ijkm])}{\partial x_{ik}} + \frac{\partial \mathscr{L}(-[ik\ell m])}{\partial x_{ik}} = 0.$$

Explicitly, they read

$$\frac{1}{x_{ij}} \log|E_{ij}| = 0 \quad \text{and} \quad \frac{1}{x_{ik}} \log|E_{ik}| = 0, \tag{12}$$

where

$$E_{ij} := \frac{x_{ij}x_{k\ell} + x_{i\ell}x_{jk}}{x_{ij}x_{k\ell} - x_{ik}x_{j\ell}} \cdot \frac{x_{ij}x_{km} - x_{ik}x_{jm}}{x_{ij}x_{km} + x_{im}x_{jk}} \cdot \frac{x_{ij}x_{\ell m} + x_{im}x_{j\ell}}{x_{ij}x_{\ell m} - x_{i\ell}x_{jm}}$$

and

$$E_{ik} := \frac{x_{ik}x_{j\ell} - x_{ij}x_{k\ell}}{x_{ik}x_{j\ell} - x_{i\ell}x_{jk}} \cdot \frac{x_{ik}x_{jm} - x_{im}x_{jk}}{x_{ik}x_{jm} - x_{ij}x_{km}} \cdot \frac{x_{ik}x_{\ell m} - x_{i\ell}x_{km}}{x_{ik}x_{\ell m} + x_{im}x_{k\ell}}.$$

For every corner equation (12) there are two classes of solutions, because any solution can either solve $E_{ij} = -1$ or $E_{ij} = 1$. Hereafter, we only consider solutions, where all fields x_{ij} are non-zero (we call such solutions non-singular).

Theorem 3.1 *Every solution of the system* (4) *solves either the system*

$$\begin{aligned} &E_{ij}=-1, \quad E_{ik}=-1, \quad E_{i\ell}=-1, \quad E_{im}=-1, \quad E_{jk}=-1,\\ &E_{j\ell}=-1, \quad E_{jm}=-1, \quad E_{k\ell}=-1, \quad E_{km}=-1, \quad E_{\ell m}=-1 \end{aligned} \tag{13}$$

or the system

$$\begin{aligned} &E_{ij}=1, \quad E_{ik}=1, \quad E_{i\ell}=1, \quad E_{im}=1, \quad E_{jk}=1,\\ &E_{j\ell}=1, \quad E_{jm}=1, \quad E_{k\ell}=1, \quad E_{km}=1, \quad E_{\ell m}=1. \end{aligned} \tag{14}$$

Furthermore, the system (13) *is equivalent to the system* (7) *(that is dKP on the corresponding black 4-ambo-simplex). The system* (14) *is equivalent to the system*

$$\begin{aligned} &x_{ik}x_{i\ell}x_{jk}x_{j\ell}-x_{ij}x_{i\ell}x_{jk}x_{k\ell}+x_{ij}x_{ik}x_{j\ell}x_{k\ell}=0,\\ &x_{j\ell}x_{jm}x_{k\ell}x_{km}-x_{jk}x_{jm}x_{k\ell}x_{\ell m}+x_{jk}x_{j\ell}x_{km}x_{\ell m}=0,\\ &x_{km}x_{ik}x_{\ell m}x_{i\ell}-x_{k\ell}x_{ik}x_{\ell m}x_{im}+x_{k\ell}x_{km}x_{i\ell}x_{im}=0,\\ &x_{i\ell}x_{j\ell}x_{im}x_{jm}-x_{\ell m}x_{j\ell}x_{im}x_{ij}+x_{\ell m}x_{i\ell}x_{jm}x_{ij}=0,\\ &x_{jm}x_{km}x_{ij}x_{ik}-x_{im}x_{km}x_{ij}x_{jk}+x_{im}x_{jm}x_{ik}x_{jk}=0, \end{aligned} \tag{15}$$

which is the system (7) *after the transformation* $x \mapsto x^{-1}$ *of fields (that is dKP*$^-$ *on the corresponding black 4-ambo-simplex).*

Proof Consider a solution x of (4) that solves $E_{ij}=-1$ and $E_{jk}=-1$. We set

$$a_{ij} := x_{\ell m}x_{ij}-x_{i\ell}x_{jm}+x_{j\ell}x_{im}, \tag{16}$$
$$a_{ik} := x_{k\ell}x_{im}-x_{km}x_{i\ell}+x_{ik}x_{\ell m}, \tag{17}$$

and

$$a_{jk} := x_{jk}x_{\ell m}-x_{j\ell}x_{km}+x_{jm}x_{k\ell}, \tag{18}$$

and use these equations to substitute x_{ij}, x_{ik} and x_{jk} in $E_{ij}=-1$ and $E_{jk}=-1$. Writing down the result in polynomial form, we get

$$x_{\ell m}^2(a_{ij}+x_{i\ell}x_{jm}-x_{im}x_{j\ell})e_{ij}=0$$

and

$$x_{\ell m}^2(a_{jk}+x_{j\ell}x_{km}-x_{jm}x_{k\ell})e_{jk}=0,$$

where e_{ij} and e_{jk} are certain polynomials. Since for every solutions of (4) all fields are non-zero this leads us to $e_{ij}=0$ and $e_{jk}=0$. Computing the difference of the latter two equations we get

$$a_{ij}x_{k\ell}x_{km}(a_{ij}+x_{i\ell}x_{jm}-x_{im}x_{j\ell})-a_{jk}x_{i\ell}x_{im}(a_{jk}+x_{j\ell}x_{km}-x_{jm}x_{k\ell})=0$$

and, with the use of (16) and (18),

$$x_{\ell m}(a_{ij}x_{ij}x_{k\ell}x_{km}-a_{jk}x_{jk}x_{i\ell}x_{im})=0,$$

which depends on seven independent fields, i.e., no subset of six fields belong to one octahedron. Then comparing coefficients leads to $a_{ij}=a_{jk}=0$. Substituting

$$x_{ij}=\frac{x_{i\ell}x_{jm}-x_{im}x_{j\ell}}{x_{\ell m}} \quad \text{and} \quad x_{jk}=\frac{x_{j\ell}x_{km}-x_{jm}x_{k\ell}}{x_{\ell m}}$$

into $E_{ij}=-1$ and solving the resulting equation with respect to x_{ik}, we get

$$x_{ik}=\frac{x_{i\ell}x_{km}-x_{im}x_{k\ell}}{x_{\ell m}}.$$

Substituting x_{ij}, x_{ik} and x_{jk} in E_{ik} by using the last three equations, we get $E_{ik}=-1$.

Analogously, one can prove that, for a solution x of (4) which solves $E_{ij}=-1$ and $E_{ik}=-1$, we have $E_{jk}=-1$, and for a solution x of (4) which solves $E_{ik}=-1$ and $E_{i\ell}=-1$, we have $E_{k\ell}=-1$. Therefore, for every solution x of (4) and for every white triangle $\{x_\alpha, x_\beta, x_\gamma\}$ on the black 4-ambo-simplex $\lfloor ijk\ell m\rfloor$ we proved the following: if $E_\alpha=-1$ and $E_\beta=-1$ then $E_\gamma=-1$, too.

On the other hand, one can easily see that x solves $E_{ij}=1$ or $E_{jk}=1$ if and only if x^{-1} solves $E_{ij}=-1$ or $E_{ik}=-1$, respectively. Therefore, we also know that, if $E_\alpha=1$ and $E_\beta=1$ then $E_\gamma=1$, too.

Summarizing, we proved that every solution x of (4) solves either (13) and then also (7) or (14) and then also (15).

Consider a non-singular solution x of the system (7). Then

$$\begin{aligned}E_{ij}&=\frac{x_{ij}x_{k\ell}+x_{i\ell}x_{jk}}{x_{ij}x_{k\ell}-x_{ik}x_{j\ell}}\cdot\frac{x_{ij}x_{km}-x_{ik}x_{jm}}{x_{ij}x_{km}+x_{im}x_{jk}}\cdot\frac{x_{ij}x_{\ell m}+x_{im}x_{j\ell}}{x_{ij}x_{\ell m}-x_{i\ell}x_{jm}}\\&=\frac{x_{ik}x_{j\ell}}{-x_{i\ell}x_{jk}}\cdot\frac{-x_{im}x_{jk}}{x_{ik}x_{jm}}\cdot\frac{x_{i\ell}x_{jm}}{-x_{im}x_{j\ell}}=-1\end{aligned}$$

and

$$\begin{aligned}E_{ik}&=\frac{x_{ik}x_{j\ell}-x_{ij}x_{k\ell}}{x_{ik}x_{j\ell}-x_{i\ell}x_{jk}}\cdot\frac{x_{ik}x_{jm}-x_{im}x_{jk}}{x_{ik}x_{jm}-x_{ij}x_{km}}\cdot\frac{x_{ik}x_{\ell m}-x_{i\ell}x_{km}}{x_{ik}x_{\ell m}+x_{im}x_{k\ell}}\\&=\frac{x_{i\ell}x_{jk}}{x_{ij}x_{k\ell}}\cdot\frac{x_{ij}x_{km}}{x_{im}x_{jk}}\cdot\frac{x_{im}x_{k\ell}}{-x_{i\ell}x_{km}}=-1.\end{aligned}$$

This proves the equivalence of (13) and (7) and also the equivalence of (14) and (15) since x solves $E_{ij}=-1$ or (7) if and only if x^{-1} solves $E_{ij}=1$ or (15), respectively. □

We will present the closure relation which can be seen as a criterion of integrability:

Theorem 3.2 *(Closure relation) There holds:*

$$\underline{S}^{ijk\ell m} \pm \frac{\pi^2}{4} = 0$$

on all solutions of (13) *and* (14)*, respectively. Therefore, one can redefine the 3-form* $\mathscr{L}$ *as*

$$\tilde{\mathscr{L}}([ijk\ell]) := \mathscr{L}([ijk\ell]) \pm \frac{\pi^2}{4}$$

in order to get $\underline{S}^{ijk\ell m} = 0$ *on all solutions of* (13) *and* (14)*, respectively.*

Proof The set of solutions $\mathscr{S}^+$ of (13), as well as the set of solutions $\mathscr{S}^-$ (14), is a connected seven-dimensional algebraic manifold which can be parametrized by the set of variables $\{x_{ij}, x_{ik}, x_{i\ell}, x_{im}, x_{jk}, x_{j\ell}, x_{jm}\}$. We want to show that the directional derivatives of $\underline{S}^{ijk\ell m}$ along tangent vectors of $\mathscr{S}^\pm$ vanish. It is easy to see that the stronger property grad$\underline{S}^{ijk\ell m} = 0$ on $\mathscr{S}^\pm$, where we $\underline{S}^{ijk\ell m}$ is considered as a function of ten variables x_{ij}, is a consequence of (13), respectively (14). Therefore, the function $\underline{S}^{ijk\ell m}$ is constant on $\mathscr{S}^\pm$.

To determine the value of $\underline{S}^{ijk\ell m}$ on solutions of (13), we consider the constant solution of (7)

$$\begin{aligned} x_{ij} = x_{jk} = x_{k\ell} = x_{\ell m} = x_{im} = a, \\ x_{ik} = x_{j\ell} = x_{km} = x_{i\ell} = x_{jm} = -1, \end{aligned} \tag{19}$$

where

$$a := \frac{1}{2} - \frac{\sqrt{5}}{2}.$$

(Indeed, for this point every equation from (7) looks like $a^2 - 1 - a = 0$.) Therefore, this point satisfies (13), because (7) and (13) are equivalent.

Consider the dilogarithm as defined in (11) and suppose that $z > 1$. According to [10], we derive:

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2(z^{-1}) - \frac{1}{2}\log^2 z + \frac{\pi^2}{3} - i\pi \log z$$

and

$$\begin{aligned} \mathrm{Re}\,\mathrm{Li}_2(z) =& \mathrm{Re}\,\mathrm{Li}_2(ze^{i0}) = -\frac{1}{2}\int_0^z \frac{\log(1 - 2x\cos 0 + x^2)}{x} dx \\ =& -\frac{1}{2}\int_0^z \frac{\log(1-x)^2}{x} dx = -\int_0^z \frac{\log|1-x|}{x} dx = \lambda(z), \end{aligned}$$

where $\lambda(z)$ is the same function as in (9). Therefore, we have

$$\lambda(z) = \begin{cases} \mathrm{Li}_2(z), & z \le 1, \\ -\mathrm{Li}_2(z^{-1}) - \dfrac{1}{2}\log^2 z + \dfrac{\pi^2}{3}, & z > 1. \end{cases}$$

By using the following special values [10]

$$\begin{aligned} &\mathrm{Li}_2(a^2) = \tfrac{\pi^2}{15} - \log^2(-a), &&\mathrm{Li}_2(-a) = \tfrac{\pi^2}{10} - \log^2(-a), \\ &\mathrm{Li}_2(a) = -\tfrac{\pi^2}{15} + \tfrac{1}{2}\log^2(-a), &&\mathrm{Li}_2(a^{-1}) = -\tfrac{\pi^2}{10} - \log^2(-a). \end{aligned}$$

a straightforward computation gives

$$\begin{aligned} \mathscr{L}([ijk\ell]) =& \mathscr{L}(-[ijkm]) = \mathscr{L}([ij\ell m]) = \mathscr{L}(-[ik\ell m]) = \mathscr{L}([jk\ell m]) \\ =& \frac{1}{2}(\Lambda(a^2) + \Lambda(-a^{-1}) + \Lambda(a^{-1})) = -\frac{\pi^2}{20} \end{aligned}$$

and

$$\begin{aligned} \underline{S}^{ijk\ell m} =& \mathscr{L}([ijk\ell]) + \mathscr{L}(-[ijkm]) + \mathscr{L}([ij\ell m]) + \mathscr{L}(-[ik\ell m]) + \mathscr{L}([jk\ell m]) \\ =& -\frac{\pi^2}{4}. \end{aligned}$$

This is, because the expression for $\mathscr{L}([ijk\ell])$ (see (9)) changes the sign under the cyclic permutation of indices $(ijk\ell)$ and the solution is invariant under cyclic permutation of indices $(ijk\ell m)$.

Let us now consider the second branch of solutions: one can easily see that

$$\begin{aligned} x_{ij} = x_{jk} = x_{k\ell} = x_{\ell m} = x_{im} = a^{-1}, \\ x_{ik} = x_{j\ell} = x_{km} = x_{i\ell} = x_{jm} = -1 \end{aligned} \tag{20}$$

with

$$a = \frac{1}{2} - \frac{\sqrt{5}}{2}$$

is a solution of (14) and (15), because (19) is a solution of (13) and (7). Therefore, on the solution (20) as well as on all other solutions of (14), we have

$$\underline{S}^{ijk\ell m} = \frac{\pi^2}{4},$$

where we used $\Lambda(z) = \lambda(z) - \lambda(z^{-1})$, and, therefore, $\Lambda(z^{-1}) = -\Lambda(z)$. $\square$

Analogously, we get similar results for the white 4-ambo-simplex $\lceil ijk\ell m\rceil$. Here, the corner equations are:

$$\frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{ijk}} = \frac{\partial \mathscr{L}(T_k[ij\ell m])}{\partial x_{ijk}} + \frac{\partial \mathscr{L}(-T_j[ik\ell m])}{\partial x_{ijk}} + \frac{\partial \mathscr{L}(T_i[jk\ell m])}{\partial x_{ijk}} = 0$$

and

$$\frac{\partial \bar{S}^{ijk\ell m}}{\partial x_{ij\ell}} = \frac{\partial \mathscr{L}(-T_\ell[ijkm])}{\partial x_{ij\ell}} + \frac{\partial \mathscr{L}(-T_j[ik\ell m])}{\partial x_{ij\ell}} + \frac{\partial \mathscr{L}(T_i[jk\ell m])}{\partial x_{ij\ell}} = 0.$$

Explicitly, they read

$$\frac{1}{x_{ijk}} \log |E_{ijk}| = 0 \quad \text{and} \quad \frac{1}{x_{ij\ell}} \log |E_{ij\ell}| = 0, \tag{21}$$

where

$$E_{ijk} := \frac{x_{ijk}x_{k\ell m} + x_{ikm}x_{jk\ell}}{x_{ijk}x_{k\ell m} - x_{ik\ell}x_{jkm}} \cdot \frac{x_{ijk}x_{j\ell m} - x_{ij\ell}x_{jkm}}{x_{ijk}x_{j\ell m} + x_{ijm}x_{jk\ell}} \cdot \frac{x_{ijk}x_{i\ell m} + x_{ijm}x_{ik\ell}}{x_{ijk}x_{i\ell m} - x_{ij\ell}x_{ikm}}$$

and

$$E_{ij\ell} := \frac{x_{ij\ell}x_{k\ell m} - x_{ik\ell}x_{j\ell m}}{x_{ij\ell}x_{k\ell m} + x_{i\ell m}x_{jk\ell}} \cdot \frac{x_{ij\ell}x_{jkm} - x_{ijm}x_{jk\ell}}{x_{ij\ell}x_{jkm} - x_{ijk}x_{j\ell m}} \cdot \frac{x_{ij\ell}x_{ikm} - x_{ijk}x_{i\ell m}}{x_{ij\ell}x_{ikm} - x_{ijm}x_{ik\ell}}.$$

The analogue of Theorem 3.1 reads:

Theorem 3.3 *Every solution of the system* (5) *solves either the system*

$$\begin{aligned} E_{ijk} &= -1, \quad E_{ij\ell} = -1, \quad E_{ijm} = -1, \quad E_{ik\ell} = -1, \quad E_{ikm} = -1, \\ E_{i\ell m} &= -1, \quad E_{jk\ell} = -1, \quad E_{jkm} = -1, \quad E_{j\ell m} = -1, \quad E_{k\ell m} = -1 \end{aligned} \tag{22}$$

or the system

$$\begin{aligned} E_{ijk} &= 1, \quad E_{ij\ell} = 1, \quad E_{ijm} = 1, \quad E_{ik\ell} = 1, \quad E_{ikm} = 1, \\ E_{i\ell m} &= 1, \quad E_{jk\ell} = 1, \quad E_{jkm} = 1, \quad E_{j\ell m} = 1, \quad E_{k\ell m} = 1. \end{aligned} \tag{23}$$

Furthermore the system (22) *is equivalent to the system* (8) *(that is dKP on the corresponding white 4-ambo-simplex). The system* (23) *is equivalent to the system*

$$\begin{aligned}
&x_{ikm}x_{i\ell m}x_{jkm}x_{j\ell m} - x_{ijm}x_{i\ell m}x_{jkm}x_{k\ell m} + x_{ijm}x_{ikm}x_{j\ell m}x_{k\ell m} = 0,\\
&x_{ij\ell}x_{ijm}x_{ik\ell}x_{ikm} - x_{ijk}x_{ijm}x_{ik\ell}x_{i\ell m} + x_{ijk}x_{ij\ell}x_{ik\ell}x_{i\ell m} = 0,\\
&x_{jkm}x_{ijk}x_{j\ell m}x_{ij\ell} - x_{jk\ell}x_{ijk}x_{j\ell m}x_{ijm} + x_{jk\ell}x_{jkm}x_{j\ell m}x_{ijm} = 0,\\
&x_{ik\ell}x_{jk\ell}x_{ikm}x_{jkm} - x_{k\ell m}x_{jk\ell}x_{ikm}x_{ijk} + x_{k\ell m}x_{ik\ell}x_{ikm}x_{ijk} = 0,\\
&x_{j\ell m}x_{k\ell m}x_{ij\ell}x_{ik\ell} - x_{i\ell m}x_{k\ell m}x_{ij\ell}x_{jk\ell} + x_{i\ell m}x_{j\ell m}x_{ij\ell}x_{jk\ell} = 0,
\end{aligned} \tag{24}$$

which is the system (8) *after the transformation* $x \mapsto x^{-1}$ *of fields (that is dKP*$^-$ *on the corresponding white 4-ambo-simplex).*

The analogue of Theorem 3.2 reads:

Theorem 3.4 *(Closure relation) There holds:*

$$\bar{S}^{ijk\ell m} \pm \frac{\pi^2}{4} = 0$$

on all solutions of (22) *and* (23)*, respectively. Therefore, one can redefine the 3-form* $\mathscr{L}$ *as*

$$\tilde{\mathscr{L}}([ijk\ell]) := \mathscr{L}([ijk\ell]) \pm \frac{\pi^2}{4}$$

in order to get $\bar{S}^{ijk\ell m} = 0$ *on all solutions of* (22) *and* (23)*, respectively.*

4 The Cubic Lattice $\mathbb{Z}^N$

We will now consider the relation between the elementary cells of the root lattice $Q(A_N)$ and the cubic lattice $\mathbb{Z}^N$. The points of $Q(A_N)$ and of $\mathbb{Z}^N$ are in a one-to-one correspondence via

$$P_i : Q(A_N) \to \mathbb{Z}^N, \quad x(n_0, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_N) \mapsto x(n_0, \ldots, n_{i-1}, n_{i+1}, \ldots, n_N).$$

In the present paper, we will always apply P_i with $i < j, k, \ell, \ldots$

We denote by

$$\{jk\ell\} := \{x, x_j, x_k, x_\ell, x_{jk}, x_{j\ell}, x_{k\ell}, x_{jk\ell}\}$$

the oriented 3D cubes of $\mathbb{Z}^N$. We say that the 3D cube $\{jk\ell\}$ is positively oriented if $j < k < \ell$. Any permutation of two indices changes the orientation to the opposite one. Also in this case, we always write the letters in the brackets in increasing order, so, e.g., in writing $\{jk\ell\}$ we assume that $j < k < \ell$ and avoid the notation $\{kj\ell\}$ or $\{j\ell k\}$ for the negatively oriented 3D cube $-\{jk\ell\}$.

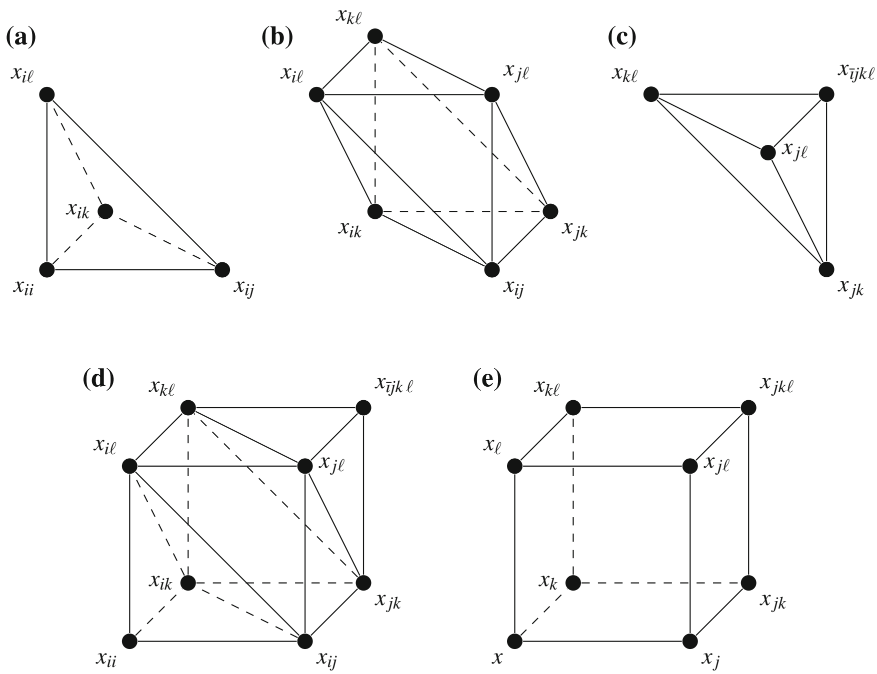

Fig. 2 Three adjacent 3-cells of the lattice $Q(A_N)$: **a** black tetrahedron $-T_i\lfloor ijk\ell\rfloor$, **b** octahedron $[ijk\ell]$, **c** white tetrahedron $-T_{\bar{\imath}}\lceil ijk\ell\rceil$. The sum **d** of these 3-cells corresponds to a 3D cube **e**

The object in $Q(A_N)$ which corresponds to the 3D cube $\{jk\ell\}$ is the sum of three adjacent 3-cells, namely

- the black tetrahedron $-T_i\lfloor ijk\ell\rfloor$ (see Fig. 2a),
- the octahedron $[ijk\ell]$ (see Fig. 2b),
- and the white tetrahedron $-T_{\bar{\imath}}\lceil ijk\ell\rceil$ (see Fig. 2c).

It contains sixteen triangles and to every quadrilateral face of $\{jkl\}$ there corresponds a pair of these triangles containing one black and one white triangle. Here, the map P_i reads as follows:

$$x_{ii} \mapsto x, \quad x_{ij} \mapsto x_j, \quad x_{jk} \mapsto x_{jk}, \quad \text{and} \quad x_{\bar{\imath}jk\ell} \mapsto x_{jk\ell}.$$

As a four-dimensional elementary cell of $\mathbb{Z}^N$, we consider an oriented 4D cube

$$\{jk\ell m\} := \{x, x_j, x_k, x_\ell, x_m, x_{jk}, x_{j\ell}, x_{jm}, x_{k\ell}, x_{km}, x_{\ell m}, x_{jk\ell}, x_{jkm}, x_{j\ell m}, x_{k\ell m}, x_{jk\ell m}\}.$$

The 4D cube $\{jk\ell m\}$ corresponds to the sum of four 4-cells in $Q(A_N)$:

- the black 4-simplex $-T_i\llbracket ijk\ell m\rrbracket$,
- the black 4-ambo-simplex $\lfloor ijk\ell m\rfloor$,

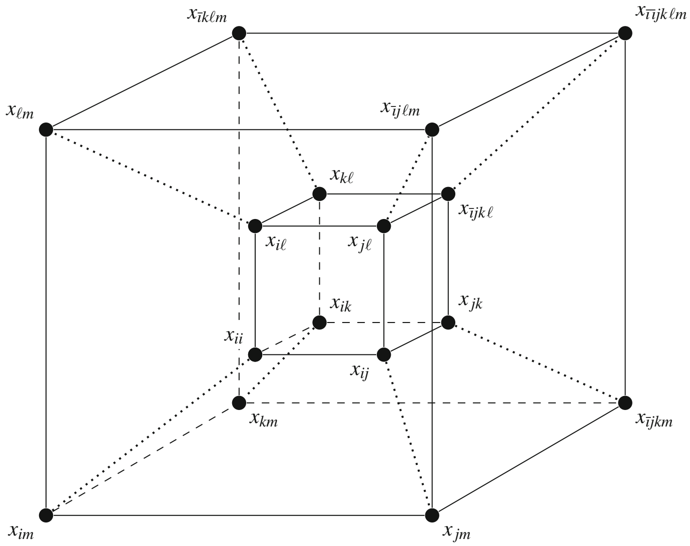

Fig. 3 The sum of the black 4-simplex $-T_i\lfloor\!\lfloor ijk\ell m\rfloor\!\rfloor$, the adjacent black 4-ambo-simplex $\lfloor ijk\ell m\rfloor$, the adjacent white 4-ambo-simplex $-T_{\bar{\imath}}\lceil ijk\ell m\rceil$, and the adjacent white 4-simplex $T_{\bar{\imath}}T_{\bar{\imath}}\lceil\!\lceil ijk\ell m\rceil\!\rceil$ corresponds to the 4D cube $\{jk\ell m\}$

- the white 4-ambo-simplex $-T_{\bar{\imath}}\lceil ijk\ell m\rceil$, and
- the white 4-simplex $T_{\bar{\imath}}T_{\bar{\imath}}\lceil\!\lceil ijk\ell m\rceil\!\rceil$

(see Fig. 3). It contains sixteen tetrahedra (eight black and eight white ones) and eight octahedra. Here, the map P_i reads as follows:

$$x_{ii} \mapsto x, \quad x_{ij} \mapsto x_j, \quad x_{jk} \mapsto x_{jk}, \quad x_{\bar{\imath}jk\ell} \mapsto x_{jk\ell}, \quad \text{and} \quad x_{\bar{\imath}\bar{\imath}jk\ell m} \mapsto x_{jk\ell m}.$$

Also in the cubic case there is an easy recipe to obtain the orientation of the facets of an (oriented) 4D cube: on every index between the brackets we put alternately a "+" and a "−" starting with a "+" on the last index. Then we get each facet by deleting one index and putting the corresponding sign in front of the bracket. For instance., the 4D cube

$$\{\overset{-}{j}\ \overset{+}{k}\ \overset{-}{\ell}\ \overset{+}{m}\}$$

has the eight 3D facets: $\{jk\ell\}$, $-\{jkm\}$, $\{j\ell m\}$, $-\{k\ell m\}$ and the opposite ones $-T_m\{jk\ell\}$, $T_\ell\{jkm\}$, $-T_k\{j\ell m\}$, and $T_j\{k\ell m\}$.

As a consequence of Definition 2.2, in each flower in $\mathbb{Z}^N$, every 3D cube has exactly four adjacent 3D cubes.

We will now prove the analogue of Theorem 2.5. This proof is easier than the one for $Q(A_N)$, because of the simpler combinatorial structure.

Theorem 4.1 *The flower at any interior vertex of any 3-manifold in $\mathbb{Z}^N$ can be represented as a sum of 4D corners in $\mathbb{Z}^{N+1}$.*

Proof Set $M := N + 1$ and consider the flower of an interior vertex x of an arbitrary 3-manifold in $\mathbb{Z}^N$. Over each 3D corner $\{jk\ell\}$ (petal) of the flower, we can build a 4D corner adjacent to x on the 4D cube $\{jk\ell M\}$. Then the vertical 3D cubes coming from two successive petals of the flower carry opposite orientations, so that all vertical squares cancel away from the sum of the 4D corners. □

Let $\mathfrak{L}$ be a discrete 3-form on $\mathbb{Z}^N$. The *exterior derivative* $d\mathfrak{L}$ is a discrete 4-form whose value at any 4D cube in $\mathbb{Z}^N$ is the action functional of $\mathfrak{L}$ on the 3-manifold consisting of the facets of the 4D cube:

$$\begin{aligned} S^{jk\ell m} :=& d\mathfrak{L}(\{jk\ell m\}) = \mathfrak{L}(\{jk\ell\}) + \mathfrak{L}(-\{jkm\}) + \mathfrak{L}(\{j\ell m\}) + \mathfrak{L}(-\{k\ell m\}) \\ &+ \mathfrak{L}(-T_m\{jk\ell\}) + \mathfrak{L}(T_\ell\{jkm\}) + \mathfrak{L}(-T_k\{j\ell m\}) + \mathfrak{L}(T_j\{k\ell m\}). \end{aligned}$$

Accordingly, the Euler-Lagrange equations on the 4D cube $\{jk\ell m\}$ are given by

$$\begin{aligned} &\frac{\partial S^{jk\ell m}}{\partial x} = 0, \\ &\frac{\partial S^{jk\ell m}}{\partial x_j} = 0, \quad \frac{\partial S^{jk\ell m}}{\partial x_k} = 0, \quad \frac{\partial S^{jk\ell m}}{\partial x_\ell} = 0, \quad \frac{\partial S^{jk\ell m}}{\partial x_m} = 0, \\ &\frac{\partial S^{jk\ell m}}{\partial x_{jk}} = 0, \quad \frac{\partial S^{jk\ell m}}{\partial x_{j\ell}} = 0, \quad \frac{\partial S^{jk\ell m}}{\partial x_{jm}} = 0, \quad \frac{\partial S^{jk\ell m}}{\partial x_{k\ell}} = 0, \quad \frac{\partial S^{jk\ell m}}{\partial x_{km}} = 0, \quad \frac{\partial S^{jk\ell m}}{\partial x_{\ell m}} = 0, \\ &\frac{\partial S^{jk\ell m}}{\partial x_{jk\ell}} = 0, \quad \frac{\partial S^{jk\ell m}}{\partial x_{jkm}} = 0, \quad \frac{\partial S^{jk\ell m}}{\partial x_{j\ell m}} = 0, \quad \frac{\partial S^{jk\ell m}}{\partial x_{k\ell m}} = 0, \\ &\frac{\partial S^{jk\ell m}}{\partial x_{jk\ell m}} = 0. \end{aligned} \tag{25}$$

They are called *corner equations*.

The following statement is an immediate consequence of Theorem 4.1:

Theorem 4.2 *For every discrete 3-form on $\mathbb{Z}^N$ and every 3-manifold in $\mathbb{Z}^N$ all corresponding Euler-Lagrange equations can be written as a sum of corner equations.*

5 The dKP Equation on $\mathbb{Z}^N$

On the 3D cube $\{jk\ell\}$ in $\mathbb{Z}^3$ $(j < k < \ell)$ we put the equation

$$x_j x_{k\ell} - x_k x_{j\ell} + x_\ell x_{jk} = 0. \tag{26}$$

We can extend this system in a consistent way (see [1]) to the four-dimensional cubic lattice $\mathbb{Z}^4$ and its higher-dimensional analogues, such that the eight facets $\{jk\ell\}$,

$-\{jkm\}$, $\{j\ell m\}$, $-\{k\ell m\}$, $-T_m\{jk\ell\}$, $T_\ell\{jkm\}$, $-T_k\{j\ell m\}$, $T_j\{k\ell m\}$ of a 4D cube $\{jk\ell m\}$ carry the equations

$$\begin{aligned}
&x_j x_{k\ell} - x_k x_{j\ell} + x_\ell x_{jk} = 0, && x_{jm}x_{k\ell m} - x_{km}x_{j\ell m} + x_{\ell m}x_{jkm} = 0,\\
&x_j x_{km} - x_k x_{jm} + x_m x_{jk} = 0, && x_{jk}x_{k\ell m} - x_{k\ell}x_{jkm} + x_{km}x_{jk\ell} = 0,\\
&x_j x_{\ell m} - x_\ell x_{jm} + x_m x_{j\ell} = 0, && x_{j\ell}x_{k\ell m} - x_{k\ell}x_{j\ell m} + x_{\ell m}x_{jk\ell} = 0,\\
&x_k x_{\ell m} - x_\ell x_{km} + x_m x_{k\ell} = 0, && x_{jk}x_{j\ell m} - x_{j\ell}x_{jkm} + x_{jm}x_{jk\ell} = 0.
\end{aligned} \tag{27}$$

Note that, in the four equations in the left column, the fields with one index always appear with increasing order of indices. The equations in the right column are shifted copies of the ones in the left column. One can derive the system (27) from the system of dKP equations (7) on the black 4-ambo-simplex $\lfloor ijk\ell m\rfloor$ and the system of dKP equations (8) on the white 4-ambo-simplex $T_{\bar{i}}\lceil ijkm\ell\rceil$, by removing the equations on the octahedra $[jk\ell m]$ and $[jkm\ell]$, respectively, from both systems and applying the transformation P_i to the fields in the remaining eight equations.

We propose the discrete 3-form $\mathfrak{L}$ defined as

$$\mathfrak{L} := (P_i)_\star \mathscr{L},$$

where $\mathscr{L}$ is the discrete 3-form on the root lattice $Q(A_N)$ (see (9)). Therefore, $\mathfrak{L}$ evaluated at the 3D cube $\{jk\ell\}$ reads as

$$\begin{aligned}
\mathfrak{L}(\{jk\ell\}) &= ((P_i)_\star \mathscr{L})(P_i(-T_i\lfloor ijk\ell\rfloor + [ijk\ell] - T_{\bar{i}}\lceil ijk\ell\rceil))\\
&= (P_i)_\star(\underbrace{\mathscr{L}(-T_i\lfloor ijk\ell\rfloor)}_{=0} + \mathscr{L}([ijk\ell]) - \underbrace{\mathscr{L}(-T_{\bar{i}}\lceil ijk\ell\rceil)}_{=0}) = (P_i)_\star \mathscr{L}([ijk\ell]).
\end{aligned}$$

For this discrete 3-form, there are no corner equations on the 4D cube $\{jk\ell m\}$ centered at x and $x_{jk\ell m}$ since $S^{jk\ell m}$ does not depend on these two variables. The remaining corner equations from (25) are given by

$$\begin{aligned}
\frac{\partial S^{jk\ell m}}{\partial x_j} =& \frac{\partial \mathfrak{L}(\{jk\ell\})}{\partial x_j} + \frac{\partial \mathfrak{L}(-\{jkm\})}{\partial x_j} + \frac{\partial \mathfrak{L}(\{j\ell m\})}{\partial x_j} + \underbrace{\frac{\partial \mathfrak{L}(T_j\{k\ell m\})}{\partial x_j}}_{\equiv 0}\\
=& (P_i)_\star \left(\frac{\partial \mathscr{L}([ijk\ell])}{\partial x_{ij}} + \frac{\partial \mathscr{L}(-[ijkm])}{\partial x_{ij}} + \frac{\partial \mathscr{L}([ij\ell m])}{\partial x_{ij}} \right) = \frac{1}{x_j}\log|\mathscr{E}_j| = 0,\\
\frac{\partial S^{jk\ell m}}{\partial x_{jk}} =& \frac{\partial \mathfrak{L}(\{jk\ell\})}{\partial x_{jk}} + \frac{\partial \mathfrak{L}(-\{jkm\})}{\partial x_{jk}} + \frac{\partial \mathfrak{L}(-T_k\{j\ell m\})}{\partial x_{jk}} + \frac{\partial \mathfrak{L}(T_j\{k\ell m\})}{\partial x_{jk}}\\
=& (P_i)_\star \left(\frac{\partial \mathscr{L}([ijk\ell])}{\partial x_{jk}} + \frac{\partial \mathscr{L}(-[ijkm])}{\partial x_{jk}} + \frac{\partial \mathscr{L}(-T_{\bar{i}}T_k[ij\ell m])}{\partial x_{jk}} \right.\\
&\left. + \frac{\partial \mathscr{L}(T_{\bar{i}}T_j[ik\ell m])}{\partial x_{jk}} \right)\\
=& \frac{1}{x_{jk}}\log\left|\frac{\mathscr{E}_{jk}}{\bar{\mathscr{E}}_{jk}}\right| = 0,
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial S^{jk\ell m}}{\partial x_{jk\ell}} =& \underbrace{\frac{\partial \mathfrak{L}(\{jk\ell\})}{\partial x_{jk\ell}}}_{\equiv 0} + \frac{\partial \mathfrak{L}(T_\ell\{jkm\})}{\partial x_{jk\ell}} + \frac{\partial \mathfrak{L}(-T_k\{j\ell m\})}{\partial x_{jk\ell}} + \frac{\partial \mathfrak{L}(T_j\{k\ell m\})}{\partial x_{jk\ell}} \\
=& (P_i)_\star \left(\frac{\partial \mathscr{L}(T_{\bar{\imath}} T_\ell [ijkm])}{\partial x_{\bar{\imath} jk\ell}} + \frac{\partial \mathscr{L}(-T_{\bar{\imath}} T_k [ij\ell m])}{\partial x_{\bar{\imath} jk\ell}} + \frac{\partial \mathscr{L}(T_{\bar{\imath}} T_j [ik\ell m])}{\partial x_{\bar{\imath} jk\ell}} \right) \\
=& \frac{1}{x_{jk\ell}} \log \left| \frac{1}{\mathscr{E}_{jk\ell}} \right| = 0,
\end{aligned}
\tag{28}
$$

where

$$
\mathscr{E}_j := (P_i)_\star E_{ij}, \quad \underline{\mathscr{E}}_{jk} := (P_i)_\star E_{jk}, \quad \bar{\mathscr{E}}_{jk} := (P_i)_\star E_{ijk}, \quad \text{and} \quad \mathscr{E}_{jk\ell} := (P_i)_\star E_{jk\ell}.
$$

Hereafter, we only consider solutions, where all fields are non-zero (we call these solutions non-singular). As in the case of the root lattice $Q(A_N)$ every corner equation has two classes of solutions.

Theorem 5.1 *Every solution of the system* (25) *solves either the system*

$$
\begin{aligned}
&\mathscr{E}_j = -1, \quad \mathscr{E}_k = -1, \quad \mathscr{E}_\ell = -1, \quad \mathscr{E}_m = -1, \\
&\underline{\mathscr{E}}_{jk} = -1, \quad \underline{\mathscr{E}}_{j\ell} = -1, \quad \underline{\mathscr{E}}_{jm} = -1, \quad \underline{\mathscr{E}}_{k\ell} = -1, \quad \underline{\mathscr{E}}_{km} = -1, \quad \underline{\mathscr{E}}_{\ell m} = -1, \\
&\bar{\mathscr{E}}_{jk} = -1, \quad \bar{\mathscr{E}}_{j\ell} = -1, \quad \bar{\mathscr{E}}_{jm} = -1, \quad \bar{\mathscr{E}}_{k\ell} = -1, \quad \bar{\mathscr{E}}_{km} = -1, \quad \bar{\mathscr{E}}_{\ell m} = -1, \\
&\mathscr{E}_{jk\ell} = -1, \quad \mathscr{E}_{jkm} = -1, \quad \mathscr{E}_{j\ell m} = -1, \quad \mathscr{E}_{k\ell m} = -1
\end{aligned}
\tag{29}
$$

or the system

$$
\begin{aligned}
&\mathscr{E}_j = 1, \quad \mathscr{E}_k = 1, \quad \mathscr{E}_\ell = 1, \quad \mathscr{E}_m = 1, \\
&\underline{\mathscr{E}}_{jk} = 1, \quad \underline{\mathscr{E}}_{j\ell} = 1, \quad \underline{\mathscr{E}}_{jm} = 1, \quad \underline{\mathscr{E}}_{k\ell} = 1, \quad \underline{\mathscr{E}}_{km} = 1, \quad \underline{\mathscr{E}}_{\ell m} = 1, \\
&\bar{\mathscr{E}}_{jk} = 1, \quad \bar{\mathscr{E}}_{j\ell} = 1, \quad \bar{\mathscr{E}}_{jm} = 1, \quad \bar{\mathscr{E}}_{k\ell} = 1, \quad \bar{\mathscr{E}}_{km} = 1, \quad \bar{\mathscr{E}}_{\ell m} = 1, \\
&\mathscr{E}_{jk\ell} = 1, \quad \mathscr{E}_{jkm} = 1, \quad \mathscr{E}_{j\ell m} = 1, \quad \mathscr{E}_{k\ell m} = 1.
\end{aligned}
\tag{30}
$$

Furthermore the system (29) *is equivalent to the system* (27) *(this is dKP on the corresponding 4D cube). The system* (30) *is equivalent to the system*

$$
\begin{aligned}
&x_k x_\ell x_{jk} x_{j\ell} - x_j x_\ell x_{jk} x_{k\ell} + x_j x_k x_{j\ell} x_{k\ell} = 0, \\
&x_k x_m x_{jk} x_{jm} - x_j x_m x_{jk} x_{km} + x_j x_k x_{jm} x_{km} = 0, \\
&x_\ell x_m x_{j\ell} x_{jm} - x_j x_m x_{j\ell} x_{\ell m} + x_j x_\ell x_{jm} x_{\ell m} = 0, \\
&x_\ell x_m x_{k\ell} x_{km} - x_k x_m x_{k\ell} x_{\ell m} + x_k x_\ell x_{km} x_{\ell m} = 0, \\
&x_{km} x_{\ell m} x_{jkm} x_{j\ell m} - x_{jm} x_{\ell m} x_{jkm} x_{k\ell m} + x_{jm} x_{km} x_{j\ell m} x_{k\ell m} = 0, \\
&x_{k\ell} x_{\ell m} x_{jk\ell} x_{j\ell m} - x_{j\ell} x_{\ell m} x_{jk\ell} x_{k\ell m} + x_{j\ell} x_{k\ell} x_{j\ell m} x_{k\ell m} = 0, \\
&x_{k\ell} x_{km} x_{jk\ell} x_{jkm} - x_{jk} x_{km} x_{jk\ell} x_{k\ell m} + x_{jk} x_{k\ell} x_{jkm} x_{k\ell m} = 0, \\
&x_{j\ell} x_{jm} x_{jk\ell} x_{jkm} - x_{jk} x_{jm} x_{jk\ell} x_{j\ell m} + x_{jk} x_{j\ell} x_{jkm} x_{j\ell m} = 0,
\end{aligned}
\tag{31}
$$

which is the system (27) after the transformation $x \mapsto x^{-1}$ of fields (this is dKP$^-$ on the corresponding 4D cube).

Proof Let x be a solution of the system (25) such that $\mathscr{E}_j = -1$ and $\mathscr{E}_k = -1$. Then we know from the proof of Theorem 3.1 that

$$\begin{gathered}
\mathscr{E}_j = -1, \quad \mathscr{E}_k = -1, \quad \mathscr{E}_\ell = -1, \quad \mathscr{E}_m = -1, \\
\underline{\mathscr{E}}_{jk} = -1, \quad \underline{\mathscr{E}}_{j\ell} = -1, \quad \underline{\mathscr{E}}_{jm} = -1, \quad \underline{\mathscr{E}}_{k\ell} = -1, \quad \underline{\mathscr{E}}_{km} = -1, \quad \underline{\mathscr{E}}_{\ell m} = -1
\end{gathered}$$

and that the latter system is equivalent to

$$\begin{aligned}
&x_j x_{k\ell} - x_k x_{j\ell} + x_\ell x_{jk} = 0, \\
&x_j x_{km} - x_k x_{jm} + x_m x_{jk} = 0, \\
&x_j x_{\ell m} - x_\ell x_{jm} + x_m x_{j\ell} = 0, \\
&x_k x_{\ell m} - x_\ell x_{km} + x_m x_{k\ell} = 0, \\
&x_{jk} x_{\ell m} - x_{j\ell} x_{km} + x_{jm} x_{k\ell} = 0.
\end{aligned}$$

On the other hand, if we consider a solution x of (25) such that $\mathscr{E}_j = 1$ and $\mathscr{E}_k = 1$, we know from the proof of Theorem 3.1 that

$$\begin{gathered}
\mathscr{E}_j = 1, \quad \mathscr{E}_k = 1, \quad \mathscr{E}_\ell = 1, \quad \mathscr{E}_m = 1, \\
\underline{\mathscr{E}}_{jk} = 1, \quad \underline{\mathscr{E}}_{j\ell} = 1, \quad \underline{\mathscr{E}}_{jm} = 1, \quad \underline{\mathscr{E}}_{k\ell} = 1, \quad \underline{\mathscr{E}}_{km} = 1, \quad \underline{\mathscr{E}}_{\ell m} = 1
\end{gathered}$$

and that the latter system is equivalent to

$$\begin{aligned}
&x_k x_\ell x_{jk} x_{j\ell} - x_j x_\ell x_{jk} x_{k\ell} + x_j x_k x_{j\ell} x_{k\ell} = 0, \\
&x_k x_m x_{jk} x_{jm} - x_j x_m x_{jk} x_{km} + x_j x_k x_{jm} x_{km} = 0, \\
&x_\ell x_m x_{j\ell} x_{jm} - x_j x_m x_{j\ell} x_{\ell m} + x_j x_\ell x_{jm} x_{\ell m} = 0, \\
&x_\ell x_m x_{k\ell} x_{km} - x_k x_m x_{k\ell} x_{\ell m} + x_k x_\ell x_{km} x_{\ell m} = 0, \\
&x_{j\ell} x_{jm} x_{k\ell} x_{km} - x_{jk} x_{jm} x_{k\ell} x_{\ell m} + x_{jk} x_{j\ell} x_{km} x_{\ell m} = 0.
\end{aligned}$$

Now, let x be a solution of the system (25) such that $\mathscr{E}_{jk\ell} = -1$ and $\mathscr{E}_{jkm} = -1$. Then we know from the proof of Theorem 3.3 that

$$\begin{gathered}
\bar{\mathscr{E}}_{jk} = 1, \quad \bar{\mathscr{E}}_{j\ell} = 1, \quad \bar{\mathscr{E}}_{jm} = 1, \quad \bar{\mathscr{E}}_{k\ell} = 1, \quad \bar{\mathscr{E}}_{km} = 1, \quad \bar{\mathscr{E}}_{\ell m} = 1, \\
\mathscr{E}_{jk\ell} = 1, \quad \mathscr{E}_{jkm} = 1, \quad \mathscr{E}_{j\ell m} = 1, \quad \mathscr{E}_{k\ell m} = 1
\end{gathered}$$

and that the latter system is equivalent to

$$\begin{aligned}
x_{\ell m}x_{jkm} - x_{km}x_{j\ell m} + x_{jm}x_{k\ell m} &= 0,\\
x_{km}x_{jk\ell} - x_{k\ell}x_{jkm} + x_{jk}x_{k\ell m} &= 0,\\
x_{\ell m}x_{jk\ell} - x_{k\ell}x_{j\ell m} + x_{j\ell}x_{k\ell m} &= 0,\\
x_{jm}x_{jk\ell} - x_{j\ell}x_{jkm} + x_{jk}x_{j\ell m} &= 0,\\
x_{jk}x_{\ell m} - x_{j\ell}x_{km} + x_{jm}x_{k\ell} &= 0.
\end{aligned}$$

On the other hand, if we consider a solution x of (25) such that $\mathscr{E}_j = 1$ and $\mathscr{E}_k = 1$, we know from the proof of Theorem 3.3 that

$$\begin{aligned}
&\bar{\mathscr{E}}_{jk} = 1, \quad \bar{\mathscr{E}}_{j\ell} = 1, \quad \bar{\mathscr{E}}_{jm} = 1, \quad \bar{\mathscr{E}}_{k\ell} = 1, \quad \bar{\mathscr{E}}_{km} = 1, \quad \bar{\mathscr{E}}_{\ell m} = 1,\\
&\mathscr{E}_{jk\ell} = 1, \quad \mathscr{E}_{jkm} = 1, \quad \mathscr{E}_{j\ell m} = 1, \quad \mathscr{E}_{k\ell m} = 1
\end{aligned}$$

and that the latter system is equivalent to

$$\begin{aligned}
x_{km}x_{\ell m}x_{jkm}x_{j\ell m} - x_{jm}x_{\ell m}x_{jkm}x_{k\ell m} + x_{jm}x_{km}x_{j\ell m}x_{k\ell m} &= 0,\\
x_{k\ell}x_{\ell m}x_{jk\ell}x_{j\ell m} - x_{j\ell}x_{\ell m}x_{jk\ell}x_{k\ell m} + x_{j\ell}x_{k\ell}x_{j\ell m}x_{k\ell m} &= 0,\\
x_{k\ell}x_{km}x_{jk\ell}x_{jkm} - x_{jk}x_{km}x_{jk\ell}x_{k\ell m} + x_{jk}x_{k\ell}x_{jkm}x_{k\ell m} &= 0,\\
x_{j\ell}x_{jm}x_{jk\ell}x_{jkm} - x_{jk}x_{jm}x_{jk\ell}x_{j\ell m} + x_{jk}x_{j\ell}x_{jkm}x_{j\ell m} &= 0,\\
x_{j\ell}x_{jm}x_{k\ell}x_{km} - x_{jk}x_{jm}x_{k\ell}x_{\ell m} + x_{jk}x_{j\ell}x_{km}x_{\ell m} &= 0.
\end{aligned}$$

Since a solution x of (25) cannot solve

$$x_{jk}x_{\ell m} - x_{j\ell}x_{km} + x_{jm}x_{k\ell} = 0$$

and

$$x_{j\ell}x_{jm}x_{k\ell}x_{km} - x_{jk}x_{jm}x_{k\ell}x_{\ell m} + x_{jk}x_{j\ell}x_{km}x_{\ell m} = 0$$

at the same time, this proves the theorem. □

Theorem 5.2 *(Closure relation) There holds $S^{jk\ell m} = 0$ on all solutions of* (25).

Proof Let x be a solution of (29) or (30). Then

$$\begin{aligned}
S^{jk\ell m} &= d\mathfrak{L}(\{jk\ell\}) = (P_i)_\star(d\mathscr{L}(\lfloor ijk\ell m\rfloor) + d\mathscr{L}(-\lceil ijk\ell m\rceil)) = \underline{S}^{ijk\ell m} - \bar{S}^{ijk\ell m}\\
&= \pm\frac{\pi^2}{4} \mp \frac{\pi^2}{4} = 0
\end{aligned}$$

due to Theorems 3.2 and 3.4 since every solution of (29) solves (13) and (22) after the transformation P_i of variables and every solution of (30) solves (14) and (23) after the transformation P_i of variables. □

6 Conclusion

The fact that the three-dimensional (hyperbolic) dKP equation is, in a sense, equivalent to the Euler-Lagrange equations of the corresponding action is rather surprising since for the two-dimensional (hyperbolic) quad-equations an analogous statement is not true (see [4, 6] for more details). On the other hand, in the continuous situation there is an example of a 2-form whose Euler-Lagrange equations are equivalent to the set of equations consisting of the (hyperbolic) sine-Gordon equation and the (evolutionary) modified Korteweg-de Vries equation (see [16] for more details). So, the general picture remains unclear.

In particular, the variational formulation for the other equations of octahedron type in the classification of [1] is still an open problem.

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Appendix 1: Facets of N-Cells of the Root Lattice $Q(A_N)$

Facets of 3-cells:

Black tetrahedra $\lfloor ijk\ell\rfloor$:	four black triangles $\lfloor ijk\rfloor$, $-\lfloor ij\ell\rfloor$, $\lfloor ik\ell\rfloor$, and $-\lfloor jk\ell\rfloor$;
Octahedra $[ijk\ell]$:	four black triangles $T_\ell\lfloor ijk\rfloor$, $-T_k\lfloor ij\ell\rfloor$, $T_j\lfloor ik\ell\rfloor$, and $-T_i\lfloor jk\ell\rfloor$, four white triangles $\lceil ijk\rceil$, $-\lceil ij\ell\rceil$, $\lceil ik\ell\rceil$, and $-\lceil jk\ell\rceil$;
White tetrahedra $\lceil ijk\ell\rceil$:	four white triangles $T_\ell\lceil ijk\rceil$, $-T_k\lceil ij\ell\rceil$, $T_j\lceil ik\ell\rceil$, and $-T_i\lceil jk\ell\rceil$;

Facets of 4-cells:

Black 4-simplices $\llbracket ijk\ell m\rrbracket$:	$five black tetrahedra$ $\lfloor ijk\ell\rfloor$, $-\lfloor ijkm\rfloor$, $\lfloor ij\ell m\rfloor$, $-\lfloor ik\ell m\rfloor$, and $\lfloor jk\ell m\rfloor$;
Black 4-ambo-simplices $\lfloor ijk\ell m\rfloor$:	five black tetrahedra $T_m\lfloor ijk\ell\rfloor$, $-T_\ell\lfloor ijkm\rfloor$, $T_k\lfloor ij\ell m\rfloor$, $-T_j\lfloor ik\ell m\rfloor$, and $T_i\lfloor jk\ell m\rfloor$, and five octahedra $[ijk\ell]$, $-[ijkm]$, $[ij\ell m]$, $-[ik\ell m]$, and $[jk\ell m]$;
White 4-ambo-simplices $\lceil ijk\ell m\rceil$:	five octahedra $T_m[ijk\ell]$, $-T_\ell[ijkm]$, $T_k[ij\ell m]$, $-T_j[ik\ell m]$, and $T_i[jk\ell m]$, and five white tetrahedra $\lceil ijk\ell\rceil$, $-\lceil ijkm\rceil$, $\lceil ij\ell m\rceil$, $-\lceil ik\ell m\rceil$, and $\lceil jk\ell m\rceil$;
White 4-simplices $\llbracket ijk\ell m\rrbracket$:	five white tetrahedra $T_m\lceil ijk\ell\rceil$, $-T_\ell\lceil ijkm\rceil$, $T_k\lceil ij\ell m\rceil$, $-T_j\lceil ik\ell m\rceil$, and $T_i\lceil jk\ell m\rceil$.

Appendix 2: 4D Corners on 4-Cells of the Root Lattice $Q(A_N)$

Black 4-simplex $\llbracket ijk\ell m \rrbracket$:
The 4D corner with center vertex x_i contains

- the four black tetrahedra $\lfloor ijk\ell \rfloor$, $-\lfloor ijkm \rfloor$, $\lfloor ij\ell m \rfloor$, and $-\lfloor ik\ell m \rfloor$;

Black 4-ambo-simplex $\lfloor ijk\ell m \rfloor$:
The 4D corner with center vertex x_{ij} contains

- the two black tetrahedra $-T_j \lfloor ik\ell m \rfloor$, and $T_i \lfloor jk\ell m \rfloor$,
- and the three octahedra $[ijk\ell]$, $-[ijkm]$, and $[ij\ell m]$;

White 4-ambo-simplex $\lceil ijk\ell m \rceil$:
The 4D corner with center vertex x_{ijk} contains

- the three octahedra $T_k[ij\ell m]$, $-T_j[ik\ell m]$, and $T_i[jk\ell m]$,
- and the two white tetrahedra $\lceil ijk\ell \rceil$, and $-\lceil ijkm \rceil$;

White 4-simplex $\llbracket ijk\ell m \rrbracket$:
The 4D corner with center vertex $x_{ijk\ell}$ contains

- the four white tetrahedra $-T_\ell \lceil ijkm \rceil$, $T_k \lceil ij\ell m \rceil$, $-T_j \lceil ik\ell m \rceil$, and $T_i \lceil jk\ell m \rceil$.

Appendix 3: Proof of Theorem 2.4

Set $M := N + 1$ and $L := N + 2$. Then, for the construction of the sum Σ of 4D corners representing the flower σ centered in X, we use the following algorithm:

(i) For every black tetrahedron $\pm\lfloor ijk\ell \rfloor \in \sigma$ at the interior vertex X we add the 4D corner with center vertex X on the black 4-simplex $\pm\llbracket ijk\ell M \rrbracket$ to Σ.
(ii) For every octahedron $\pm[ijk\ell] \in \sigma$ we add the 4D corner with center vertex X on the black 4-ambo-simplex $\pm\lfloor ijk\ell M \rfloor$ to Σ.
(iii) For every white tetrahedron $\pm\lceil ijk\ell \rceil \in \sigma$ we add the 4D corner with center vertex X on the white 4-ambo-simplex $\pm\lceil ijk\ell M \rceil$ to Σ.
(iv) For every white tetrahedron $\pm\lceil ijkM \rceil \in \Sigma \setminus \sigma$ which appeared in Σ during the previous step we add the 4D corner with center vertex X on the white 4-simplex $\mp T_{\bar{L}} \llbracket ijkML \rrbracket$ to Σ.

Therefore, we have to prove that $\Sigma = \sigma$.

Assume that $X = x_i$. Then for each black tetrahedron $\pm\lfloor ijk\ell \rfloor \in \sigma$ we added the three black tetrahedra $\mp\lfloor ijkM \rfloor$, $\pm\lfloor ij\ell M \rfloor$, and $\mp\lfloor ik\ell M \rfloor$ to Σ which do not belong to σ. Moreover, $\pm\lfloor ijk\ell \rfloor$ has three black triangular facets adjacent to x_i, namely $\pm\lfloor ijk \rfloor$, which is the common triangle with $\mp\lfloor ijkM \rfloor$ (up to orientation), $\mp\lfloor ij\ell \rfloor$, which is the common triangle with $\pm\lfloor ij\ell M \rfloor$, and $\pm\lfloor ik\ell \rfloor$, which is the common triangle with $\mp\lfloor ik\ell M \rfloor$. Therefore, each of these black tetrahedra has to