Chapter 6

Pseudo-differential operators on the Heisenberg group

The Heisenberg group was introduced in Example 1.6.4. It was our primal example of a stratified Lie group, see Section 3.1.1. Due to the importance of the Heisenberg group and of its many realisations, we start this chapter by sketching various descriptions of the Heisenberg group. We also describe its dual via the well known Schrödinger representations. Eventually, we particularise our general approach given in Chapter 5 to the Heisenberg group. Among other things, we show that using the (Euclidean) Weyl quantization, the analysis of pseudo-differential operators on the Heisenberg group can be reduced to considering scalar-valued symbols parametrised not only by the elements of the Heisenberg group but also by a parameter $\lambda \in \mathbb{R}\backslash\{0\}$; such symbols will be called λ-symbols. The corresponding classes of symbols are of Shubin-type but with an interesting dependence on λ which we explore in detail in this chapter; such classes will be called λ-Shubin classes. Some results of this chapter have been announced in the authors' paper [FR14b], this chapter contains their proofs.

In [BFKG12a], a pseudo-differential calculus on the Heisenberg group was developed with a different approach (but related results) from our work presented here.

There is an important change of notation concerning the Heisenberg group in this chapter. In Example 1.6.4, where the Heisenberg group $\mathbb{H}_{n_o}$ was introduced, we used the index n_o as its subscript because the index n was already used to denote quantities associated with the homogeneous groups. However, throughout Chapter 6, general groups will hardly appear, so we can simplify the notation by denoting the Heisenberg group by $\mathbb{H}_n$ instead of $\mathbb{H}_{n_o}$, so that the notation change is

$$\boxed{\mathbb{H}_{n_o} \longrightarrow \mathbb{H}_n}$$

V. Fischer, M. Ruzhansky, *Quantization on Nilpotent Lie Groups*, Progress in Mathematics 314,
DOI 10.1007/978-3-319-29558-9_6

We emphasise that n is the index here (not the dimension): the topological dimension on $\mathbb{H}_n$ is $2n+1$, and its homogeneous dimension is $2n+2$.

6.1 Preliminaries

In this section, we discuss several aspects of the Heisenberg group, hopefully shedding some light on its importance and general structure.

6.1.1 Descriptions of the Heisenberg group

We remind the reader that the Heisenberg group $\mathbb{H}_n$ was defined in Example 1.6.4 in the following way: the *Heisenberg group* $\mathbb{H}_n$ is the manifold $\mathbb{R}^{2n+1}$ endowed with the law

$$(x,y,t)(x',y',t') := (x+x', y+y', t+t' + \frac{1}{2}(xy' - x'y)), \tag{6.1}$$

where (x,y,t) and (x',y',t') are in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \sim \mathbb{H}_n$.

In the formula above as in the whole chapter, we adopt the following convention: if x and y are two vectors in $\mathbb{R}^n$ for some $n \in \mathbb{N}$, then xy denotes their standard scalar product

$$xy = \sum_{j=1}^{n} x_j y_j \quad \text{if} \quad x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n).$$

First we remark that the factor $\frac{1}{2}$ in the group law given by (6.1) is irrelevant in the following sense. Let $\alpha \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$. Consider the group $\mathbb{H}_n^{(\alpha)}$ endowed with the law

$$(x,y,t)(x',y',t') := (x+x', y+y', t+t' + \frac{1}{\alpha}(xy' - x'y)).$$

Then the groups $\mathbb{H}_n^{(\alpha)}$ and $\mathbb{H}_n = \mathbb{H}_n^{(2)}$ are isomorphic via

$$\left\{ \begin{array}{rcl} \mathbb{H}_n & \longrightarrow & \mathbb{H}_n^{(\alpha)} \\ (x,y,t) & \longmapsto & (x,y,\frac{2}{\alpha}t) \end{array} \right. .$$

In the same way, consider the *polarised Heisenberg group* $\tilde{\mathbb{H}}_n$ (or $\mathbb{H}_n^{pol}$) endowed with the law

$$(x,y,t)(x',y',t') := (x+x', y+y', t+t' + xy').$$

Then the groups $\tilde{\mathbb{H}}_n$ and $\mathbb{H}_n$ are isomorphic via

$$\left\{ \begin{array}{rcl} \mathbb{H}_n & \longrightarrow & \tilde{\mathbb{H}}_n \\ (x,y,t) & \longmapsto & (x,y,t+\frac{1}{2}xy) \end{array} \right. .$$

Note that the Heisenberg group $\mathbb{H}_n$ can be also viewed as a matrix group. For simplicity, we consider $n = 1$, in which case the group $\tilde{\mathbb{H}}_1$ is isomorphic to T_3, the group of 3-by-3 upper triangular real matrices with 1 on the diagonal:

$$\begin{cases} \tilde{\mathbb{H}}_1 & \longrightarrow & T_3 \\ (x, y, t) & \longmapsto & \begin{bmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \end{cases}.$$

All the statements above can be readily checked by a straightforward computation. Combining two isomorphisms above, we obtain the identification $\mathbb{H}_1 \longrightarrow \tilde{\mathbb{H}}_1 \longrightarrow T_3$ given by

$$\begin{cases} \mathbb{H}_1 & \longrightarrow & T_3 \\ (x, y, t) & \longmapsto & \begin{bmatrix} 1 & x & t + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \end{cases}.$$

Although we will not use it, let us mention a couple of other important appearances of the Heisenberg group. The Heisenberg group can be also realised as a group of transformations; for example, for each

$$h = (x, y, t) \in \mathbb{H}_1,$$

the affine (holomorphic) map given by

$$\phi_h : \mathbb{C} \times \mathbb{C} \ni (z_1, z_2) \longmapsto (z_1 + x + iy, z_2 + t + 2iz_1(x - iy) + i(x^2 + y^2)) \in \mathbb{C} \times \mathbb{C},$$

sends the (Siegel) domain

$$\mathscr{U} := \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \ : \ \operatorname{Im} z_2 > |z_1|^2\} \quad (= \mathrm{SU}(2,1)/\mathrm{U}(2))$$

to itself, and the (Shilov) boundary of $\mathscr{U}$,

$$b\mathscr{U} := \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \ : \ \operatorname{Im} z_2 = |z_1|^2\},$$

also to itself. One can check that $\mathbb{H}_1 \ni h \mapsto \phi_h$ defines an action of $\mathbb{H}_1$ on $\mathscr{U}$ and on $b\mathscr{U}$. Furthermore, the action of $\mathbb{H}_1$ on $b\mathscr{U}$ is simply transitive. A Cayley type transform

$$(w_1, w_2) \longmapsto (z_1, z_2) \quad \text{with} \quad z_1 = \frac{w_1}{1 + w_2}, \quad z_2 = i\frac{1 - w_2}{1 + w_2},$$

is a biholomorphic bijective mapping which sends $\mathscr{U}$ onto the unit complex ball of $\mathbb{C}^2$. It also send $b\mathscr{U}$ to the unit complex sphere $\mathbb{S}^3$, more precisely onto $\mathbb{S}^3\backslash\{S\}$ where $S = (0, -1)$ is the south pole (which may be viewed as the image of ∞). Hence the Heisenberg group acts simply transitively on $\mathbb{S}^3\backslash\{S\}$.

We can also mention here that the group $\mathrm{U}(n)$ acts naturally by automorphisms on $\mathbb{H}_n$ leading to the interpretation of $(\mathrm{U}(n), \mathbb{H}_n)$ as a nilpotent Gelfand pair with strong relation to the theory of commutative convolution algebras. For example, such analysis can be used to characterise Gelfand (spherical) transforms of K-invariant Schwartz functions on $\mathbb{H}_n$ for a group $K \subset \mathrm{U}(n)$ ([BJR98]), or view them as Schwartz functions on the Gelfand spectrum ([ADBR09]).

6.1.2 Heisenberg Lie algebra and the stratified structure

The Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$ is identified with the vector space of left-invariant vector fields. Its canonical basis is given by the left-invariant vector fields

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \ j = 1, \ldots, n, \quad \text{and } T = \partial_t. \tag{6.2}$$

For comparison, the corresponding right-invariant vector fields are

$$\tilde{X}_j = \partial_{x_j} + \frac{y_j}{2}\partial_t, \ \tilde{Y}_j = \partial_{y_j} - \frac{x_j}{2}\partial_t, \ j = 1, \ldots, n, \text{ and } \tilde{T} = \partial_t. \tag{6.3}$$

The canonical commutation relations are

$$[X_j, Y_j] = T, \quad j = 1, \ldots, n,$$

and T is the centre of $\mathfrak{h}_n$. This shows that the Lie algebra $\mathfrak{h}_n$ and the Lie group $\mathbb{H}_n$ are nilpotent of step 2. Hence the Heisenberg group $\mathbb{H}_n$ described above in Section 6.1.1, that is, $\mathbb{R}^{2n+1}$ endowed with the group law given in (6.1), is the connected simply connected (step-two nilpotent) Lie group whose Lie algebra is $\mathfrak{h}_n$ and which is realised via the exponential mapping together with the canonical basis. This means that the element $(x, y, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ of $\mathbb{H}_n$ can be written as

$$(x, y, t) = \exp_{\mathbb{H}_n}(x_1X_1 + \ldots + x_nX_n + y_1Y_1 + \ldots + y_nY_n + tT).$$

We fix

$$dxdydt = dx_1 \ldots dx_n dy_1 \ldots dy_n dt$$

as the Lebesgue measure on $\mathbb{H}_n$, see Proposition 1.6.6. Therefore, we may be free to write formulae like

$$\int_{\mathbb{H}_n} \cdots \ dxdydt = \int_{\mathbb{R}^{2n+1}} \cdots \ dxdydt.$$

The Heisenberg Lie algebra is stratified via $\mathfrak{h}_n = V_1 \oplus V_2$, where V_1 is linearly spanned by the X_j's and Y_j's, while $V_2 = \mathbb{R}T$. Since the Heisenberg Lie algebra is stratified via $\mathfrak{h}_n = V_1 \oplus V_2$, the natural dilations on the Lie algebra are given by

$$D_r(X_j) = rX_j \quad \text{and} \quad D_r(Y_j) = rY_j, \quad j = 1, \ldots, n, \quad \text{and} \quad D_r(T) = r^2T, \tag{6.4}$$

see Section 3.1.2. We keep the same notation D_r for the dilations on the group $\mathbb{H}_n$. They are therefore given by

$$D_r(x,y,t) = r(x,y,t) = (rx, ry, r^2 t), \quad (x,y,t) \in \mathbb{H}_n,\ r>0.$$

We also keep the same notation D_r for the dilations on the universal enveloping algebra $\mathfrak{U}(\mathfrak{h}_n)$ induced by Property (6.4).

Note that the homogeneous dimension of $\mathbb{H}_n$ is $Q = 2n+2$. This is also the homogeneous degree of the Lebesgue measure $dxdydt$.

Example 6.1.1. The sub-Laplacian

$$\begin{aligned} \mathcal{L} &:= \sum_{j=1}^{n} (X_j^2 + Y_j^2) \qquad (6.5) \\ &= \sum_{j=1}^{n} \left(\partial_{x_j} - \frac{y_j}{2}\partial_t\right)^2 + \left(\partial_{y_j} + \frac{x_j}{2}\partial_t\right)^2, \end{aligned}$$

is homogeneous of degree 2 since

$$D_r(\mathcal{L}) = r^2 \mathcal{L}.$$

Remark 6.1.2. The 'canonical' positive Rockland operator in this setting is

$$\mathcal{R} = -\mathcal{L}.$$

We will also use the mapping $\Theta : \mathbb{H}_n \to \mathbb{H}_n$ given by

$$\Theta(x,y,t) := (x, -y, -t).$$

One checks easily that for any $(x,y,t), (x',y',t') \in \mathbb{H}_n$, we have

$$\Theta\big((x,y,t)(x',y',t')\big) = \Theta(x,y,t)\ \Theta(x',y',t') \quad \text{and} \quad \Theta\big(\Theta(x,y,t)\big) = (x,y,t).$$

Therefore, Θ is a group automorphism and an involution. Furthermore, it is clear that it commutes with the dilations:

$$\forall r > 0 \qquad \Theta \circ D_r = D_r \circ \Theta.$$

We keep the same notation for the corresponding Lie algebra morphism and we have

$$\Theta(X_j) = X_j,\ \Theta(Y_j) = -Y_j,\ j = 1, \ldots, n,\ \Theta(T) = -T. \qquad (6.6)$$

6.2 Dual of the Heisenberg group

In this section we will analyse the unitary dual of the Heisenberg group $\mathbb{H}_n$. For our purposes, it will be more convenient to work with the Schrödinger representations. This will lead to the group Fourier transform parametrised by λ in (6.19). Such group Fourier transforms yield operators acting on the representation space $L^2(\mathbb{R}^n)$. The latter can be, in turn, analysed using the Weyl quantization on $\mathbb{R}^n$ that appears naturally.

6.2.1 Schrödinger representations π_λ

The Schrödinger representations of the Heisenberg group $\mathbb{H}_n$ are the infinite dimensional unitary representations of $\mathbb{H}_n$, where, as usual, we allow ourselves to identify unitary representations with their unitary equivalence classes. They are parametrised by the co-adjoint orbits (see Section 1.8.1) and more concretely by $\lambda \in \mathbb{R}\backslash\{0\}$. We denote these representations π_λ. Each π_λ acts on the Hilbert space

$$\mathcal{H}_{\pi_\lambda} = L^2(\mathbb{R}^n)$$

in the way we now describe. An element of $L^2(\mathbb{R}^n)$ will very often be denoted as a function h of the variable $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$.

First let us define π_1 corresponding to $\lambda = 1$. It is the representation of the group $\mathbb{H}_n$ acting on $L^2(\mathbb{R}^n)$ via

$$\pi_1(x, y, t)h(u) := e^{i(t+\frac{1}{2}xy)} e^{iyu} h(u + x),$$

for $h \in L^2(\mathbb{R}^n)$ and $(x, y, t) \in \mathbb{H}_n$. Here xy denotes the scalar product in $\mathbb{R}^n$ of x and y, and similarly for yu. Consequently its infinitesimal representation (see Section 1.7) is given by

$$\begin{cases} \pi_1(X_j) &= \partial_{u_j} \quad \text{(differentiate with respect to } u_j), \quad j = 1, \ldots, n, \\ \pi_1(Y_j) &= iu_j, \quad \text{(multiplication by } iu_j), \quad j = 1, \ldots, n, \\ \pi_1(T) &= i\mathrm{I}, \quad \text{(multiplication by } i). \end{cases} \tag{6.7}$$

The Schrödinger representations π_λ on the group are realised in this monograph using

$$\pi_\lambda := \begin{cases} \pi_1 \circ D_{\sqrt{\lambda}} & \text{if } \lambda > 0, \\ \pi_{-\lambda} \circ \Theta & \text{if } \lambda < 0, \end{cases}$$

that is,

$$\pi_\lambda(x, y, t)h(u) = e^{i\lambda(t+\frac{1}{2}xy)} e^{i\sqrt{\lambda}yu} h(u + \sqrt{|\lambda|}x), \tag{6.8}$$

for $h \in L^2(\mathbb{R}^n)$ and $(x, y, t) \in \mathbb{H}_n$ where we use the following convention:

$$\sqrt{\lambda} := \operatorname{sgn}(\lambda)\sqrt{|\lambda|} = \begin{cases} \sqrt{\lambda} & \text{if } \lambda > 0, \\ -\sqrt{|\lambda|} & \text{if } \lambda < 0. \end{cases} \tag{6.9}$$

We observe that for any $\lambda \in \mathbb{R}\backslash\{0\}$ and $r > 0$,

$$\pi_\lambda \circ \Theta = \pi_{-\lambda} \quad \text{and} \quad \pi_\lambda \circ D_r = \pi_{r^2\lambda}, \tag{6.10}$$

and this is true for the group representation π_λ on $\mathbb{H}_n$ and for its corresponding infinitesimal representation on the Lie algebra $\mathfrak{h}_n$ and on the universal enveloping algebra $\mathfrak{U}(\mathfrak{h}_n)$. As usual we keep the same notation, here π_λ for the corresponding infinitesimal representation.

Lemma 6.2.1. *The infinitesimal representation of π_λ acts on the canonical basis of $\mathfrak{h}_n$ via*

$$\pi_\lambda(X_j) = \sqrt{|\lambda|}\partial_{u_j}, \ \pi_\lambda(Y_j) = i\sqrt{\lambda}u_j, \ j = 1, \ldots, n, \quad \text{and} \quad \pi_\lambda(T) = i\lambda \mathrm{I}, \quad (6.11)$$

using the convention in (6.9).

Proof. Formulae (6.11) can be computed easily from (6.8). Here we show that they also follow from Properties (6.7) and (6.10). Indeed we have for $\lambda > 0$

$$\begin{cases} \pi_\lambda(X_j) &= \pi_1(D_{\sqrt{\lambda}}(X_j)) = \sqrt{\lambda}\pi_1(X_j) = \sqrt{\lambda}\partial_{u_j} \quad j = 1, \ldots, n, \\ \pi_\lambda(Y_j) &= \pi_1(D_{\sqrt{\lambda}}(Y_j)) = \sqrt{\lambda}\pi_1(Y_j) = \sqrt{\lambda} i u_j, \quad j = 1, \ldots, n, \\ \pi_\lambda(T) &= \pi_1(D_{\sqrt{\lambda}}(T)) = \lambda\pi_1(T) = i\lambda, \end{cases}$$

and thus for $\lambda < 0$

$$\begin{cases} \pi_\lambda(X_j) &= \pi_{-\lambda}(\Theta(X_j)) = \pi_{-\lambda}(X_j) = \sqrt{|\lambda|}\partial_{u_j} \quad j = 1, \ldots, n, \\ \pi_\lambda(Y_j) &= \pi_{-\lambda}(\Theta(Y_j)) = -\pi_{-\lambda}(Y_j) = -\sqrt{|\lambda|} i u_j, \quad j = 1, \ldots, n, \\ \pi_\lambda(T) &= \pi_{-\lambda}(\Theta(T)) = -\pi_{-\lambda}(T) = -(-\lambda) i = i\lambda, \end{cases}$$

proving (6.11) in both cases. □

Consequently, the group Fourier transform of the sub-Laplacian

$$\mathcal{L} = \sum_{j=1}^{n} (X_j^2 + Y_j^2)$$

is

$$\pi_\lambda(\mathcal{L}) = |\lambda| \sum_{j=1}^{n} (\partial_{u_j}^2 - u_j^2). \quad (6.12)$$

A direct characterisation implies that the space of smooth vectors of π_λ is

$$\mathcal{H}_{\pi_\lambda}^\infty = \mathcal{S}(\mathbb{R}^n).$$

This is true more generally for any representation of a connected simply connected nilpotent Lie group realised on some $L^2(\mathbb{R}^m)$ via the orbit method, see [CG90, Corollary 4.1.2].

6.2.2 Group Fourier transform on the Heisenberg group

We could have realised the equivalence classes $[\pi_\lambda]$ of Schrödinger representations in various ways. For instance by composing with the unitary operator $U_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by $Uf(x) = |\lambda|^{\frac{n}{2}} f(\sqrt{\lambda}x)$, one would have obtained a slightly different, although equivalent, representation. Another realisation is with the Bargmann representations, see, e.g., [Tay86]. Our choice of representation π_λ

to represent its equivalence class will prove useful in relation with the Weyl-Shubin calculus on $\mathbb{R}^n$ later, see Section 6.5.

The group Fourier transform of a function $\kappa \in L^1(\mathbb{H}_n)$ at π_1 is

$$\mathcal{F}_{\mathbb{H}_n}(\kappa)(\pi_1) = \pi_1(\kappa) = \int_{\mathbb{H}_n} \kappa(x,y,t)\pi_1(x,y,t)^* dxdydt,$$

that is, the operator on $L^2(\mathbb{R}^n)$ given by

$$\pi_1(\kappa)h(u) = \int_{\mathbb{H}_n} \kappa(x,y,t) e^{i(-t+\frac{1}{2}xy)} e^{-iyu} h(u-x) dxdydt.$$

We now fix the notation concerning the Euclidean Fourier transform and recall some facts about the Weyl quantization on $\mathbb{R}^n$.

The Euclidean Fourier transform

In order to give a nicer expression for the operator $\mathcal{F}_{\mathbb{H}_n}(\kappa)(\pi_1)$, we adopt here the following notation for the Euclidean Fourier transform on $\mathbb{R}^N$:

$$\mathcal{F}_{\mathbb{R}^N} f(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} f(x) e^{-ix\xi} dx, \qquad (6.13)$$

where $\xi \in \mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{C}$ is for instance integrable. With our choice of notation and normalisation, the mapping $\mathcal{F}_{\mathbb{R}^N}$ extends unitarily to a mapping on $L^2(\mathbb{R}^N)$ and

$$\mathcal{F}_{\mathbb{R}^N}(f)(x) = \mathcal{F}_{\mathbb{R}^N}^{-1}(f)(-x).$$

Let us also recall the Fourier inversion formula for a (e.g. Schwartz) function $f : \mathbb{R}^n \to \mathbb{C}$:

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{i(u-v)\xi} f(v) dv d\xi = (2\pi)^N f(u). \qquad (6.14)$$

In our context N will be equal to $2n+1$.

Unfortunately, due to our choice of notation π for the representations, in the formulae in the sequel π will appear both as a representation and as the constant $\pi = 3.1415926...$ However, as powers of this 2π will appear mostly as constants in front of integrals it should not lead to major confusion.

The (Euclidean) Weyl quantization

Let us also set some notation regarding the *Weyl quantization* on $\mathbb{R}^n$. If a is a symbol, that is, a reasonable function on $\mathbb{R}^n \times \mathbb{R}^n$, then the Weyl quantization associates to a the operator

$$\mathrm{Op}^W(a) \equiv a(D, X)$$

given by

$$\mathrm{Op}^W(a)f(u) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(u-v)\xi}a(\xi,\frac{u+v}{2})f(v)dvd\xi, \tag{6.15}$$

where $f\in\mathcal{S}(\mathbb{R}^n)$ and $u\in\mathbb{R}^n$.

Example 6.2.2. Particular examples are

$$\mathrm{Op}^W(1) = \mathrm{I}, \quad \mathrm{Op}^W(\xi_j) = \frac{1}{i}\partial_{u_j}, \quad \mathrm{Op}^W(u_j) = u_j,$$

and

$$\mathrm{Op}^W(\xi_k u_j) = \frac{1}{2i}(\partial_{u_k}u_j + u_j\partial_{u_k}).$$

The composition of two Weyl-quantized operators is

$$\mathrm{Op}^W(a)\circ\mathrm{Op}^W(b) = \mathrm{Op}^W(a\star b), \tag{6.16}$$

where (see, e.g., [Ler10])

$$a\star b(\zeta,u) = (2\pi)^{-2n}4^n\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-2i\{(\xi-\zeta)(y-u)-(\eta-\zeta)(x-u)\}} a(\xi,x)\ b(\eta,y)\ d\xi d\eta dx dy,$$

and asymptotically

$$a\star b\sim\sum_{m'=0}^{\infty}c_{m',n}\sum_{|\alpha_1|+|\alpha_2|=m'}\frac{(-1)^{|\alpha_2|}}{\alpha_1!\alpha_2!}\left(\left(\frac{1}{i}\partial_\xi\right)^{\alpha_1}\partial_x^{\alpha_2}a\right)\left(\left(\frac{1}{i}\partial_\xi\right)^{\alpha_2}\partial_x^{\alpha_1}b\right), \tag{6.17}$$

with $c_{0,n_0}=1$ and, in fact,

$$a\star b\sim ab+\frac{1}{2i}\{a,b\}+\dots \quad \text{where} \quad \{a,b\} = \sum_{j=1}^n\left(\frac{\partial a}{\partial\xi_j}\frac{\partial b}{\partial u_j}-\frac{\partial a}{\partial u_j}\frac{\partial b}{\partial\xi_j}\right).$$

This formula can already be checked on the basic examples given in Example 6.2.2 and on the following property:

Lemma 6.2.3. *Let a be a symbol. Then we have*

$$\begin{aligned}(\mathrm{ad}u_j)\left(\mathrm{Op}^W(a)\right) &\equiv u_j\mathrm{Op}^W(a)-\mathrm{Op}^W(a)u_j = \mathrm{Op}^W(i\partial_{\xi_j}a),\\ \left(\mathrm{ad}\partial_{u_j}\right)\left(\mathrm{Op}^W(a)\right) &\equiv \partial_{u_j}\mathrm{Op}^W(a)-\mathrm{Op}^W(a)\partial_{u_j} = \mathrm{Op}^W(\partial_{u_j}a).\end{aligned}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$. Then we have

$$\begin{aligned}
&(\mathrm{ad}u_j)\left(\mathrm{Op}^W(a)\right) f(u) = u_j \mathrm{Op}^W(a) f(u) - \mathrm{Op}^W(a)(u_j f)(u) \\
&= u_j (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-v)\xi} a(\xi, \frac{u+v}{2}) f(v) dv d\xi \\
&\qquad - (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-v)\xi} a(\xi, \frac{u+v}{2}) v_j f(v) dv d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-v)\xi} a(\xi, \frac{u+v}{2})(u_j - v_j) f(v) dv d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{i} \partial_{\xi_j} \left\{ e^{i(u-v)\xi} \right\} \ a(\xi, \frac{u+v}{2}) f(v) dv d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-v)\xi} i \partial_{\xi_j} \left\{ a(\xi, \frac{u+v}{2}) \right\} f(v) dv d\xi,
\end{aligned}$$

after integration by parts. This shows the first equality.

For the second one, we compute

$$\partial_{u_j} \mathrm{Op}^W(a) f(u) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_{u_j} \left\{ e^{i(u-v)\xi} a(\xi, \frac{u+v}{2}) \right\} f(v) dv d\xi.$$

Since

$$\begin{aligned}
\partial_{u_j} \left\{ e^{i(u-v)\xi} \ a(\xi, \frac{u+v}{2}) \right\} \quad &= \quad - \left\{ \partial_{v_j} e^{i(u-v)\xi} \right\} a(\xi, \frac{u+v}{2}) \\
&\qquad + \frac{1}{2} e^{i(u-v)\xi} \{\partial_{u_j} a\}(\xi, \frac{u+v}{2}),
\end{aligned}$$

we compute using integration by parts

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_{u_j} \left\{ e^{i(u-v)\xi} a(\xi, \frac{u+v}{2}) \right\} f(v) dv d\xi \\
&= - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left\{ \partial_{v_j} e^{i(u-v)\xi} \right\} a(\xi, \frac{u+v}{2}) f(v) dv d\xi \\
&\qquad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-v)\xi} \frac{1}{2} \{\partial_{u_j} a\}(\xi, \frac{u+v}{2}) f(v) dv d\xi \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-v)\xi} \partial_{v_j} \left\{ a(\xi, \frac{u+v}{2}) f(v) \right\} dv d\xi \\
&\qquad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-v)\xi} \frac{1}{2} \{\partial_{u_j} a\}(\xi, \frac{u+v}{2}) f(v) dv d\xi.
\end{aligned}$$

Now

$$\partial_{v_j} \left\{ a(\xi, \frac{u+v}{2}) f(v) \right\} = \frac{1}{2} \{\partial_{u_j} a\}(\xi, \frac{u+v}{2}) f(v) + a(\xi, \frac{u+v}{2}) \partial_{v_j} f(v),$$

thus

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_{u_j}\left\{e^{i(u-v)\xi}a(\xi,\frac{u+v}{2})\right\} f(v)dvd\xi$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(u-v)\xi}\{\partial_{u_j}a\}(\xi,\frac{u+v}{2})f(v)dvd\xi$$
$$+ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(u-v)\xi}a(\xi,\frac{u+v}{2})\partial_{v_j}f(v)dvd\xi.$$

We have obtained

$$\partial_{u_j}\mathrm{Op}^W(a)f(u)$$
$$= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(u-v)\xi}\{\partial_{u_j}a\}(\xi,\frac{u+v}{2})f(v)dvd\xi$$
$$+(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(u-v)\xi}a(\xi,\frac{u+v}{2})\partial_{v_j}f(v)dvd\xi.$$

Therefore, we have

$$\left(\mathrm{ad}\partial_{u_j}\right)\left(\mathrm{Op}^W(a)\right)f(u) = \partial_{u_j}\mathrm{Op}^W(a)f(u) - \mathrm{Op}^W(a)(\partial_{u_j}f)(u)$$
$$= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(u-v)\xi}\{\partial_{u_j}a\}(\xi,\frac{u+v}{2})f(v)dvd\xi$$
$$= \mathrm{Op}^W(\partial_{u_j}a)f(u).$$

This shows the second equality. □

The operator $\mathcal{F}_{\mathbb{H}_n}(\kappa)(\pi_1)$

Going back to $\pi_1(\kappa) \equiv \widehat{\kappa}(\pi_1)$ and using the well-known properties of the Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^{2n+1}}$, for instance see (6.14), it is not difficult to turn into rigorous computations the following calculations:

$$\pi_1(\kappa)h(u) = \int_{\mathbb{R}^{2n+1}} \kappa(x,y,t)e^{i(-t+\frac{1}{2}xy)}e^{-iyu}h(u-x)dxdydt$$
$$= \int_{\mathbb{R}^{2n+1}}\int_{\mathbb{R}^{2n+1}} (2\pi)^{-\frac{2n+1}{2}}\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\xi,\eta,\tau)e^{it\tau}e^{iy\eta}e^{ix\xi}$$
$$e^{i(-t+\frac{1}{2}xy)}e^{-iyu}h(u-x)d\xi d\eta d\tau dxdydt$$
$$= \sqrt{2\pi}\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\xi,u-\frac{x}{2},1)e^{ix\xi}h(u-x)d\xi dx$$
$$= \sqrt{2\pi}\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\xi,u-\frac{u-v}{2},1)e^{i\xi(u-v)}h(v)d\xi dv,$$

after the change of variable $v = u - x$. Comparing this last expression with (6.15), we see that

$$\pi_1(\kappa)h(u) = \sqrt{2\pi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\xi(u-v)}\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\xi,\frac{u+v}{2},1)h(v)d\xi dv,$$

may be written as

$$\pi_1(\kappa) = (2\pi)^{\frac{2n+1}{2}} \operatorname{Op}^W [\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\cdot,\cdot,1)] = (2\pi)^{\frac{2n+1}{2}} \mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(D,X,1). \quad (6.18)$$

More generally, we could compute in the same way $\pi_\lambda(\kappa)$ or use the following computational remarks.

Lemma 6.2.4. *Let $\lambda \in \mathbb{R}\backslash\{0\}$. With the convention given in (6.9) we obtain*

$$\begin{aligned} \pi_\lambda(\kappa) &= |\lambda|^{-(n+1)} \pi_{\operatorname{sgn}(\lambda)1}\left(\kappa \circ D_{1/\sqrt{|\lambda|}}\right) & (6.19)\\ &= (2\pi)^{\frac{2n+1}{2}} \operatorname{Op}^W \left[\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\sqrt{|\lambda|}\,\cdot, \sqrt{\lambda}\,\cdot, \lambda)\right], & (6.20)\end{aligned}$$

or, equivalently,

$$\begin{aligned} &\pi_\lambda(\kappa)h(u)\\ =& \int_{\mathbb{R}^{2n+1}} \kappa(x,y,t) e^{i\lambda(-t+\frac{1}{2}xy)} e^{-i\sqrt{\lambda}yu} h(u-\sqrt{|\lambda|}x) dxdydt & (6.21)\\ =& (2\pi)^{\frac{2n+1}{2}} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i(u-v)\xi} \mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\sqrt{|\lambda|}\ \xi, \sqrt{\lambda}\ \frac{u+v}{2}, \lambda) h(v) dv d\xi. & (6.22)\end{aligned}$$

We also have

$$\pi_\lambda(\kappa) = \pi_{-\lambda}(\kappa \circ \Theta), \quad (6.23)$$

and for $r>0$, $Q = 2n+2$,

$$\pi_\lambda(r^Q \kappa \circ D_r) = \pi_{r^{-2}\lambda}(\kappa). \quad (6.24)$$

For any $X \in \mathfrak{U}(\mathfrak{h}_n)$ and $r>0$, we have

$$\pi_\lambda(D_{r^{-1}}X) = \pi_{r^{-2}\lambda}(X). \quad (6.25)$$

Here $\mathfrak{U}(\mathfrak{h}_n)$ stands for the universal enveloping algebra of the Lie algebra $\mathfrak{h}_n$, see Section 1.3.

Proof of Lemma 6.2.4. By (6.8), we have for $h \in L^2(\mathbb{R}^n)$ and $(x,y,t) \in \mathbb{H}_n$,

$$\begin{aligned} \pi_\lambda(x,y,t)^* h(u) &= \pi_\lambda\left((x,y,t)^{-1}\right) h(u) = \pi_\lambda(-x,-y,-t)h(u)\\ &= e^{i\lambda(-t+\frac{1}{2}xy)} e^{-i\sqrt{\lambda}yu} h(u-\sqrt{|\lambda|}x).\end{aligned}$$

Thus

$$\begin{aligned} \pi_\lambda(\kappa)h(u) &= \int_{\mathbb{H}_n} \kappa(x,y,t)\ \pi_\lambda(x,y,t)^* h(u)\ dxdydt\\ &= \int_{\mathbb{R}^{2n+1}} \kappa(x,y,t) e^{i\lambda(-t+\frac{1}{2}xy)} e^{-i\sqrt{\lambda}yu} h(u-\sqrt{|\lambda|}x) dxdydt.\end{aligned}$$

This is Formula (6.21).

For Formula (6.23), since by (6.10) we have $\pi_{-\lambda} = \pi_\lambda \circ \Theta$ for any $\lambda \in \mathbb{R}\backslash\{0\}$, we see that

$$\begin{aligned}\pi_\lambda(\kappa) &= \int_{\mathbb{H}_n} \kappa(x,y,t)\pi_\lambda(x,y,t)^* dxdydt \\ &= \int_{\mathbb{H}_n} \kappa(x,y,t)\pi_{-\lambda}\big(\Theta(x,y,t)\big)^* dxdydt \\ &= \int_{\mathbb{H}_n} \kappa\big(\Theta(x,y,t)\big)\pi_{-\lambda}(x,y,t)^* dxdydt = \pi_{-\lambda}(\kappa\circ\Theta),\end{aligned}$$

after the change of variables given by Θ, which has the Jacobian equal to 1. We proceed in the same way for formula (6.24)

$$\begin{aligned}\pi_\lambda(r^Q\kappa\circ D_r) &= \int_{\mathbb{H}_n} \kappa\circ D_r(x,y,t)\pi_\lambda(x,y,t)^* r^Q dxdydt \\ &= \int_{\mathbb{H}_n} \kappa(x,y,t)\pi_\lambda\big(D_r^{-1}(x,y,t)\big)^* dxdydt \\ &= \int_{\mathbb{H}_n} \kappa(x,y,t)\pi_{r^{-2}\lambda}(x,y,t)^* dxdydt = \pi_{r^{-2}\lambda}(\kappa),\end{aligned}$$

after the change of variable given by D_r, using (6.10).

For any $X \in \mathfrak{U}(\mathfrak{h}_n)$ and $\kappa \in \mathcal{S}(G)$, recalling $D_{r^{-1}}X$ from (6.4), then using

$$(X\kappa)\circ D_r = (D_{r^{-1}}X)(\kappa\circ D_r) \tag{6.26}$$

and (6.24), we have

$$\begin{aligned}\pi_{r^{-2}\lambda}(X)\pi_{r^{-2}\lambda}(\kappa) &= \pi_{r^{-2}\lambda}(X\kappa) \\ &= \pi_\lambda(r^Q(X\kappa)\circ D_r) \\ &= \pi_\lambda(r^Q(D_{r^{-1}}X)(\kappa\circ D_r)) \\ &= \pi_\lambda(D_{r^{-1}}X)\pi_\lambda(r^Q\kappa\circ D_r) \\ &= \pi_\lambda(D_{r^{-1}}X)\pi_{r^{-2}\lambda}(\kappa),\end{aligned}$$

and this shows (6.25).

Thus Formulae (6.25), (6.24) and (6.23) hold for any $\lambda \in \mathbb{R}\backslash\{0\}$.

Let us assume $\lambda > 0$. Using $\pi_\lambda = \pi_1 \circ D_{\sqrt{\lambda}}$ we see that

$$\begin{aligned}\pi_\lambda(\kappa) &= \int_{\mathbb{H}_n} \kappa(x,y,t)\pi_1\big(D_{\sqrt{\lambda}}(x,y,t)\big)^* dxdydt \\ &= \int_{\mathbb{H}_n} \kappa\big(D_{1/\sqrt{\lambda}}(x,y,t)\big)\pi_1(x,y,t)^*\lambda^{-(n+1)} dxdydt \\ &= \lambda^{-(n+1)}\pi_1\left(\kappa\circ D_{1/\sqrt{\lambda}}\right),\end{aligned}$$

and this gives Formula (6.19) for $\lambda > 0$. But Formula (6.18) gives here

$$\pi_1\left(\kappa \circ D_{1/\sqrt{\lambda}}\right) = (2\pi)^{n+\frac{1}{2}} \mathrm{Op}^W \left[\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa \circ D_{1/\sqrt{\lambda}})(\cdot, \cdot, 1)\right].$$

Since a simple change of variable in $\mathbb{R}^{2n+1}$ yields

$$\mathcal{F}_{\mathbb{R}^{2n+1}}\left(\kappa \circ D_{1/\sqrt{\lambda}}\right) = \lambda^{n+1}\left(\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)\right) \circ D_{\sqrt{\lambda}}, \tag{6.27}$$

we obtain Formula (6.20) for any $\lambda > 0$.

For $\lambda < 0$, we use Formula (6.23) and the case $\lambda > 0$, that is,

$$\begin{aligned} \pi_\lambda(\kappa) &= \pi_{-\lambda}(\kappa \circ \Theta) \\ &= (-\lambda)^{-(n+1)} \pi_1\left(\kappa \circ \Theta \circ D_{1/\sqrt{-\lambda}}\right) \\ &= (-\lambda)^{-(n+1)} \pi_1\left(\kappa \circ D_{1/\sqrt{-\lambda}} \circ \Theta\right) \\ &= (-\lambda)^{-(n+1)} \pi_{-1}\left(\kappa \circ D_{1/\sqrt{-\lambda}}\right). \end{aligned}$$

Hence Formula (6.19) is proved for any $\lambda < 0$. Here, Formula (6.18) and the relation $\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa \circ \Theta) = \mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa) \circ \Theta$ with (6.27) give

$$\begin{aligned} \pi_1\left(\kappa \circ \Theta \circ D_{1/\sqrt{-\lambda}}\right) &= (2\pi)^{n+\frac{1}{2}} \mathrm{Op}^W \left[\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa \circ \Theta \circ D_{1/\sqrt{-\lambda}})(\cdot, \cdot, 1)\right] \\ &= (2\pi)^{n+\frac{1}{2}} (-\lambda)^{n+1} \left(\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)\right) \circ \Theta \circ D_{\sqrt{-\lambda}}(\cdot, \cdot, 1), \end{aligned}$$

we obtain Formula (6.20) for any $\lambda < 0$. □

From Lemma 6.2.4 or from (6.11), we see that

$$\pi_\lambda(X_j) = \mathrm{Op}^W(i\sqrt{|\lambda|}\xi_j) \quad \text{and} \quad \pi_\lambda(Y_j) = \mathrm{Op}^W(i\sqrt{\lambda}u_j). \tag{6.28}$$

Remark 6.2.5. This was already noted in [Tay84, BFKG12a]. However in [Tay84], the Fourier transform on $\mathbb{R}^n$ is chosen to be non-unitarily defined by

$$\xi \longmapsto \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Remark 6.2.6. The Schwartz space on the Heisenberg group $\mathbb{H}_n$, realised as we have done, is defined as $\mathcal{S}(\mathbb{R}^{2n+1})$, see Section 3.1.9. The characterisation of the Fourier image of the (full) Schwartz space on $\mathbb{H}_n$ is a difficult problem analysed by Geller in [Gel80]. See also the more recent paper [ADBR13].

6.2.3 Plancherel measure

The dual $\widehat{\mathbb{H}}_n$ of the Heisenberg group $\mathbb{H}_n$ may be described together with its Plancherel measure by the orbit method, see Section 1.8.1. Here we obtain a concrete formula for the Plancherel measure μ of the Heisenberg group $\mathbb{H}_n$ using well known properties of Euclidean analysis together with our choice of representatives for the elements of $\widehat{\mathbb{H}}_n$, especially the Schrödinger representations π_λ.

Proposition 6.2.7. *Let $f \in \mathcal{S}(\mathbb{H}_n)$. Then for each $\lambda \in \mathbb{R}\backslash\{0\}$ the operator $\widehat{f}(\pi_\lambda)$ acting on $L^2(\mathbb{R}^n)$ is the Hilbert-Schmidt operator with integral kernel*

$$K_{f,\lambda} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C},$$

given by

$$K_{f,\lambda}(u,v) = (2\pi)^{n+\frac{1}{2}} \int_{\mathbb{R}^n} e^{i(u-v)\xi} \mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\xi, \sqrt{\lambda}\frac{u+v}{2}, \lambda) d\xi,$$

and Hilbert-Schmidt norm

$$\begin{aligned} \|\widehat{f}(\pi_\lambda)\|_{\mathtt{HS}(L^2(\mathbb{R}^n))} &= (2\pi)^{\frac{3n+1}{2}} |\lambda|^{-\frac{n}{2}} \|\mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\cdot,\cdot,\lambda)\|_{L^2(\mathbb{R}^{2n})} \\ &= (2\pi)^{\frac{3n+1}{2}} |\lambda|^{-\frac{n}{2}} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\xi,w,\lambda)|^2 d\xi dw\right)^{\frac{1}{2}}. \end{aligned}$$

Furthermore, we have

$$\int_{\mathbb{H}_n} |f(x,y,t)|^2 dxdydt = c_n \int_{\lambda\in\mathbb{R}\backslash\{0\}} \|\widehat{f}(\pi_\lambda)\|^2_{\mathtt{HS}(L^2(\mathbb{R}^n))} |\lambda|^n d\lambda,$$

where $c_n = (2\pi)^{-(3n+1)}$.

In particular, Proposition 6.2.7 implies that the Plancherel measure μ on the Heisenberg group is supported in $\{[\pi_\lambda],\ \lambda \in \mathbb{R}\backslash\{0\}\}$, see (6.29). Moreover, we have

$$d\mu(\pi_\lambda) \equiv c_n |\lambda|^n d\lambda, \quad \lambda \in \mathbb{R}\backslash\{0\}.$$

The constant c_n depends on our choice of realisation of $\pi_\lambda \in [\pi_\lambda]$.

Proof of Proposition 6.2.7. By (6.22), we have for $h \in L^2(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$,

$$\begin{aligned} \widehat{f}(\pi_\lambda)h(u) &= (2\pi)^{n+\frac{1}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(u-v)\xi} \mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\xi, \sqrt{\lambda}\frac{u+v}{2}, \lambda) h(v) dv d\xi \\ &= \int_{\mathbb{R}^n} K_{f,\lambda}(u,v) h(v) dv, \end{aligned}$$

where $K_{f,\lambda}$ is the integral kernel of $\widehat{f}(\pi_\lambda)$ hence given by

$$K_{f,\lambda}(u,v) = (2\pi)^{n+\frac{1}{2}} \int_{\mathbb{R}^n} e^{i(u-v)\xi} \mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\xi, \sqrt{\lambda}\frac{u+v}{2}, \lambda) d\xi.$$

Using the Euclidean Fourier transform (see (6.13) for our normalisation of $\mathcal{F}_{\mathbb{R}^n}$), we may rewrite this as

$$K_{f,\lambda}(u,v) = (2\pi)^{\frac{3}{2}n+\frac{1}{2}} \mathcal{F}_{\mathbb{R}^n} \left\{ \mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\,\cdot, \sqrt{\lambda}\frac{u+v}{2}, \lambda) \right\} (v-u).$$

The $L^2(\mathbb{R}^n \times \mathbb{R}^n)$-norm of the integral kernel is

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n} |K_{f,\lambda}(u,v)|^2 du dv \\
&\quad = (2\pi)^{3n+1} \int_{\mathbb{R}^n\times\mathbb{R}^n} \left|\mathcal{F}_{\mathbb{R}^n} \left\{ \mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\,\cdot, \sqrt{\lambda}\frac{u+v}{2}, \lambda) \right\} (v-u)\right|^2 du dv \\
&\quad = (2\pi)^{3n+1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\mathcal{F}_{\mathbb{R}^n} \left\{ \mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\,\cdot, w_2, \lambda) \right\} (w_1)\right|^2 |\lambda|^{-\frac{n}{2}} dw_1 dw_2,
\end{aligned}$$

after the change of variable $(w_1, w_2) = (v-u, \sqrt{\lambda}\frac{u+v}{2})$. The (Euclidean) Plancherel formula on $\mathbb{R}^n$ in the variable w_1 (with dual variable ξ_1) then yields

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n} |K_{f,\lambda}(u,v)|^2 du dv \\
&\quad = (2\pi)^{3n+1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\xi_1, w_2, \lambda)|^2 |\lambda|^{-\frac{n}{2}} d\xi_1 dw_2 \\
&\quad = (2\pi)^{3n+1} |\lambda|^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\xi, w_2, \lambda)|^2 d\xi dw_2,
\end{aligned}$$

after the change of variable $\xi = \sqrt{|\lambda|}\xi_1$. Since $f \in \mathcal{S}(\mathbb{H}_n)$, this quantity is finite. Since the integral kernel of $\widehat{f}(\pi_\lambda)$ is square integrable, the operator $\widehat{f}(\pi_\lambda)$ is Hilbert-Schmidt and its Hilbert-Schmidt norm is the L^2-norm of its integral kernel (see, e.g., [RS80, Theorem VI.23]). This shows the first part of the statement.

To finish the proof, we now integrate each side of the last equality against $|\lambda|^n d\lambda$ and then use again the (Euclidean) Plancherel formula on $\mathbb{R}^{2n+1}$ in the variable (ξ, w_2, λ). We obtain

$$\begin{aligned}
&\int_{\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}^n\times\mathbb{R}^n} |K_{f,\lambda}(u,v)|^2 du dv \ |\lambda|^n d\lambda \\
&\quad = (2\pi)^{3n+1} \int_{\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\xi, w_2, \lambda)|^2 d\xi dw_2 d\lambda \\
&\quad = (2\pi)^{3n+1} \int_{\mathbb{R}^{2n+1}} |f(x,y,t)|^2 dx dy dt.
\end{aligned}$$

This concludes the proof of Proposition 6.2.7. □

It follows from the Plancherel formula in Proposition 6.2.7 that the Schrödinger representations π_λ, $\lambda \in \mathbb{R}\setminus\{0\}$, are almost all the representations of $\mathbb{H}_n$

modulo unitary equivalence. 'Almost all' here refers to the Plancherel measure $\mu = c_n|\lambda|^n d\lambda$ on $\widehat{\mathbb{H}}_n$. The other representations are finite dimensional and in fact 1-dimensional. They are given by the unitary characters of $\mathbb{H}_n$

$$\chi_w : (x, y, t) \mapsto e^{i(xw_1+yw_2)}, \quad w = (w_1, w_2) \in \mathbb{R}^n \times \mathbb{R}^n \sim \mathbb{R}^{2n}.$$

See also Example 1.8.1 for the link with the orbit method.

We can summarise this paragraph by writing

$$\widehat{\mathbb{H}}_n = \{[\pi_\lambda],\ \lambda \in \mathbb{R}\backslash\{0\}\} \bigcup \{[\chi_w],\ w \in \mathbb{R}^{2n}\} \overset{\mu \text{ a.e.}}{=} \{[\pi_\lambda],\ \lambda \in \mathbb{R}\backslash\{0\}\}. \quad (6.29)$$

6.3 Difference operators

In this section we compute the difference operators Δ_{x_j}, Δ_{y_j}, and Δ_t which are the operators defined via

$$\begin{aligned} \Delta_{x_j}\widehat{\kappa}(\pi_\lambda) &:= \pi_\lambda(x_j\kappa), \\ \Delta_{y_j}\widehat{\kappa}(\pi_\lambda) &:= \pi_\lambda(y_j\kappa), \\ \Delta_t\widehat{\kappa}(\pi_\lambda) &:= \pi_\lambda(t\kappa). \end{aligned}$$

General properties of such difference operators have been analysed in Section 5.2.1. Here we aim at providing explicit expressions for them in the setting of the Heisenberg group $\mathbb{H}_n$.

6.3.1 Difference operators Δ_{x_j} and Δ_{y_j}

We start with the difference operators with respect to x and y.

Lemma 6.3.1. *For any $j = 1, \ldots, n$,*

$$\begin{aligned} \Delta_{x_j}|_{\pi_\lambda} &= \frac{1}{i\lambda}\operatorname{ad}(\pi_\lambda(Y_j)) = \frac{1}{\sqrt{|\lambda|}}\operatorname{ad} u_j, \\ \Delta_{y_j}|_{\pi_\lambda} &= -\frac{1}{i\lambda}\operatorname{ad}(\pi_\lambda(X_j)) = -\frac{1}{i\sqrt{\lambda}}\operatorname{ad}\partial_{u_j}. \end{aligned}$$

By this we mean that for any κ in some $\mathcal{K}_{a,b}(\mathbb{H}_n)$ such that $x_j\kappa$ is in some $\mathcal{K}_{a',b'}(\mathbb{H}_n)$ or $y_j\kappa$ in some $\mathcal{K}_{a',b'}(\mathbb{H}_n)$ for Δ_{x_j} or Δ_{y_j}, respectively, we have for all $h \in \mathcal{S}(\mathbb{R}^n)$ that

$$\begin{aligned} \left(\Delta_{x_j}\widehat{\kappa}(\pi_\lambda)\right) h\,(u) &= \frac{1}{\sqrt{|\lambda|}}\left(u_j\left(\widehat{\kappa}(\pi_\lambda)h\right)(u) - \left(\widehat{\kappa}(\pi_\lambda)(u_j h)\right)(u)\right), \\ \left(\Delta_{y_j}\widehat{\kappa}(\pi_\lambda)\right) h\,(u) &= \frac{1}{i\sqrt{\lambda}}\left(-\partial_{u_j}\{\widehat{\kappa}(\pi_\lambda)h\}\,(u) + \widehat{\kappa}(\pi_\lambda)\{\partial_{u_j}h\}\,(u)\right). \end{aligned}$$

Proof. Although we could just use direct computations, we prefer to use the following observations. Firstly we have by (6.2) and (6.3) that

$$Y_j - \tilde{Y}_j = x_j\partial_t = \partial_t x_j \quad \text{and} \quad \tilde{X}_j - X_j = y_j\partial_t = \partial_t y_j.$$

Secondly for any κ_1 in some $\mathcal{K}_{a,b}(\mathbb{H}_n)$,

$$\pi_\lambda(\partial_t\kappa_1) = \pi_\lambda(T\kappa_1) = \pi_\lambda(T)\pi_\lambda(\kappa_1) = i\lambda\pi_\lambda(\kappa_1), \tag{6.30}$$

as $T = \partial_t$ and using (6.11). Therefore, these two observations yield

$$\begin{aligned}
\pi_\lambda(x_j\kappa) &= \frac{1}{i\lambda}\pi_\lambda(\partial_t x_j\kappa) = \frac{1}{i\lambda}\pi_\lambda\left((Y_j - \tilde{Y}_j)\kappa\right) \\
&= \frac{1}{i\lambda}\left(\pi_\lambda(Y_j\kappa) - \pi_\lambda(\tilde{Y}_j\kappa)\right) \\
&= \frac{1}{i\lambda}\left(\pi_\lambda(Y_j)\pi_\lambda(\kappa) - \pi_\lambda(\kappa)\pi_\lambda(Y_j)\right),
\end{aligned}$$

and

$$\begin{aligned}
\pi_\lambda(y_j\kappa) &= \frac{1}{i\lambda}\pi_\lambda(\partial_t y_j\kappa) = \frac{1}{i\lambda}\pi_\lambda\left((\tilde{X}_j - X_j)\kappa\right) \\
&= \frac{1}{i\lambda}\left(\pi_\lambda(\kappa)\pi_\lambda(X_j) - \pi_\lambda(X_j)\pi_\lambda(\kappa)\right).
\end{aligned}$$

Using Lemma 6.2.1, we have obtained the expressions for Δ_{y_j} and Δ_{x_j} given in the statement. □

Above and also below, we use the formula for the symbols of right derivatives, for example, $\pi_\lambda(\tilde{Y}_j\kappa) = \pi_\lambda(\kappa)\pi_\lambda(Y_j)$, see Proposition 1.7.6, (iv).

Before giving some examples of applications of the difference operators Δ_{x_j} and Δ_{y_j}, let us make a couple of remarks.

Remark 6.3.2. 1. The formulae in Lemma 6.3.1 respect the properties of the automorphism Θ. Indeed, using (6.23) we have

$$\begin{aligned}
\left(\Delta_{x_j}\widehat{\kappa}(\pi)\right)|_{\pi=\pi_{-\lambda}} &= (\widehat{x_j\kappa}(\pi))\,|_{\pi=\pi_{-\lambda}} = \pi_{-\lambda}(x_j\kappa) = \pi_\lambda\left((x_j\kappa)\circ\Theta\right) \\
&= \pi_\lambda\left(x_j\ \kappa\circ\Theta\right) = \Delta_{x_j}\widehat{\kappa\circ\Theta}(\pi_\lambda) = \Delta_{x_j}\left(\widehat{\kappa}(\pi_{-\lambda})\right), \\
\left(\Delta_{y_j}\widehat{\kappa}(\pi)\right)|_{\pi=\pi_{-\lambda}} &= (\widehat{y_j\kappa}(\pi))\,|_{\pi=\pi_{-\lambda}} = \pi_{-\lambda}(y_j\kappa) = \pi_\lambda\left((y_j\kappa)\circ\Theta\right) \\
&= \pi_\lambda\left(-y_j\ \kappa\circ\Theta\right) = -\Delta_{y_j}\widehat{\kappa\circ\Theta}(\pi_\lambda) = -\Delta_{y_j}\left(\widehat{\kappa}(\pi_{-\lambda})\right).
\end{aligned}$$

This can also be viewed directly from the formulae in Lemma 6.3.1:

$$\begin{aligned}
\left(\Delta_{x_j}\widehat{\kappa}(\pi)\right)|_{\pi=\pi_{-\lambda}} &= \frac{1}{\sqrt{|-\lambda|}}\text{ad}u_j\left(\widehat{\kappa}(\pi_{-\lambda})\right) = \Delta_{x_j}\left(\widehat{\kappa}(\pi_{-\lambda})\right), \\
\left(\Delta_{y_j}\widehat{\kappa}(\pi)\right)|_{\pi=\pi_{-\lambda}} &= -\frac{1}{i\sqrt{-\lambda}}\text{ad}\partial_{u_j} = -\Delta_{y_j}\left(\widehat{\kappa}(\pi_{-\lambda})\right).
\end{aligned}$$

2. The formulae in Lemma 6.3.1 respect the properties of the dilations D_r. This time using (6.24), we have

$$\begin{aligned}\left(\Delta_{x_j}\widehat{\kappa}(\pi)\right)|_{\pi=\pi_{r^{-2}\lambda}} &= (\widehat{x_j\kappa}(\pi))\,|_{\pi=\pi_{r^{-2}\lambda}} = \pi_{r^{-2}\lambda}(x_j\kappa) = \pi_\lambda\left(r^Q(x_j\kappa)\circ D_r\right)\\ &= r\ \pi_\lambda\left(r^Q x_j\ \kappa\circ D_r\right) = r\ \Delta_{x_j}\left(\widehat{\kappa}(\pi_{r^{-2}\lambda})\right).\end{aligned}$$

This can also be viewed directly from the formulae in Lemma 6.3.1:

$$\begin{aligned}\left(\Delta_{x_j}\widehat{\kappa}(\pi)\right)|_{\pi=\pi_{r^{-2}\lambda}} &= \frac{1}{\sqrt{|r^{-2}\lambda|}}\,(\mathrm{ad}u_j)\left(\widehat{\kappa}(\pi_{r^{-2}\lambda})\right)\\ &= r\times\left(\frac{1}{\sqrt{|\lambda|}}\,(\mathrm{ad}u_j)\left(\widehat{\kappa}(\pi_{r^{-2}\lambda})\right)\right)\\ &= r\ \Delta_{x_j}\left(\widehat{\kappa}(\pi_{r^{-2}\lambda})\right).\end{aligned}$$

In exactly the same two ways we obtain for Δ_{y_j} that

$$\left(\Delta_{y_j}\widehat{\kappa}(\pi)\right)|_{\pi=\pi_{r^{-2}\lambda}} = r\Delta_{y_j}\left(\widehat{\kappa}(\pi_{r^{-2}\lambda})\right).$$

Lemmata 6.3.1 and 6.2.3 imply:

Corollary 6.3.3. *If* $\widehat{\kappa}(\pi_\lambda) = \mathrm{Op}^W(a_\lambda)$ *and* $a_\lambda = \{a_\lambda(\xi,u)\}$, *then*

$$\begin{aligned}\Delta_{x_j}\widehat{\kappa}(\pi_\lambda) &= \mathrm{Op}^W\left(\frac{i}{\sqrt{|\lambda|}}\partial_{\xi_j}a_\lambda\right),\\ \Delta_{y_j}\widehat{\kappa}(\pi_\lambda) &= \mathrm{Op}^W\left(\frac{i}{\sqrt{\lambda}}\partial_{u_j}a_\lambda\right).\end{aligned}$$

If $\widehat{\kappa}(\pi_\lambda) = \mathrm{Op}^W(a_\lambda)$ and $a_\lambda = \{a_\lambda(\xi,u)\}$ as in the statement above, we will often say that a_λ is the λ-*symbol*.

Up to now, we analysed the difference operators applied to a 'general' group Fourier transform of a distribution κ (provided that the difference operators made sense, see Definition 5.2.1 and the subsequent discussion). This is equivalent to applying difference operators acting on symbols, see Section 5.1.3. In what follows, we particularise this to some known symbols, mainly to the one in Example 5.1.26, that is, to $\pi(A)$ where A is a left-invariant differential operator such as $A = X_j, Y_j$ or T.

We now give some explicit examples.

Example 6.3.4. We already know that $\Delta_{x_j}\mathrm{I} = 0$, see Example 5.2.8. We can compute

$$\Delta_{x_j}\pi_\lambda(X_k) = -\delta_{jk}\mathrm{I},\quad \Delta_{x_j}\pi_\lambda(Y_k) = 0\quad\text{and}\quad \Delta_{x_j}\pi_\lambda(T) = 0, \tag{6.31}$$

and

$$\Delta_{x_j}\pi_\lambda(\mathcal{L}) = -2\pi_\lambda(X_j). \tag{6.32}$$

Proof. By Lemma 6.3.1,

$$\begin{aligned}\Delta_{x_j}\pi_\lambda(X_k) &= \frac{1}{i\lambda}\mathrm{ad}\,(\pi_\lambda(Y_j))\,\pi_\lambda(X_k) = \frac{1}{i\lambda}[\pi_\lambda(Y_j),\pi_\lambda(X_k)] \\ &= \frac{1}{i\lambda}\pi_\lambda[Y_j,X_k],\end{aligned}$$

since π_λ is a representation of the Lie algebra $\mathfrak{g}$. Similarly,

$$\begin{aligned}\Delta_{x_j}\pi_\lambda(Y_k) &= \frac{1}{i\lambda}\mathrm{ad}\,(\pi_\lambda(Y_j))\,\pi_\lambda(Y_k) = \frac{1}{i\lambda}\pi_\lambda[Y_j,Y_k], \\ \Delta_{x_j}\pi_\lambda(T) &= \frac{1}{i\lambda}\mathrm{ad}\,(\pi_\lambda(Y_j))\,\pi_\lambda(T) = \frac{1}{i\lambda}\pi_\lambda[Y_j,T].\end{aligned}$$

By the canonical commutation relations, we have

$$[Y_j,X_k] = -\delta_{jk}T, \quad [Y_j,Y_k]=0 \quad \text{and} \quad [Y_j,T]=0.$$

Since $\pi_\lambda(T) = i\lambda\mathrm{I}$, we obtain (6.31).

In the same way, we have

$$\Delta_{x_j}\pi_\lambda(X_k)^2 = \frac{1}{i\lambda}\pi_\lambda[Y_j,X_k^2] \quad \text{and} \quad \Delta_{x_j}\pi_\lambda(Y_k^2) = \frac{1}{i\lambda}\pi_\lambda[Y_j,Y_k^2].$$

Using the canonical commutation relations, we see that Y_j and Y_k commute in the Lie algebra $\mathfrak{g}$ thus Y_j and Y_k^2 commute in the enveloping Lie algebra $\mathfrak{U}(\mathfrak{g})$: $[Y_j,Y_k^2]=0$. Again using the canonical commutation relation we compute

$$[Y_j,X_k^2] = -2\delta_{jk}X_kT,$$

since

$$\begin{aligned}Y_jX_k^2 &= Y_jX_kX_k = (-\delta_{jk}T + X_kY_j)X_k \\ &= -\delta_{jk}TX_k + X_k(-\delta_{jk}T + X_kY_j) \\ &= -2\delta_{jk}X_kT + X_k^2Y_j.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta_{x_j}\pi_\lambda(X_k)^2 &= \frac{1}{i\lambda}\pi_\lambda(-2\delta_{jk}X_kT) = \frac{-2\delta_{jk}}{i\lambda}\pi_\lambda(X_kT) = \frac{-2\delta_{jk}}{i\lambda}\pi_\lambda(X_k)\pi_\lambda(T) \\ &= \frac{-2\delta_{jk}}{i\lambda}\pi_\lambda(X_k)(i\lambda) = -2\delta_{jk}\pi_\lambda(X_k),\end{aligned}$$

and $\Delta_{x_j}\pi_\lambda(Y_k^2)=0$. This implies (6.32). □

Example 6.3.5. We already know that $\Delta_{y_j}\mathrm{I}=0$, see Example 5.2.8. We can compute

$$\Delta_{y_j}\pi_\lambda(X_k)=0, \quad \Delta_{y_j}\pi_\lambda(Y_k) = -\delta_{jk}\mathrm{I} \quad \text{and} \quad \Delta_{y_j}\pi_\lambda(T)=0, \tag{6.33}$$

and

$$\Delta_{y_j}\pi_\lambda(\mathcal{L}) = -2\pi_\lambda(Y_j). \tag{6.34}$$

Proof. Proceeding as in the proof of Example 6.3.4, we have

$$\begin{aligned}
\Delta_{y_j}\pi_\lambda(X_k) &= -\frac{1}{i\lambda}\mathrm{ad}\,(\pi_\lambda(X_j))\,\pi_\lambda(X_k) = -\frac{1}{i\lambda}\pi_\lambda[X_j,X_k], \\
\Delta_{y_j}\pi_\lambda(Y_k) &= -\frac{1}{i\lambda}\mathrm{ad}\,(\pi_\lambda(X_j))\,\pi_\lambda(Y_k) = -\frac{1}{i\lambda}\pi_\lambda[X_j,Y_k], \\
\Delta_{y_j}\pi_\lambda(T) &= -\frac{1}{i\lambda}\mathrm{ad}\,(\pi_\lambda(X_j))\,\pi_\lambda(T) = -\frac{1}{i\lambda}\pi_\lambda[X_j,T],
\end{aligned}$$

and this together with the canonical commutation relations and $\pi_\lambda(T) = i\lambda \mathrm{I}$, yield (6.33).

For the second part of Example 6.3.5, we have

$$\Delta_{y_j}\pi_\lambda(X_k)^2 = -\frac{1}{i\lambda}\pi_\lambda[X_j,X_k^2] \quad \text{and} \quad \Delta_{y_j}\pi_\lambda(Y_k^2) = -\frac{1}{i\lambda}\pi_\lambda[X_j,Y_k^2],$$

and using the canonical commutation relations we compute $[X_j, X_k^2] = 0$ whereas

$$[X_j, Y_k^2] = 2\delta_{jk}Y_kT,$$

since

$$\begin{aligned}
X_jY_k^2 &= X_jY_kY_k = (\delta_{jk}T + Y_kX_j)Y_k \\
&= \delta_{jk}TY_k + Y_k(\delta_{jk}T + Y_kX_j) \\
&= 2\delta_{jk}Y_kT + Y_k^2X_j.
\end{aligned}$$

Therefore

$$\Delta_{y_j}\pi_\lambda(Y_k)^2 = -\frac{1}{i\lambda}\pi_\lambda(2\delta_{jk}Y_kT) = -2\delta_{jk}\pi_\lambda(Y_k) \quad \text{and} \quad \Delta_{y_j}\pi_\lambda(X_k^2) = 0.$$

This implies (6.34). □

6.3.2 Difference operator Δ_t

Naturally, very important information will be contained in the difference operator corresponding to multiplication by t.

Lemma 6.3.6. *We have*

$$\Delta_t|_{\pi_\lambda} = i\partial_\lambda + \frac{1}{2}\sum_{j=1}^n \Delta_{x_j}\Delta_{y_j}|_{\pi_\lambda} + \frac{i}{2\lambda}\sum_{j=1}^n \left\{\pi_\lambda(Y_j)\Delta_{y_j}|_{\pi_\lambda} + \Delta_{x_j}|_{\pi_\lambda}\pi_\lambda(X_j)\right\}.$$

By this we mean that for any κ in some $\mathcal{K}_{a,b}(\mathbb{H}_n)$ such that $t\kappa$ is in some $\mathcal{K}_{a',b'}(\mathbb{H}_n)$, we have

$$\begin{aligned}
\Delta_t\pi_\lambda(\kappa) \;=\; & i\partial_\lambda\pi_\lambda(\kappa) + \frac{1}{2}\sum_{j=1}^n \Delta_{x_j}\Delta_{y_j}\pi_\lambda(\kappa) \\
& + \frac{i}{2\lambda}\sum_{j=1}^n \left\{\pi_\lambda(Y_j)\Delta_{y_j}\pi_\lambda(\kappa) + \Delta_{x_j}\pi_\lambda(\kappa)\pi_\lambda(X_j)\right\},
\end{aligned}$$

or, rewriting this with the equivalent notation $\widehat{\kappa}(\pi_\lambda)$ as before,

$$\begin{aligned}\Delta_t\widehat{\kappa}(\pi_\lambda) &= i\partial_\lambda\widehat{\kappa}(\pi_\lambda)+\frac{1}{2}\sum_{j=1}^n\Delta_{x_j}\Delta_{y_j}\widehat{\kappa}(\pi_\lambda)\\ &\quad+\frac{i}{2\lambda}\sum_{j=1}^n\left\{\pi_\lambda(Y_j)\Delta_{y_j}\widehat{\kappa}(\pi_\lambda)+\Delta_{x_j}\widehat{\kappa}(\pi_\lambda)\pi_\lambda(X_j)\right\}.\end{aligned}$$

Before giving some examples of applications of the difference operator Δ_t, let us make a couple of remarks.

Remark 6.3.7. 1. This lemma shows that the difference operators act on the field of operators $\{\pi_\lambda(\kappa),\ \lambda\in\mathbb{R}\backslash\{0\}\}$, rather than on 'one' $\pi_\lambda(\kappa)$ for an individual λ, see Remark 5.2.2.

2. In a similar way as in Remark 6.3.2, the formula in Lemma 6.3.6 respects the properties of the automorphism Θ and the dilations D_r. Indeed, using (6.23) we have

$$\begin{aligned}(\Delta_t\widehat{\kappa}(\pi))\,|_{\pi=\pi_{-\lambda}} &= \left(\widehat{t\kappa}(\pi)\right)|_{\pi=\pi_{-\lambda}}=\pi_{-\lambda}(t\kappa)=\pi_\lambda\left((t\kappa)\circ\Theta\right)\\ &= \pi_\lambda\left(-t\ \kappa\circ\Theta\right)=-\Delta_t\widehat{\kappa\circ\Theta}(\pi_\lambda)=-\Delta_t\left(\widehat{\kappa}(\pi_{-\lambda})\right),\end{aligned}$$

that is

$$(\Delta_t\widehat{\kappa}(\pi))\,|_{\pi=\pi_{-\lambda}}=-\Delta_t\left(\widehat{\kappa}(\pi_{-\lambda})\right). \tag{6.35}$$

For the dilations, using (6.24), we have

$$\begin{aligned}(\Delta_t\widehat{\kappa}(\pi))\,|_{\pi=\pi_{r^{-2}\lambda}} &= \left(\widehat{t\kappa}(\pi)\right)|_{\pi=\pi_{r^{-2}\lambda}}=\pi_{r^{-2}\lambda}(t\kappa)=\pi_\lambda\left(r^Q(t\kappa)\circ D_r\right)\\ &= r^2\pi_\lambda\left(r^Qt\ \kappa\circ D_r\right)=r^2\Delta_t\left(\widehat{\kappa}(\pi_{r^{-2}\lambda})\right).\end{aligned}$$

that is

$$(\Delta_t\widehat{\kappa}(\pi))\,|_{\pi=\pi_{r^{-2}\lambda}}=r^2\Delta_t\left(\widehat{\kappa}(\pi_{r^{-2}\lambda})\right). \tag{6.36}$$

Formulae (6.35) and (6.36) can also be viewed directly from the formula in Lemma 6.3.6:

$$\begin{aligned}(\Delta_t\widehat{\kappa}(\pi))\,|_{\pi=\pi_{-\lambda}} &= i\partial_{\lambda_1=-\lambda}\{\pi_{\lambda_1}(\kappa)\}+\frac{1}{2}\sum_{j=1}^n\{\Delta_{x_j}\Delta_{y_j}\pi(\kappa)\}_{\pi=\pi_{-\lambda}}\\ &\quad+\frac{i}{-2\lambda}\sum_{j=1}^n\{\pi(Y_j)\Delta_{y_j}\pi(\kappa)+\Delta_{x_j}\pi(\kappa)\pi(X_j)\}_{\pi=\pi_{-\lambda}},\end{aligned} \tag{6.37}$$

$$\begin{aligned}(\Delta_t\widehat{\kappa}(\pi))\,|_{\pi=\pi_{r^{-2}\lambda}} &= i\partial_{\lambda_1=r^{-2}\lambda}\{\pi_{\lambda_1}(\kappa)\}+\frac{1}{2}\sum_{j=1}^n\{\Delta_{x_j}\Delta_{y_j}\pi(\kappa)\}_{\pi=\pi_{r^{-2}\lambda}}\\ &\quad+\frac{i}{2r^{-2}\lambda}\sum_{j=1}^n\{\pi(Y_j)\Delta_{y_j}\pi(\kappa)+\Delta_{x_j}\pi(\kappa)\pi(X_j)\}_{\pi=\pi_{r^{-2}\lambda}}.\end{aligned} \tag{6.38}$$

For the first terms in the right hand side in (6.37) and (6.38) we have easily that

$$\begin{aligned} \partial_{\lambda_1=-\lambda}\pi_{\lambda_1}(\kappa) &= -\partial_\lambda\{\pi_{-\lambda}(\kappa)\}, \\ \partial_{\lambda_1=r^{-2}\lambda}\pi_{\lambda_1}(\kappa) &= r^2\partial_\lambda\{\pi_{r^{-2}\lambda}(\kappa)\}. \end{aligned}$$

From Remark 6.3.2 we know that

$$\left\{ \begin{aligned} \left(\Delta_{x_j}\widehat{\kappa}(\pi)\right)|_{\pi=\pi_{-\lambda}} &= \Delta_{x_j}\left(\widehat{\kappa}(\pi_{-\lambda})\right) \\ \left(\Delta_{y_j}\widehat{\kappa}(\pi)\right)|_{\pi=\pi_{-\lambda}} &= -\Delta_{y_j}\left(\widehat{\kappa}(\pi_{-\lambda})\right) \\ \left(\Delta_{x_j}\widehat{\kappa}(\pi)\right)|_{\pi=\pi_{r^{-2}\lambda}} &= r\Delta_{x_j}\left(\widehat{\kappa}(\pi_{r^{-2}\lambda})\right) \\ \left(\Delta_{y_j}\widehat{\kappa}(\pi)\right)|_{\pi=\pi_{r^{-2}\lambda}} &= r\Delta_{y_j}\left(\widehat{\kappa}(\pi_{r^{-2}\lambda})\right) \end{aligned} \right. \tag{6.39}$$

so we have for the second term of the right hand side in (6.37) and (6.38) respectively:

$$\begin{aligned} \sum_{j=1}^n\{\Delta_{x_j}\Delta_{y_j}\pi(\kappa)\}_{\pi=\pi_{-\lambda}} &= -\sum_{j=1}^n \Delta_{x_j}\Delta_{y_j}\left(\widehat{\kappa}(\pi_{-\lambda})\right), \\ \sum_{j=1}^n\{\Delta_{x_j}\Delta_{y_j}\pi(\kappa)\}_{\pi=\pi_{r^{-2}\lambda}} &= r^2\sum_{j=1}^n \Delta_{x_j}\Delta_{y_j}\left(\widehat{\kappa}(\pi_{r^{-2}\lambda})\right). \end{aligned}$$

Now viewing X_j and Y_j as elements of the Lie algebra and left invariant vector fields, we see using (6.23) and (6.6) that

$$\begin{aligned} \pi_{-\lambda}(X_j) &= \pi_{-\lambda}(\Theta(X_j)) = \pi_{-\lambda}(X_j\circ\Theta) = \pi_\lambda(X_j), \\ \pi_{-\lambda}(Y_j) &= -\pi_{-\lambda}(\Theta(Y_j)) = -\pi_{-\lambda}(Y_j\circ\Theta) = -\pi_\lambda(Y_j), \end{aligned}$$

and, using (6.25) and (6.4), we obtain

$$\begin{aligned} \pi_{r^{-2}\lambda}(X_j) &= \pi_\lambda(D_{r^{-1}}X_j) = r^{-1}\pi_\lambda(X_j), \\ \pi_{r^{-2}\lambda}(Y_j) &= \pi_\lambda(D_{r^{-1}}Y_j) = r^{-1}\pi_\lambda(Y_j). \end{aligned}$$

So from this and (6.39) we obtain for the third terms of the right hand side in (6.35) and in (6.36) that

$$\begin{aligned} &\frac{i}{-2\lambda}\sum_{j=1}^n\{\pi(Y_j)\Delta_{y_j}\pi(\kappa)+\Delta_{x_j}\pi(\kappa)\pi(X_j)\}_{\pi=\pi_{-\lambda}} \\ &\qquad = -\frac{i}{2\lambda}\sum_{j=1}^n \pi_{-\lambda}(Y_j)\Delta_{y_j}\pi_{-\lambda}(\kappa)+\Delta_{x_j}\pi_{-\lambda}(\kappa)\pi_{-\lambda}(X_j), \\ &\frac{i}{2r^{-2}\lambda}\sum_{j=1}^n\{\pi(Y_j)\Delta_{y_j}\pi(\kappa)+\Delta_{x_j}\pi(\kappa)\pi(X_j)\}_{\pi=\pi_{r^{-2}\lambda}} \\ &\qquad = r^2\frac{i}{2\lambda}\sum_{j=1}^n \pi_{r^{-2}\lambda}(Y_j)\Delta_{y_j}\pi_{r^{-2}\lambda}(\kappa)+\Delta_{x_j}\pi_{r^{-2}\lambda}(\kappa)\pi_{-\lambda}(X_j). \end{aligned}$$

Collecting the new expressions for the three terms of the right hand sides in (6.35) and in (6.36) we obtain a new proof for Equalities (6.35) and (6.36).

Proof of Lemma 6.3.6. Let κ be in some $\mathcal{K}_{a,b}(\mathbb{H}_n)$ and $h \in \mathcal{S}(\mathbb{R}^n)$. We start by differentiating with respect to λ the expression from Lemma 6.2.4:

$$\pi_\lambda(\kappa)h(u) = \int_{\mathbb{H}_n} \kappa(x,y,t) e^{i\lambda(-t+\frac{1}{2}xy)} e^{-i\sqrt{\lambda}yu} h(u - \sqrt{|\lambda|}x) dxdydt,$$

and obtain

$$\partial_\lambda \{\pi_\lambda(\kappa)h(u)\} = \int_{\mathbb{H}_n} \kappa(x,y,t) e^{i\lambda(-t+\frac{1}{2}xy)} e^{-i\sqrt{\lambda}yu}$$
$$\left(\left[i(-t + \frac{1}{2}xy) - i\frac{yu}{2\sqrt{|\lambda|}} \right] h(u - \sqrt{|\lambda|}x) - \frac{1}{2\sqrt{\lambda}} x\nabla h(u - \sqrt{|\lambda|}x) \right) dxdydt;$$

indeed with our convention we have

$$x\nabla h = \sum_{j=1}^n x_j \partial_{u_j} h, \quad \text{and} \quad \partial_\lambda\{\sqrt{\lambda}\} = \frac{1}{2\sqrt{|\lambda|}}, \quad \partial_\lambda\{\sqrt{|\lambda|}\} = \frac{1}{2\sqrt{\lambda}}.$$

We can now interpret the formula above in the light of difference operators as

$$\begin{aligned} \partial_\lambda \pi_\lambda(\kappa) &= i\pi_\lambda((-t + \frac{1}{2}xy)\kappa) + \sum_{j=1}^n \left\{ -\frac{iu_j}{2\sqrt{|\lambda|}} \pi_\lambda(y_j\kappa) - \frac{1}{2\sqrt{\lambda}} \pi_\lambda(x_j\kappa)\partial_{u_j} \right\} \\ &= -i\Delta_t \pi_\lambda(\kappa) + \frac{i}{2} \sum_{j=1}^n \Delta_{x_j}\Delta_{y_j}\pi_\lambda(\kappa) \\ &\qquad -\frac{1}{2\lambda} \sum_{j=1}^n \left\{ \pi_\lambda(Y_j) \left(\Delta_{y_j}\pi_\lambda(\kappa)\right) + \left(\Delta_{x_j}\pi_\lambda(\kappa)\right) \pi_\lambda(X_j) \right\}, \end{aligned}$$

using (6.11). □

We already know that

$$\Delta_t \mathrm{I} = 0 \quad \text{and} \quad \Delta_t \pi_\lambda(X_k) = \Delta_t \pi_\lambda(Y_k) = 0, \tag{6.40}$$

see Example 5.2.8 and Lemma 5.2.9, but we can also test it with the formula given in Lemma 6.3.6. We also obtain the following (more substantial) examples:

Example 6.3.8. We can compute

$$\Delta_t \pi_\lambda(T) = -\mathrm{I}, \tag{6.41}$$

and

$$\Delta_t \pi_\lambda(\mathcal{L}) = 0. \tag{6.42}$$

Proof. Since

$$\pi_\lambda(T) = i\lambda \mathrm{I}$$

(see Lemma 6.2.1), we compute directly $\partial_\lambda \pi_\lambda(T) = i\mathrm{I}$. By (6.31) and (6.33), we know

$$\Delta_{y_j}\pi_\lambda(T) = \Delta_{x_j}\pi_\lambda(T) = 0,$$

thus we have obtained (6.41) by Lemma 6.3.6. Furthermore, by (6.12), we have

$$\partial_\lambda \pi_\lambda(\mathcal{L}) = \mathrm{sgn}(\lambda) \sum_{j=1}^n \left(\partial_{u_j}^2 - u_j^2\right) = \frac{1}{\lambda}\pi_\lambda(\mathcal{L})$$

and by (6.32) and (6.34)

$$\begin{aligned} &\sum_{j=1}^n \left\{\pi_\lambda(Y_j)\Delta_{y_j}\pi_\lambda(\mathcal{L}) + \Delta_{x_j}\pi_\lambda(\mathcal{L})\pi_\lambda(X_j)\right\} \\ &\qquad = -\sum_{j=1}^n \left\{\pi_\lambda(Y_j)2\pi_\lambda(Y_j) + 2\pi_\lambda(X_j)\pi_\lambda(X_j) = -2\pi_\lambda(\mathcal{L})\right\}, \end{aligned}$$

and also by Example 6.3.4, we get

$$\Delta_{x_j}\Delta_{y_j}\pi_\lambda(\mathcal{L}) = -\Delta_{x_j}2\pi_\lambda(Y_j) = 0.$$

Combining all these equalities together with Lemma 6.3.6 yields (6.42). □

Note that (6.42) can also be obtained from (6.40) and the Leibniz formula (in the sense of (5.28)) for Δ_t.

In terms of λ-symbols, we obtain

Corollary 6.3.9. *If* $\widehat{\kappa}(\pi_\lambda) \equiv \pi_\lambda(\kappa) = \mathrm{Op}^W(a_\lambda)$ *with* $a_\lambda = \{a_\lambda(\xi, u)\}$, *then*

$$\Delta_t\widehat{\kappa}\ (\pi_\lambda) = i\mathrm{Op}^W\left(\tilde{\partial}_{\lambda,\xi,u} a_\lambda\right),$$

where

$$\tilde{\partial}_{\lambda,\xi,u} := \partial_\lambda - \frac{1}{2\lambda}\sum_{j=1}^n \left(u_j\partial_{u_j} + \xi_j\partial_{\xi_j}\right). \tag{6.43}$$

Proof. Using formulae (6.28), Corollary 6.3.3 and the properties of the Weyl calculus (see especially the composition formula in (6.16)), we obtain easily that

$$\begin{aligned} \pi_\lambda(Y_j)\Delta_{y_j}\pi_\lambda(\kappa) &= \mathrm{Op}^W\left(i\sqrt{\lambda}u_j\right)\mathrm{Op}^W\left(\frac{-1}{i\sqrt{\lambda}}\partial_{u_j}a_\lambda\right) \\ &= -\mathrm{Op}^W(u_j)\,\mathrm{Op}^W\left(\partial_{u_j}a_\lambda\right) \\ &= -\mathrm{Op}^W\left(u_j\partial_{u_j}a_\lambda - \frac{1}{2i}\partial_{\xi_j}\partial_{u_j}a_\lambda\right), \end{aligned}$$

and

$$\begin{aligned}
\Delta_{x_j}\pi_\lambda(\kappa)\pi_\lambda(X_j) &= \mathrm{Op}^W\left(\frac{-1}{i\sqrt{|\lambda|}}\partial_{\xi_j}a_\lambda\right)\mathrm{Op}^W\left(i\sqrt{|\lambda|}\xi_j\right)\\
&= -\mathrm{Op}^W\left(\partial_{\xi_j}a_\lambda\right)\mathrm{Op}^W\left(\xi_j\right)\\
&= -\mathrm{Op}^W\left((\partial_{\xi_j}a_\lambda)\xi_j-\frac{1}{2i}\partial_{u_j}\partial_{\xi_j}a_\lambda\right),
\end{aligned}$$

thus

$$\begin{aligned}
&\pi_\lambda(Y_j)\Delta_{y_j}\pi_\lambda(\kappa)+\Delta_{x_j}\pi_\lambda(\kappa)\pi_\lambda(X_j)\\
&\quad= -\mathrm{Op}^W\left(u_j\partial_{u_j}a_\lambda-\frac{1}{2i}\partial_{\xi_j}\partial_{u_j}a_\lambda\right)-\mathrm{Op}^W\left((\partial_{\xi_j}a_\lambda)\xi_j-\frac{1}{2i}\partial_{u_j}\partial_{\xi_j}a_\lambda\right).\\
&\quad= \mathrm{Op}^W\left(-u_j\partial_{u_j}a_\lambda-\xi_j\partial_{\xi_j}a_\lambda+\frac{1}{i}\partial_{\xi_j}\partial_{u_j}a_\lambda\right).
\end{aligned}$$

We also have

$$\begin{aligned}
\Delta_{x_j}\Delta_{y_j}\pi_\lambda(\kappa) &= \mathrm{Op}^W\left(\frac{-1}{i\sqrt{|\lambda|}}\partial_{\xi_j}\frac{-1}{i\sqrt{\lambda}}\partial_{u_j}a_\lambda\right)\\
&= -\frac{1}{\lambda}\mathrm{Op}^W\left(\partial_{\xi_j}\partial_{u_j}a_\lambda\right). \qquad (6.44)
\end{aligned}$$

Bringing these equalities in the formula for Δ_t in Lemma 6.3.6, we obtain

$$\begin{aligned}
\Delta_t\pi_\lambda(\kappa) &= i\partial_\lambda\pi_\lambda(\kappa)+\frac{1}{2}\sum_{j=1}^n\Delta_{x_j}\Delta_{y_j}\pi_\lambda(\kappa)\\
&\qquad+\frac{i}{2\lambda}\sum_{j=1}^n\left\{\pi_\lambda(Y_j)\Delta_{y_j}\pi_\lambda(\kappa)+\Delta_{x_j}\pi_\lambda(\kappa)\pi_\lambda(X_j)\right\}\\
&= i\mathrm{Op}^W(\partial_\lambda a_\lambda)+\frac{1}{2}\sum_{j=1}^n-\frac{1}{\lambda}\mathrm{Op}^W\left(\partial_{\xi_j}\partial_{u_j}a_\lambda\right)\\
&\qquad+\frac{i}{2\lambda}\sum_{j=1}^n\mathrm{Op}^W\left(-u_j\partial_{u_j}a_\lambda-\xi_j\partial_{\xi_j}a_\lambda+\frac{1}{i}\partial_{\xi_j}\partial_{u_j}a_\lambda\right)\\
&= \mathrm{Op}^W\left(i\partial_\lambda a_\lambda-\frac{i}{2\lambda}\sum_{j=1}^n\left(u_j\partial_{u_j}a_\lambda+\xi_j\partial_{\xi_j}a_\lambda\right)\right).
\end{aligned}$$

This completes the proof. □

6.3.3 Formulae

Here we summarise the formulae obtained so far in Sections 6.3.1 and 6.3.2. Let us recall our convention regarding square roots (6.9) setting

$$\sqrt{\lambda} := \operatorname{sgn}(\lambda)\sqrt{|\lambda|} = \begin{cases} \sqrt{\lambda} & \text{if } \lambda > 0 \\ -\sqrt{|\lambda|} & \text{if } \lambda < 0 \end{cases}.$$

For the Schrödinger infinitesimal representation we have obtained (see (6.11), (6.12) and (6.28)) that

$$\boxed{\begin{array}{rcl} \pi_\lambda(X_j) &=& \sqrt{|\lambda|}\partial_{u_j} &=& \mathrm{Op}^W\left(i\sqrt{|\lambda|}\xi_j\right) \\ \pi_\lambda(Y_j) &=& i\sqrt{\lambda}u_j &=& \mathrm{Op}^W\left(i\sqrt{\lambda}u_j\right) \\ \pi_\lambda(T) &=& i\lambda \mathrm{I} &=& \mathrm{Op}^W(i\lambda) \\ \pi_\lambda(\mathcal{L}) &=& |\lambda|\sum_j(\partial_{u_j}^2 - u_j^2) &=& \mathrm{Op}^W\left(|\lambda|\sum_j(-\xi_j^2 - u_j^2)\right) \end{array}}$$

while for difference operators (cf. Lemmata 6.3.1 and 6.3.6) we have

$$\boxed{\begin{array}{rcl} \Delta_{x_j}|_{\pi_\lambda} &=& \frac{1}{i\lambda}\mathrm{ad}\,(\pi_\lambda(Y_j)) &=& \frac{1}{\sqrt{|\lambda|}}\mathrm{ad}u_j \\ \Delta_{y_j}|_{\pi_\lambda} &=& -\frac{1}{i\lambda}\mathrm{ad}\,(\pi_\lambda(X_j)) &=& -\frac{1}{i\sqrt{\lambda}}\mathrm{ad}\partial_{u_j} \\ \Delta_t|_{\pi_\lambda} &=& i\partial_\lambda + \frac{1}{2}\sum_{j=1}^n \Delta_{x_j}\Delta_{y_j}|_{\pi_\lambda} + \frac{i}{2\lambda}\sum_{j=1}^n\{\pi_\lambda(Y_j)|_{\pi_\lambda}\Delta_{y_j} + \Delta_{x_j}|_{\pi_\lambda}\pi_\lambda(X_j)\} \end{array}}$$

and in terms of λ-symbols, that is, with

$$\widehat{\kappa}(\pi_\lambda) \equiv \pi_\lambda(\kappa) = \mathrm{Op}^W(a_\lambda) \text{ and } a_\lambda = \{a_\lambda(\xi, u)\},$$

(cf. Corollaries 6.3.3 and 6.3.9):

$$\boxed{\begin{array}{rcl} \Delta_{x_j}\pi_\lambda(\kappa) &=& i\mathrm{Op}^W\left(\frac{1}{\sqrt{|\lambda|}}\partial_{\xi_j}a_\lambda\right) \\ \Delta_{y_j}\pi_\lambda(\kappa) &=& i\mathrm{Op}^W\left(\frac{1}{\sqrt{\lambda}}\partial_{u_j}a_\lambda\right) \\ \Delta_t\pi_\lambda(\kappa) &=& i\mathrm{Op}^W\left(\tilde{\partial}_{\lambda,\xi,u}a_\lambda\right) \\ &=& i\mathrm{Op}^W\left((\partial_\lambda - \frac{1}{2\lambda}\sum_{j=1}^n\{u_j\partial_{u_j} + \xi_j\partial_{\xi_j}\})a_\lambda\right) \end{array}} \tag{6.45}$$

In Examples 6.3.4, 6.3.5, 6.3.8 together with (6.40), we have also obtained

	$\pi_\lambda(X_k)$	$\pi_\lambda(Y_k)$	$\pi_\lambda(T)$	$\pi_\lambda(\mathcal{L})$
Δ_{x_j}	$-\delta_{j=k}$	0	0	$-2\pi_\lambda(X_j)$
Δ_{y_j}	0	$-\delta_{j=k}$	0	$-2\pi_\lambda(Y_j)$
Δ_t	0	0	$-\mathrm{I}$	0

The equalities given in the following lemma concern another normalisation of the Weyl symbol which is motivated by (6.20) and by the fact that the expressions of the right-hand sides in (6.45), in particular for the operator $\tilde{\partial}_{\lambda,\xi,u}$, become then very simple:

Lemma 6.3.10. *Let $a_\lambda = \{a_\lambda(\xi, u)\}$ be a family of Weyl symbols depending smoothly on $\lambda \neq 0$. If $\tilde{a}_\lambda$ is the renormalisation obtained via*

$$a_\lambda(\xi, u) = \tilde{a}_\lambda(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u), \tag{6.46}$$

then

$$\begin{aligned}
\{\tilde{\partial}_{\lambda,\xi,u} a_\lambda\}(\xi, u) &= \{\partial_\lambda \tilde{a}_\lambda\}(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u), \\
\frac{1}{\sqrt{|\lambda|}}\{\partial_{\xi_j} a_\lambda\}(\xi, u) &= \{\partial_{\xi_j} \tilde{a}_\lambda\}(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u), \\
\frac{1}{\sqrt{\lambda}}\{\partial_{u_j} a_\lambda\}(\xi, u) &= \{\partial_{u_j} \tilde{a}_\lambda\}(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u).
\end{aligned}$$

Proof. We see that

$$\tilde{a}_\lambda(\xi, u) = a_\lambda\left(\frac{1}{\sqrt{|\lambda|}}\xi, \frac{1}{\sqrt{\lambda}}u\right),$$

thus

$$\begin{aligned}
\partial_\lambda \tilde{a}_\lambda(\xi, u) &= (\partial_\lambda a_\lambda)\left(\frac{1}{\sqrt{|\lambda|}}\xi, \frac{1}{\sqrt{\lambda}}u\right) \\
&\quad - \sum_{j=1}^n \frac{\xi_j}{2\lambda\sqrt{|\lambda|}}\left(\partial_{\xi_j} a_\lambda\right)\left(\frac{1}{\sqrt{|\lambda|}}\xi, \frac{1}{\sqrt{\lambda}}u\right) \\
&\quad - \sum_{j=1}^n \frac{u_j}{2|\lambda|\sqrt{|\lambda|}}\left(\partial_{u_j} a_\lambda\right)\left(\frac{1}{\sqrt{|\lambda|}}\xi, \frac{1}{\sqrt{\lambda}}u\right),
\end{aligned}$$

and

$$\begin{aligned}
\{\partial_\lambda \tilde{a}_\lambda\}\left(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u\right) &= (\partial_\lambda a_\lambda)(\xi, u) \\
&\quad - \sum_{j=1}^n \left(\frac{\sqrt{|\lambda|}\xi_j}{2\lambda\sqrt{|\lambda|}}\partial_{\xi_j} a_\lambda(\xi, u) + \frac{\sqrt{\lambda}u_j}{2|\lambda|\sqrt{|\lambda|}}\partial_{u_j} a_\lambda(\xi, u)\right) \\
&= \partial_\lambda a_\lambda(\xi, u) - \frac{1}{2\lambda}\sum_{j=1}^n \left(\xi_j \partial_{\xi_j} a_\lambda(\xi, u) + u_j \partial_{u_j} a_\lambda(\xi, u)\right) \\
&= \tilde{\partial}_{\lambda,\xi,u} a_\lambda(\xi, u).
\end{aligned}$$

This shows the first stated equality. The other two are easy. □

Lemma 6.3.10 and the formulae already obtained yield

$$\begin{aligned}
\Delta_{x_j}\pi_\lambda(\kappa) &= i\mathrm{Op}^W\left(\partial_{\xi_j}\tilde{a}_\lambda\right), \\
\Delta_{y_j}\pi_\lambda(\kappa) &= i\mathrm{Op}^W\left(\partial_{u_j}\tilde{a}_\lambda\right), \\
\Delta_t\pi_\lambda(\kappa) &= i\mathrm{Op}^W\left(\partial_\lambda\tilde{a}_\lambda\right),
\end{aligned}$$

where the λ-symbol a_λ of $\pi_\lambda(\kappa)$, that is, $\pi_\lambda(\kappa) = \mathrm{Op}^W(a_\lambda)$, has been rescaled via (6.46), i.e.

$$a_\lambda(\xi, u) = \tilde{a}_\lambda(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u).$$

Recall that

$$a_\lambda(\xi, u) = (2\pi)^{\frac{2n+1}{2}} \mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u, \lambda),$$

see (6.20), so

$$\tilde{a}_\lambda(\xi, u) = (2\pi)^{\frac{2n+1}{2}} \mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\xi, u, \lambda).$$

The above formulae in terms of the rescaled λ-symbols look neat. The drawback of using this rescaling is that one rescales the Weyl quantization:

$$\widehat{\kappa}(\pi_\lambda) = \mathrm{Op}^W(a_\lambda) = \mathrm{Op}^W\left(\tilde{a}_\lambda\left(\sqrt{|\lambda|}\,\cdot, \sqrt{\lambda}\,\cdot\right)\right).$$

Since our aim is to study the group Fourier transform on $\mathbb{H}_n$, it is more natural to study the Weyl-symbol a_λ without any rescaling.

In fact, the following two sections are devoted to understanding $\widehat{\kappa} \equiv \{\pi_\lambda(\kappa)\}$ as a family of Weyl pseudo-differential operators parametrised by $\lambda \in \mathbb{R}\backslash\{0\}$. The Weyl quantization will force us to work on the λ-symbol a_λ directly, and not on its rescaling $\tilde{a}_\lambda$.

This will lead to defining a family of symbol classes parametrised by $\lambda \in \mathbb{R}\backslash\{0\}$ for the λ-symbols a_λ. This will be done via a family of Hörmander metrics parametrised by $\lambda \in \mathbb{R}\backslash\{0\}$. Importantly the structural bounds of these metrics will be uniform with respect to λ. The resulting symbol classes will be called λ-Shubin classes.

6.4 Shubin classes

In this Section, we recall elements of the Weyl-Hörmander pseudo-differential calculus and the associated Sobolev spaces, and we apply this to obtain the Shubin classes of symbols and the associated Sobolev spaces. The dependence in a parameter λ will be of particular importance to us. We will call the resulting symbol classes the λ-Shubin classes.

6.4.1 Weyl-Hörmander calculus

Here we present the main elements of the Weyl-Hörmander calculus that will be relevant for our analysis. For more details on the underlying general theory, we can refer, for instance, to [Ler10].

We consider $\mathbb{R}^n$ and identify its cotangent bundle $T^*\mathbb{R}^n$ with $\mathbb{R}^{2n}$. The canonical symplectic form on $\mathbb{R}^{2n}$ is ω defined by

$$\omega(T, T') = x \cdot \xi' - x' \cdot \xi, \quad T = (\xi, x),\ T' = (\xi', x') \in \mathbb{R}^{2n}.$$

Definition 6.4.1. If q is a positive quadratic form on $\mathbb{R}^{2n}$, then we define its *conjugate* q^ω by

$$\forall T \in \mathbb{R}^{2n} \quad q^\omega(T) := \sup_{T' \in \mathbb{R}^{2n}\backslash\{0\}} \frac{|\omega(T,T')|^2}{q(T')},$$

and its *gain factor* by

$$\Lambda_q := \inf_{T \in \mathbb{R}^{2n}\backslash\{0\}} \frac{q^\omega(T)}{q(T)}.$$

Definition 6.4.2. A *metric* is a family of positive quadratic forms

$$g = \{g_X, X \in \mathbb{R}^{2n}\}$$

depending smoothly on $X \in \mathbb{R}^{2n}$.

- The metric g is *uncertain* when $\forall X \in \mathbb{R}^{2n}$, $\Lambda_{g_X} \geq 1$.
- The metric g is *slowly varying* when there exists a constant $\bar{C} > 0$ such that we have for any $X, X' \in \mathbb{R}^{2n}$:

$$g_X(X - X') \leq \bar{C}^{-1} \Longrightarrow \sup_{T \in \mathbb{R}^{2n}\backslash\{0\}} \left(\frac{g_X(T)}{g_{X'}(T)} + \frac{g_{X'}(T)}{g_X(T)} \right) \leq \bar{C}.$$

- The metric g is *temperate* when there are constants $\bar{C} > 0$ and $\bar{N} > 0$ such that we have for any $X, X' \in \mathbb{R}^{2n}$ and $T \in \mathbb{R}^{2n}\backslash\{0\}$:

$$\frac{g_X(T)}{g_{X'}(T)} \leq \bar{C}(1 + g_X^\omega(X - X'))^{\bar{N}}.$$

A metric g is of *Hörmander type* if it is uncertain, slowly varying and temperate. In this case the constants $\bar{C}$ and $\bar{N}$ appearing above and any constant depending only on them are called *structural*.

Proposition 6.4.3. *A metric $g = \{g_X, X \in \mathbb{R}^{2n}\}$ is* slowly varying *if and only if there exist constants $C, r > 0$ such that we have for any $X, Y \in \mathbb{R}^{2n}$ that*

$$g_X(Y - X) \leq r^2 \Longrightarrow \forall T \quad g_Y(T) \leq C g_X(T). \tag{6.47}$$

Proof. If g is slowly varying then it satisfies (6.47). Conversely, let us assume (6.47). Necessarily $C \geq 1$ since we can take $X = Y$ in (6.47). If $g_X(Y - X) \leq C^{-1} r^2$, then $g_X(Y - X) \leq r^2$ and, applying (6.47) with $T = Y - X$, we obtain

$$g_Y(Y - X) \leq C g_X(Y - X) \leq r^2,$$

thus re-applying (6.47) (but at g_Y), we have $g_X(T) \leq C g_Y(T)$ for all T. This shows that g is slowly varying. □

Remark 6.4.4. If g satisfies (6.47) with constant $C > 1$ and $r > 0$ then g is slowly varying with a constant $\bar{C} = \min(C^{-1}r^2, 2C)$.

Example 6.4.5. Let ϕ be a positive smooth function on $\mathbb{R}^{2n}$ which is Lipschitz on $\mathbb{R}^{2n}$. We denote by $T \mapsto |T|^2$ the canonical (Euclidean) quadratic form on $\mathbb{R}^{2n}$. The metric g given by

$$g_X(T) = \phi(X)^{-2}|T|^2$$

is slowly varying.

Proof. Let us assume $g_X(Y - X) \leq r^2$ for a constant $r > 0$ to be determined. This means $|Y - X| \leq r\phi(X)$. Since ϕ is Lipschitz on $\mathbb{R}^{2n}$, denoting by L its Lipschitz constant, we have

$$\phi(X) \leq \phi(Y) + L|X - Y| \leq \phi(Y) + Lr\phi(X),$$

thus

$$(1 - Lr)\phi(X) \leq \phi(Y).$$

Hence if we choose $r > 0$ so that $1 - Lr > 0$, we have obtained

$$\forall T \qquad g_Y(T) \leq Cg_X(T),$$

with $C = (1 - Lr)^{-1}$. This shows that g_X satisfies (6.47) and is therefore slowly varying. □

Remark 6.4.6. If ϕ is L-Lipschitz then g given in Example 6.4.5 satisfies (6.47) with any $r \in (0, L^{-1})$ and a corresponding $C = (1 - Lr)^{-1}$.

Definition 6.4.7. Let g be a metric of Hörmander type. A positive function M defined on $\mathbb{R}^{2n}$ is a *g-weight* when there are structural constants $\bar{C}'$ and $\bar{N}'$ satisfying for any $X, Y \in \mathbb{R}^{2n}$:

$$g_X(X - Y) \leq \bar{C}'^{-1} \Longrightarrow \frac{M(X)}{M(Y)} + \frac{M(Y)}{M(X)} \leq \bar{C}',$$

and

$$\frac{M(X)}{M(Y)} \leq \bar{C}(1 + g_X^{\omega}(X - Y))^{\bar{N}'}.$$

It is easy to check that the set of g-weights forms a group for the usual multiplication of positive functions.

Definition 6.4.8 (Hörmander symbol class $S(M, g)$)**.** Let g be a metric of Hörmander type and M a g-weight on $\mathbb{R}^{2n}$. The symbol class $S(M, g)$ is the set of functions $a \in C^\infty(\mathbb{R}^{2n})$ such that for each integer $\ell \in \mathbb{N}_0$, the quantity

$$\|a\|_{S(M,g),\ell} := \sup_{\substack{\ell' \leq \ell, X \in \mathbb{R}^{2n} \\ g_X(T_{\ell'}) \leq 1}} \frac{|\partial_{T_1} \ldots \partial_{T_{\ell'}} a(X)|}{M(X)}$$

is finite.

Here $\partial_T a$ denotes the quantity (da, T).

The following properties are well known [Ler10, Chapters 1 and 2]:

Theorem 6.4.9. *Let g be a metric of Hörmander type and let M, M_1, M_2 be g-weights.*

1. *The symbol class $S(M, g)$ is a vector space endowed with a Fréchet topology via the family of seminorms $\|\cdot\|_{S(M,g),\ell}$, $\ell \in \mathbb{N}_0$.*
2. *If $a \in S(M, g)$ then the symbol b defined by*
$$\mathrm{Op}^W b = \left(\mathrm{Op}^W a\right)^*$$
is in $S(M, g)$ as well. Furthermore, for any $\ell \in \mathbb{N}_0$ there exist a constant $C > 0$ and a integer $\ell' \in \mathbb{N}_0$ such that
$$\|b\|_{S(M,g),\ell} \leq C\|a\|_{S(M,g),\ell'}.$$
The constant C and the integer ℓ' may be chosen to depend on ℓ and on the structural constants and to be independent of g, M and a.
3. *If $a_1 \in S(M_1, g)$ and $a_2 \in S(M_2, g)$ then the symbol b defined by*
$$\mathrm{Op}^W b = \left(\mathrm{Op}^W a_1\right)\left(\mathrm{Op}^W a_2\right),$$
is in $S(M_1M_2, g)$. Furthermore, for any $\ell \in \mathbb{N}_0$ there exist a constant $C > 0$ and two integers $\ell_1, \ell_2 \in \mathbb{N}_0$ such that
$$\|b\|_{S(M_1M_2,g),\ell} \leq C\|a_1\|_{S(M_1,g),\ell_1}\|a_2\|_{S(M_2,g),\ell_2}.$$
The constant C and the integers ℓ_1, ℓ_2 may be chosen to depend on ℓ and on the structural constants and to be independent of g, M_1, M_2 and a_1, a_2.

Definition 6.4.10 (Sobolev spaces $H(M, g)$)**.** Let g be a metric of Hörmander type and M a g-weight on $\mathbb{R}^{2n}$. We denote by $H(M, g)$ the set of all tempered distributions f on $\mathbb{R}^n$ such that for any symbol $a \in S(M, g)$ we have $\mathrm{Op}^W(a)f \in L^2(\mathbb{R}^n)$.

Theorem 6.4.11. *Let g be a metric of Hörmander type on $\mathbb{R}^{2n}$.*

1. *The space $H(1, g)$ coincides with $L^2(\mathbb{R}^n)$. Furthermore, there exist a structural constant $C > 0$ and a structural integer $\ell \in \mathbb{N}_0$ such that for any symbol $a \in S(1, g)$, we have*
$$\|\mathrm{Op}^W(a)\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \leq C\|a\|_{S(1,g),\ell}.$$
2. *Let M_1, M_2 be g-weights. For any $a \in S(M_1, g)$, the operator $\mathrm{Op}^W(a)$ maps continuously $H(M_2, g)$ to $H(M_2M_1^{-1}, g)$. Furthermore, there exist a constant $C > 0$ and an integer $\ell \in \mathbb{N}_0$ such that*
$$\|\mathrm{Op}^W(a)\|_{\mathscr{L}(H(M_2,g),H(M_2M_1^{-1},g))} \leq C\|a\|_{S(M_1,g),\ell}.$$
The constant C and the integers ℓ may be chosen to depend only on the structural constants of g, M_1, M_2 and to be independent of g, M and a.

6.4.2 Shubin classes $\Sigma_\rho^m(\mathbb{R}^n)$ and the harmonic oscillator

It is well known (and can be readily checked) that the metric

$$\frac{d\xi^2 + du^2}{(1+|u|^2+|\xi|^2)^\rho},$$

is of Hörmander type with corresponding weights $(1+|u|^2+|\xi|^2)^{m/2}$ for $m \in \mathbb{R}$. This will be also shown later in the proof of Proposition 6.4.21. For $m \in \mathbb{R}$ and $\rho \in (0,1]$, we denote by $\Sigma_\rho^m(\mathbb{R}^n)$ the corresponding symbol class, often called the Shubin classes of symbols on $\mathbb{R}^n$:

$$\Sigma_\rho^m(\mathbb{R}^n) := S\left((1+|u|^2+|\xi|^2)^{m/2}, \frac{d\xi^2 + du^2}{(1+|u|^2+|\xi|^2)^\rho}\right).$$

This means that a symbol $a \in C^\infty(\mathbb{R}^{2n})$ is in $\Sigma_\rho^m(\mathbb{R}^n)$ if and only if for any $\alpha, \beta \in \mathbb{N}_0^n$ there exists a constant $C = C_{\alpha,\beta} > 0$ such that

$$\forall(\xi,u) \in \mathbb{R}^{2n} \qquad |\partial_\xi^\alpha \partial_u^\beta a(\xi,u)| \le C\left(1+|\xi|^2+|u|^2\right)^{\frac{m-\rho(|\alpha|+|\beta|)}{2}}.$$

The class $\Sigma_\rho^m(\mathbb{R}^n)$ is a vector subspace of $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which becomes a Fréchet space when endowed with the family of seminorms

$$\|a\|_{\Sigma_\rho^m,N} = \sup_{\substack{(\xi,u)\in\mathbb{R}^n\times\mathbb{R}^n \\ |\alpha|,|\beta|\le N}} \left(1+|\xi|^2+|u|^2\right)^{-\frac{m-\rho(|\alpha|+|\beta|)}{2}} |\partial_\xi^\alpha \partial_u^\beta a(\xi,u)|,$$

where $N \in \mathbb{N}_0$. We denote by

$$\Psi\Sigma_\rho^m(\mathbb{R}^n) := \mathrm{Op}^W(\Sigma_\rho^m(\mathbb{R}^n))$$

the corresponding class of operators and by $\|\cdot\|_{\Psi\Sigma_\rho^m,N}$ the corresponding seminorms.

We have the inclusions

$$\rho_1 \ge \rho_2 \quad \text{and} \quad m_1 \le m_2 \Longrightarrow \Psi\Sigma_{\rho_1}^{m_1}(\mathbb{R}^n) \subset \Psi\Sigma_{\rho_2}^{m_2}(\mathbb{R}^n).$$

Example 6.4.12. The operators $\partial_{u_j} = \mathrm{Op}^W(i\xi_j)$, $j = 1, \dots, n$, or multiplication by $u_k = \mathrm{Op}^W(u_k)$, $k = 1, \dots, n$, are two operators in $\Psi\Sigma_1^1(\mathbb{R}^n)$.

Standard computations also show:

Example 6.4.13. For each $m \in \mathbb{R}$, the symbol b^m, where

$$b(\xi,u) = \sqrt{1+|u|^2+|\xi|^2},$$

is in $\Sigma_1^m(\mathbb{R}^n)$.

The following is well known and can be viewed more generally as a consequence of the Weyl-Hörmander calculus (see Theorem 6.4.9)

Theorem 6.4.14. • *The class of operators $\cup_{m\in\mathbb{R}}\Psi\Sigma_\rho^m(\mathbb{R}^n)$ forms an algebra of operators stable by taking the adjoint. Furthermore, the operations*

$$\begin{array}{ccc}\Psi\Sigma_\rho^m(\mathbb{R}^n) & \longrightarrow & \Psi\Sigma_\rho^m(\mathbb{R}^n)\\ A & \longmapsto & A^*\end{array}$$

and

$$\begin{array}{ccc}\Psi\Sigma_\rho^{m_1}(\mathbb{R}^n)\times\Psi\Sigma_\rho^{m_2}(\mathbb{R}^n) & \longrightarrow & \Psi\Sigma_\rho^{m_1+m_2}(\mathbb{R}^n)\\ (A,B) & \longmapsto & AB\end{array}$$

are continuous.

- *The operators in $\Psi\Sigma_\rho^0(\mathbb{R}^n)$ extend boundedly to $L^2(\mathbb{R}^n)$. Furthermore, there exist $C>0$ and $N\in\mathbb{N}$ such that if $A\in\Psi\Sigma_\rho^0(\mathbb{R}^n)$ then*

$$\|A\|_{\mathscr{L}(L^2(\mathbb{R}^n))}\leq C\|A\|_{\Psi\Sigma_\rho^m,N}.$$

From Example 6.4.12, it follows that the (positive) *harmonic oscillator*

$$Q:=\sum_{j=1}^n(-\partial_{u_j}^2+u_j^2), \tag{6.48}$$

is in $\Psi\Sigma_1^2(\mathbb{R}^n)$.

Note that from now on Q denotes the harmonic oscillator and not the homogeneous dimension as in all previous chapters.

We keep the same notation for Q and for its self-adjoint extension as an unbounded operator on $L^2(\mathbb{R}^n)$. The harmonic oscillator Q is a positive (unbounded) operator on $L^2(\mathbb{R}^n)$. Its spectrum is

$$\{2|\ell|+n,\ell\in\mathbb{N}_0^n\},$$

where $|\ell|=\ell_1+\ldots+\ell_n$. The eigenfunctions associated with the eigenvalues $2|\ell|+n$ are

$$h_\ell: x=(x_1,\ldots,x_n)\longmapsto h_{\ell_1}(x_1)\ldots h_{\ell_n}(x_n),$$

where each h_j, $j=0,1,2\ldots$, is a Hermite function, that is,

$$h_j(\tau)=(-1)^j\frac{e^{\frac{\tau^2}{2}}}{\sqrt{2^j j!\sqrt{\pi}}}\frac{d^j}{d\tau^j}e^{-\tau^2},\qquad \tau\in\mathbb{R}.$$

The Hermite functions are Schwartz, i.e. $h_j\in\mathcal{S}(\mathbb{R})$. With our choice of normalisation, the functions h_j, $j=0,1,\ldots$, form an orthonormal basis of $L^2(\mathbb{R})$. Therefore, the functions h_ℓ form an orthonormal basis of $L^2(\mathbb{R}^n)$. For each $s\in\mathbb{R}$, we define the operator $(\mathrm{I}+Q)^{s/2}$ using the functional calculus, that is, in this case, the domain of $(\mathrm{I}+Q)^{s/2}$ is the space of functions

$$\mathrm{Dom}(\mathrm{I}+Q)^{s/2} = \{h \in L^2(\mathbb{R}^n) : \sum_{\ell\in\mathbb{N}_0^n} (2|\ell|+n)^s |(h_\ell, h)_{L^2(\mathbb{R}^n)}|^2 < \infty\},$$

and if $h \in \mathrm{Dom}(\mathrm{I}+Q)^{s/2}$ then

$$(\mathrm{I}+Q)^{s/2} h = \sum_{\ell\in\mathbb{N}_0^n} (2|\ell|+n)^{s/2} (h_\ell, h)_{L^2(\mathbb{R}^n)} h_\ell.$$

6.4.3 Shubin Sobolev spaces

In this section, we study Shubin Sobolev spaces. Many of their properties, especially their equivalent characterisations, are well known. Their proofs are quite easy but often omitted in the literature. Thus we have chosen to sketch their demonstrations.

The Shubin Sobolev spaces below are a special case of Sobolev spaces for measurable fields on representation spaces, see Definition 5.1.6.

Our starting point will be the following definition for the Shubin Sobolev spaces:

Definition 6.4.15. Let $s \in \mathbb{R}$. The *Shubin Sobolev space* $\mathcal{Q}_s(\mathbb{R}^n)$ is the subspace of $\mathcal{S}'(\mathbb{R}^n)$ which is the completion of $\mathrm{Dom}(\mathrm{I}+Q)^{s/2}$ for the norm

$$\|h\|_{\mathcal{Q}_s} := \|(\mathrm{I}+Q)^{s/2} h\|_{L^2(\mathbb{R}^n)}.$$

They satisfy the following properties:

Theorem 6.4.16. *1. The space $\mathcal{Q}_s(\mathbb{R}^n)$ is a Hilbert space endowed with the sesquilinear form*

$$(g,h)_{\mathcal{Q}_s} = \Big((\mathrm{I}+Q)^{s/2} g, (\mathrm{I}+Q)^{s/2} h\Big)_{L^2(\mathbb{R}^n)}.$$

We have the inclusions

$$\mathcal{S}(\mathbb{R}^n) \subset \mathcal{Q}_{s_1}(\mathbb{R}^n) \subset \mathcal{Q}_{s_2}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n), \quad s_1 > s_2.$$

We also have

$$L^2(\mathbb{R}^n) = \mathcal{Q}_0(\mathbb{R}^n) \quad \text{and} \quad \mathcal{S}(\mathbb{R}^n) = \bigcap_{s\in\mathbb{R}} \mathcal{Q}_s(\mathbb{R}^n).$$

2. The dual of $\mathcal{Q}_s(\mathbb{R}^n)$ may be identified with $\mathcal{Q}_{-s}(\mathbb{R}^n)$ via the distributional duality form $\langle g,h\rangle = \int_{\mathbb{R}^n} gh$.

3. *If $s \in \mathbb{N}_0$, $\mathcal{Q}_s(\mathbb{R}^n)$ coincides with*

$$\mathcal{Q}_s(\mathbb{R}^n) = \{h \in L^2(\mathbb{R}^n) : u^\alpha \partial_u^\beta h \in L^2(\mathbb{R}^n) \quad \forall \alpha, \beta \in \mathbb{N}_0^n,\ |\alpha| + |\beta| \leq s\}.$$

Furthermore, the norm given by

$$\|h\|_{\mathcal{Q}_s}^{(int)} = \sum_{|\alpha|+|\beta|\leq s} \|u^\alpha \partial_u^\beta h\|_{L^2(\mathbb{R}^n)},$$

is equivalent to $\|\cdot\|_{\mathcal{Q}_s}$.

4. *For any $s \in \mathbb{R}$, $\mathcal{Q}_s(\mathbb{R}^n)$ coincides with the completion (in $\mathcal{S}'(\mathbb{R}^n)$) of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ for the norm*

$$\|h\|_{\mathcal{Q}_s}^{(b)} = \|\mathrm{Op}^W\,(b^s)\,h\|_{L^2(\mathbb{R}^n)},$$

where b was given in Example 6.4.13. The norm $\|\cdot\|_{\mathcal{Q}_s}^{(b)}$ extended to $\mathcal{Q}_s(\mathbb{R}^n)$ is equivalent to $\|\cdot\|_{\mathcal{Q}_s}$.

5. *For any $s \in \mathbb{R}$, the Shubin Sobolev space $\mathcal{Q}_s(\mathbb{R}^n)$ coincides with the Sobolev space associated with the following metric weight (see Definition 6.4.10)*

$$\mathcal{Q}_s(\mathbb{R}^n) = H\left((1 + |u|^2 + |\xi|^2)^{s/2}, \frac{d\xi^2 + du^2}{1 + |u|^2 + |\xi|^2}\right).$$

6. *For any $s \in \mathbb{R}$, the operators $\mathrm{Op}^W(b^{-s})(\mathrm{I} + Q)^{s/2}$ and $(\mathrm{I} + Q)^{s/2}\mathrm{Op}^W(b^{-s})$ are bounded and invertible on $L^2(\mathbb{R}^n)$.*

7. *The complex interpolation between the spaces $\mathcal{Q}_{s_0}(\mathbb{R}^n)$ and $\mathcal{Q}_{s_1}(\mathbb{R}^n)$ is*

$$(\mathcal{Q}_{s_0}(\mathbb{R}^n), \mathcal{Q}_{s_1}(\mathbb{R}^n))_\theta = \mathcal{Q}_{s_\theta}(\mathbb{R}^n), \qquad s_\theta = (1-\theta)s_0 + \theta s_1,\ \theta \in (0,1).$$

Before giving the proof of Theorem 6.4.16, let us recall the definition of complex interpolation:

Definition 6.4.17 (Complex interpolation)**.** Let X_0 and X_1 be two subspaces of a vector space Z. We assume that X_0 and X_1 are Banach spaces with norms denoted by $|\cdot|_j$, $j = 0, 1$.

Let $\mathscr{X}$ be the space of the functions f defined on the strip $\bar{S} = \{0 \leq \mathrm{Re}\, z \leq 1\}$ and valued in $X_0 + X_1$ such that f is continuous on $\bar{S}$ and holomorphic in $S = \{0 < \mathrm{Re}\, z < 1\}$. For $f \in \mathscr{X}$ we define the quantity (possibly infinite)

$$\|f\|_{\mathscr{X}} := \sup_{y\in\mathbb{R}}\{|f(iy)|_0, |f(1+iy)|_1\}.$$

The complex interpolation space of exponent $\theta \in (0,1)$ is the space $(X_0, X_1)_\theta$ of vectors $v \in X_0 + X_1$ such that there exists $f \in \mathscr{X}$ satisfying $f(\theta) = v$ and $\|f\|_{\mathscr{X}} < \infty$.

The space $(X_0, X_1)_\theta$ is a subspace of Z; it is a Banach space when endowed with the norm given by

$$|v|_\theta := \inf\{\|f\|_{\mathscr{X}} : f \in \mathscr{X} \quad \text{and} \quad f(\theta) = v\}.$$

We also refer to Appendix A.6 for the notion of analytic interpolation.

Proof of Theorem 6.4.16. From Definition 6.4.15, it is easy to prove that the space $\mathcal{Q}_s(\mathbb{R}^n)$ is a Hilbert space, that it is included in $\mathcal{S}'(\mathbb{R}^n)$ and that $\mathcal{Q}_0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. It is a routine exercise left to the reader that the dual of $\mathcal{Q}_s(\mathbb{R}^n)$ is $\mathcal{Q}_{-s}(\mathbb{R}^n)$ via the distributional duality (Part (2)) and that the spaces $\mathcal{Q}_s(\mathbb{R}^n)$ decrease with $s \in \mathbb{R}$.

Let us prove the complex interpolation property of Part (7). We may assume $s_1 > s_0$. For $h \in \mathcal{Q}_{s_\theta}$, we consider the function

$$f(z) := (\mathrm{I} + Q)^{\frac{-(zs_1+(1-z)s_0)+s_\theta}{2}} h,$$

and we check easily that

$$f(\theta) = h, \quad \|f(iy)\|_{\mathcal{Q}_{s_0}} = \|f(1+iy)\|_{\mathcal{Q}_{s_1}} = \|h\|_{\mathcal{Q}_{s_\theta}} \quad \forall y \in \mathbb{R}.$$

This shows that $\mathcal{Q}_{s_\theta}$ is continuously included in $(\mathcal{Q}_{s_0}(\mathbb{R}^n), \mathcal{Q}_{s_1}(\mathbb{R}^n))_\theta$. By duality of the complex interpolation and of the $\mathcal{Q}_s(\mathbb{R}^n)$, we obtain the reverse inclusion and Part (7) is proved.

Let us prove Part (4). For any $s \in \mathbb{R}$, the operator $\mathrm{Op}^W(b^s)$ maps $\mathcal{S}(\mathbb{R}^n)$ to itself and the mapping $\|\cdot\|_{\mathcal{Q}_s}^{(b)}$ as defined in Part (4) is a norm on $\mathcal{S}(\mathbb{R}^n)$. We denote its completion in $\mathcal{S}'(\mathbb{R}^n)$ by $\mathcal{Q}_s^{(b)}(\mathbb{R}^n)$. From the properties of the calculus it is again a routine exercise left to the reader that the dual of $\mathcal{Q}_s^{(b)}(\mathbb{R}^n)$ is $\mathcal{Q}_{-s}^{(b)}(\mathbb{R}^n)$ via the distributional duality and that the spaces $\mathcal{Q}_s^{(b)}(\mathbb{R}^n)$ decrease with $s \in \mathbb{R}$.

We can prove the following property about interpolation between the $\mathcal{Q}^{(b)}(\mathbb{R}^n)$ spaces which is analogous to Part (7):

$$(\mathcal{Q}_{s_0}^{(b)}(\mathbb{R}^n), \mathcal{Q}_{s_1}^{(b)}(\mathbb{R}^n))_\theta = \mathcal{Q}_{s_\theta}^{(b)}(\mathbb{R}^n), \qquad s_\theta = (1-\theta)s_0 + \theta s_1, \ \theta \in (0,1). \quad (6.49)$$

Indeed we may assume $s_1 > s_0$. For $h \in \mathcal{Q}_{s_\theta}^{(b)}$, we consider the function

$$f(z) = e^{z(s_z - s_\theta)} \mathrm{Op}^W\left(b^{-s_z+s_\theta}\right) h \quad \text{where} \quad s_z = (1-z)s_0 + zs_1.$$

Clearly $f(\theta) = h$. Furthermore,

$$\begin{aligned} \|f(iy)\|_{\mathcal{Q}_{s_1}}^{(b)} &= |e^{iy(s_{iy}-s_\theta)}| \|\mathrm{Op}^W(b^{s_1})\mathrm{Op}^W\left(b^{-s_{iy}+s_\theta}\right) h\|_{L^2(\mathbb{R}^n)} \\ &\leq e^{-y^2(s_1-s_0)} \|\mathrm{Op}^W(b^{s_1})\mathrm{Op}^W\left(b^{-s_{iy}+s_\theta}\right)\mathrm{Op}^W(b^{-s_\theta})\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \\ &\qquad \|h\|_{\mathcal{Q}_{s_\theta}}^{(b)}, \end{aligned} \quad (6.50)$$

and

$$\begin{aligned} \|f(1+iy)\|_{\mathcal{Q}_{s_0}}^{(b)} &= |e^{(1+iy)(s_{1+iy}-s_\theta)}| \|\mathrm{Op}^W(b^{s_0})\mathrm{Op}^W\left(b^{-s_{1+iy}+s_\theta}\right) h\|_{L^2(\mathbb{R}^n)} \\ &\leq e^{s_1-s_\theta-y^2(s_1-s_0)} \|\mathrm{Op}^W(b^{s_0})\mathrm{Op}^W\left(b^{-s_{1+iy}+s_\theta}\right)\mathrm{Op}^W(b^{-s_\theta})\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \\ &\qquad \|h\|_{\mathcal{Q}_{s_\theta}}^{(b)}. \end{aligned} \quad (6.51)$$

From the calculus we obtain that the two operator norms on $L^2(\mathbb{R}^n)$ in (6.50) and (6.51) are bounded by a constant of the form $C(1+|y|)^N$ where $C>0$ and $N\in\mathbb{N}_0$ are independent of y. This shows that $\mathcal{Q}^{(b)}_{s_\theta}$ is continuously included in $(\mathcal{Q}^{(b)}_{s_0}(\mathbb{R}^n),\mathcal{Q}^{(b)}_{s_1}(\mathbb{R}^n))_\theta$. By duality of the complex interpolation and of the spaces $\mathcal{Q}_s(\mathbb{R}^n)$, we obtain the reverse inclusion and (6.49) is proved.

Let us show that the spaces $\mathcal{Q}^{(b)}_s(\mathbb{R}^n)$ and $\mathcal{Q}_s(\mathbb{R}^n)$ coincide. First let us assume $s\in 2\mathbb{N}_0$. We have for any $h\in\mathcal{Q}^{(b)}_s(\mathbb{R}^n)$:

$$\|h\|_{\mathcal{Q}_s}\leq \|(\mathrm{I}+\mathcal{Q})^{s/2}\mathrm{Op}^W(b^{-s})\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\|h\|^{(b)}_{\mathcal{Q}_s}.$$

As $\mathcal{Q}\in\Psi\Sigma^2_1(\mathbb{R}^n)$, by Theorem 6.4.14, the operator $(\mathrm{I}+\mathcal{Q})^{s/2}\mathrm{Op}^W(b^{-s})$ is in $\Psi\Sigma^0_1$ and thus is bounded on $L^2(\mathbb{R}^n)$. We have obtained a continuous inclusion of $\mathcal{Q}^{(b)}_s(\mathbb{R}^n)$ into $\mathcal{Q}_s(\mathbb{R}^n)$. Conversely, we have for any $h\in\mathcal{Q}_s(\mathbb{R}^n)$ that

$$\|h\|^{(b)}_{\mathcal{Q}_s}\leq \|\mathrm{Op}^W(b^s)(\mathrm{I}+\mathcal{Q})^{-s/2}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\|h\|_{\mathcal{Q}_s}.$$

The inverse of $\mathrm{Op}^W(b^s)(\mathrm{I}+\mathcal{Q})^{-s/2}$ is $(\mathrm{I}+\mathcal{Q})^{s/2}(\mathrm{Op}^W(b^s))^{-1}$ since the operators $\mathrm{I}+\mathcal{Q}$ and $\mathrm{Op}^W(b^s)$ are invertible. Moreover, for the same reason as above, $(\mathrm{I}+\mathcal{Q})^{s/2}(\mathrm{Op}^W(b^s))^{-1}$ is bounded on $L^2(\mathbb{R}^n)$. By the inverse mapping theorem, $\mathrm{Op}^W(b^s)(\mathrm{I}+\mathcal{Q})^{-s/2}$ is bounded on $L^2(\mathbb{R}^n)$. This shows the reverse continuous inclusion. We have proved

$$\mathcal{Q}^{(b)}_s(\mathbb{R}^n)=\mathcal{Q}_s(\mathbb{R}^n)$$

with equivalence of norms for $s\in 2\mathbb{N}_0$ and this implies that this is true for any $s\in\mathbb{R}$ by the properties of duality and interpolation for $\mathcal{Q}^{(b)}_s(\mathbb{R}^n)$ and $\mathcal{Q}_s(\mathbb{R}^n)$. This shows Part (4) and implies Parts (5) and (6).

Let us show that, for each $s\in\mathbb{N}_0$, the space $\mathcal{Q}_s(\mathbb{R}^n)$ coincides with the space $\mathcal{Q}^{(int)}_s(\mathbb{R}^n)$ of functions $h\in L^2(\mathbb{R}^n)$ such that the tempered distributions $u^\alpha\partial_u^\beta h$ are in $L^2(\mathbb{R}^n)$ for every $\alpha,\beta\in\mathbb{N}_0^n$ such that $|\alpha|+|\beta|\leq s$. Endowed with the norm $\|\cdot\|^{(int)}_{\mathcal{Q}_s}$ defined in Part (3), $\mathcal{Q}^{(int)}_s(\mathbb{R}^n)$ is a Banach space. We have for any $h\in\mathcal{Q}_s(\mathbb{R}^n)=\mathcal{Q}^{(b)}_s(\mathbb{R}^n)$

$$\|h\|^{(int)}_{\mathcal{Q}_s}\leq\sum_{|\alpha|+|\beta|\leq s}\|u^\alpha\partial_u^\beta\mathrm{Op}^W(b^{-s})\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\|h\|^{(b)}_{\mathcal{Q}_s}.$$

Since the operators $u^\alpha\partial_u^\beta\mathrm{Op}^W(b^{-s})$ are in $\Psi\Sigma_1^{|\alpha|+|\beta|-s}(\mathbb{R}^n)$ thus continuous on $L^2(\mathbb{R}^n)$ when $|\alpha|+|\beta|\leq s$, we see that $\mathcal{Q}_s(\mathbb{R}^n)$ is continuously included in $\mathcal{Q}^{(int)}_s(\mathbb{R}^n)$. For the converse, we separate the cases s even and odd. If $s\in 2\mathbb{N}_0$ then we have easily that

$$\begin{aligned}\|h\|_{\mathcal{Q}_s} &= \|\big(\mathrm{I}+\sum_j(-\partial_{u_j}^2+u_j^2)\big)^{s/2}h\|_{L^2(\mathbb{R}^n)}\\ &\leq C_s\sum_{|\alpha|+|\beta|\leq s}\|u^\alpha\partial_u^\beta h\|_{L^2(\mathbb{R}^n)}=C_s\|h\|^{(int)}_{\mathcal{Q}_s}.\end{aligned}$$

Now if $s \in 2\mathbb{N}_0 + 1$, we have, since $\mathrm{Op}^W(b^{-1})(\mathrm{I}+Q)^{1/2}$ is bounded and invertible (see Part (6) already proven),

$$\begin{aligned}\|h\|_{\mathcal{Q}_s} &= \|(\mathrm{I}+Q)^{s/2}h\|_{L^2(\mathbb{R}^n)} \leq C\|\mathrm{Op}^W(b^{-1})(\mathrm{I}+Q)^{1/2}(\mathrm{I}+Q)^{s/2}h\|_{L^2(\mathbb{R}^n)} \\ &\leq C\|\mathrm{Op}^W(b^{-1})(\mathrm{I}+\sum_j -\partial_{u_j}^2 + u_j^2)^{(s+1)/2}h\|_{L^2(\mathbb{R}^n)} \\ &\leq C_s \sum_{|\alpha|+|\beta|\leq s+1} \|\mathrm{Op}^W(b^{-1})x^\alpha\partial_x^\beta h\|_{L^2(\mathbb{R}^n)} \\ &\leq C_s \sum_{|\alpha'|+|\beta'|\leq s} \|u^{\alpha'}\partial_u^{\beta'}h\|_{L^2(\mathbb{R}^n)} = C_s\|h\|_{\mathcal{Q}_s}^{(int)},\end{aligned}$$

by the property of the calculus. Therefore, for s even and odd, $\mathcal{Q}_s^{(int)}(\mathbb{R}^n)$ is continuously included in $\mathcal{Q}_s(\mathbb{R}^n)$. As we have already proven the reverse inclusion, the equality holds and Part (3) is proved. This implies

$$\bigcap_{s\in\mathbb{R}} \mathcal{Q}_s(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$$

and Part (1) is now completely proved. □

These Sobolev spaces enable us to characterise the operators in the calculus. We allow ourselves to use the shorthand notation

$$(\mathrm{ad}u)^{\alpha_1} := (\mathrm{ad}u_1)^{\alpha_{11}} \dots (\mathrm{ad}u_n)^{\alpha_{1n}},$$

and

$$(\mathrm{ad}\partial_u)^{\alpha_2} := (\mathrm{ad}\partial_{u_1})^{\alpha_{21}} \dots (\mathrm{ad}\partial_{u_n})^{\alpha_{2n}}.$$

Theorem 6.4.18. *We assume that $\rho \in (0,1]$. Let $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a linear continuous operator such that all the operators*

$$(\mathrm{ad}u)^{\alpha_1}(\mathrm{ad}\partial_u)^{\alpha_2}A, \quad \alpha_1, \alpha_2 \in \mathbb{N}_0^n,$$

are in $\mathscr{L}(L^2(\mathbb{R}^n), \mathcal{Q}_{-m+\rho(|\alpha_1|+|\alpha_2|)})$ in the sense that they extend to continuous operators from $L^2(\mathbb{R}^n)$ to $\mathcal{Q}_{-m+\rho(|\alpha_1|+|\alpha_2|)}$. Then $A \in \Psi\Sigma_\rho^m(\mathbb{R}^n)$. Moreover, for any $\ell \in \mathbb{N}$, there exist a constant C and an integer ℓ', both independent of A, such that

$$\|A\|_{\Psi\Sigma_\rho^m,\ell} \leq C \sum_{|\alpha_1|+|\alpha_2|\leq\ell'} \|(\mathrm{ad}u)^{\alpha_1}(\mathrm{ad}\partial_u)^{\alpha_2}A\|_{\mathscr{L}(L^2(\mathbb{R}^n),\mathcal{Q}_{-m+\rho(|\alpha_1|+|\alpha_2|)})}.$$

Note that the converse is true, that is, given $A \in \Psi\Sigma_\rho^m$ then

$$\forall\alpha_1,\alpha_2 \in \mathbb{N}_0^n \qquad (\mathrm{ad}u)^{\alpha_1}(\mathrm{ad}\partial_u)^{\alpha_2}A \in \mathscr{L}(L^2(\mathbb{R}^n), \mathcal{Q}_{-m+\rho(|\alpha_1|+|\beta|)},).$$

This is just a consequence of the properties of the calculus.

The proof of Theorem 6.4.18 relies on the following characterisation of the class of symbols

$$\Sigma_0^0(\mathbb{R}^n) := S(1, d\xi^2 + du^2).$$

Theorem 6.4.19 (Beals' characterisation of $\Sigma_0^0(\mathbb{R}^n)$)**.** *Let $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a linear continuous operator such that all the operators*

$$(\text{ad}u)^{\alpha_1}(\text{ad}\partial_u)^{\alpha_2} A, \quad \alpha_1, \alpha_2 \in \mathbb{N}_0^n,$$

are in $\mathscr{L}(L^2(\mathbb{R}^n))$ in the sense that they extend to continuous operators on $L^2(\mathbb{R}^n)$. Then there exits a unique function $a = \{a(\xi, x)\} \in \Sigma_0^0(\mathbb{R}^n)$ such that $A = \text{Op}^W(a)$. Moreover, for any $\ell \in \mathbb{N}$, there exist a constant C and an integer ℓ', both independent of A, such that

$$\|a\|_{\Sigma_0^0,\ell} \leq C \sum_{|\alpha_1|+|\alpha_2|\leq \ell'} \|(\text{ad}u)^{\alpha_1}(\text{ad}\partial_u)^{\alpha_2} A\|_{\mathscr{L}(L^2(\mathbb{R}^n))}.$$

The converse is true, that is, given $a \in \Sigma_0^0(\mathbb{R}^n)$ then $A = \text{Op}^W(a)$ satisfies

$$\forall \alpha_1, \alpha_2 \in \mathbb{N}_0^n \qquad (\text{ad}u)^{\alpha_1}(\text{ad}\partial_u)^{\alpha_2} A \in \mathscr{L}(L^2(\mathbb{R}^n)).$$

We admit Beals' theorem stated in Theorem 6.4.19, see the original article [Bea77a] for the proof.

For the sake of completeness we prove Theorem 6.4.18. This proof can also be found in [Hel84a, Théorème 1.21.1].

Sketch of the proof of Theorem 6.4.18. Let A be as in the statement and b as in Example 6.4.13. We write

$$B_s := \text{Op}^W(b^s)$$

and

$$A_{\alpha_1,\alpha_2} := (\text{ad}u)^{\alpha_1}(\text{ad}\partial_u)^{\alpha_2} A, \quad \alpha_1, \alpha_2 \in \mathbb{N}_0^n.$$

We set $s := m - \rho(|\alpha_1| + |\alpha_2|)$. Then $B_s^{-1} A_{\alpha_1,\alpha_2} \in \mathscr{L}(L^2(\mathbb{R}^n))$. Moreover, we have

$$\text{ad}\partial_{u_1}\left(B_s^{-1} A_{\alpha_1,\alpha_2}\right) = \left(\text{ad}\partial_{u_1}\left(B_s^{-1}\right)\right) A_{\alpha_1,\alpha_2} + B_s^{-1}\text{ad}\partial_{u_1}\left(A_{\alpha_1,\alpha_2}\right);$$

the first operator of the right-hand side is in $\mathscr{L}(L^2(\mathbb{R}^n), \mathcal{Q}_1(\mathbb{R}^n))$ whereas the second is in $\mathscr{L}(L^2(\mathbb{R}^n), \mathcal{Q}_\rho(\mathbb{R}^n))$. Proceeding recursively, we obtain that the operator $B^{-1}_{m-\rho(|\alpha_1|+|\alpha_2|)} A_{\alpha_1,\alpha_2}$ satisfies the hypothesis of Beals' Theorem (Theorem 6.4.19). Therefore, there exists $c_{\alpha_1,\alpha_2} \in \Sigma_0^0(\mathbb{R}^n)$ such that

$$B^{-1}_{m-\rho(|\alpha_1|+|\alpha_2|)} A_{\alpha_1,\alpha_2} = \text{Op}^W(c_{\alpha_1,\alpha_2})$$

or, equivalently,

$$A_{\alpha_1,\alpha_2} = \mathrm{Op}^W(a_{\alpha_1,\alpha_2}) \quad \text{with} \quad a_{\alpha_1,\alpha_2} = b_{m-\rho(|\alpha_1|+|\alpha_2|)} \star c_{\alpha_1,\alpha_2}.$$

We have $A = \mathrm{Op}^W(a_{0,0})$ and

$$\begin{aligned} \mathrm{Op}^W(a_{\alpha_1,\alpha_2}) &= A_{\alpha_1,\alpha_2} = (\mathrm{ad}u)^{\alpha_1}(\mathrm{ad}\partial_u)^{\alpha_2}A \\ &= (\mathrm{ad}u)^{\alpha_1}(\mathrm{ad}\partial_u)^{\alpha_2}\mathrm{Op}^W(a_{0,0}) \\ &= \mathrm{Op}^W\left(i^{|\alpha_1|}\partial_\xi^{\alpha_1}\partial_u^{\alpha_2}a_{0,0}\right), \end{aligned}$$

by Lemma 6.2.3, thus

$$a_{\alpha_1,\alpha_2} = i^{|\alpha_1|}\partial_\xi^{\alpha_1}\partial_u^{\alpha_2}a_{0,0}.$$

Consequently $a \in \Sigma_\rho^m$. □

Looking back at the proof, we see that it can be slightly improved in the following way:

Corollary 6.4.20. *We assume that $\rho \in (0,1]$. Let $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a linear continuous operator.*

The operator A is in $\Psi\Sigma_\rho^m(\mathbb{R}^n)$ if and only if there exists $\gamma_o \in \mathbb{R}$ such that for each $\alpha_1, \alpha_2 \in \mathbb{N}_0^n$ we have

$$(\mathrm{ad}u)^{\alpha_1}(\mathrm{ad}\partial_u)^{\alpha_2}A \in \mathscr{L}(\mathcal{Q}_{\gamma_o}(\mathbb{R}^n), \mathcal{Q}_{-m+\rho(|\alpha_1|+|\alpha_2|)+\gamma_o}).$$

In this case this property is true for every $\gamma \in \mathbb{R}$, that is, for each $\gamma \in \mathbb{R}$ and $\alpha_1, \alpha_2 \in \mathbb{N}_0^n$, we have

$$(\mathrm{ad}u)^{\alpha_1}(\mathrm{ad}\partial_u)^{\alpha_2}A \in \mathscr{L}(\mathcal{Q}_{\gamma}(\mathbb{R}^n), \mathcal{Q}_{-m+\rho(|\alpha_1|+|\alpha_2|)+\gamma}).$$

Moreover, for any $\ell \in \mathbb{N}$, there exist a constant C and an integer ℓ', both independent of A, such that

$$\|A\|_{\Psi\Sigma_\rho^m,\ell} \leq C \sum_{|\alpha_1|+|\alpha_2|\leq\ell'} \|(\mathrm{ad}u)^{\alpha_1}(\mathrm{ad}\partial_u)^{\alpha_2}A\|_{\mathscr{L}(\mathcal{Q}_\gamma(\mathbb{R}^n), \mathcal{Q}_{-m+\rho(|\alpha_1|+|\alpha_2|)+\gamma})}.$$

Sketch of the proof of Corollary 6.4.20. We keep the notation of the proof of Theorem 6.4.18. Let A be as in the statement and let $s := m - \rho(|\alpha_1| + |\alpha_2|)$. Then $B_{s+\gamma_o}^{-1}A_{\alpha_1,\alpha_2}B_{\gamma_o} \in \mathscr{L}(L^2(\mathbb{R}^n))$. Moreover, we have

$$\begin{aligned} \mathrm{ad}\partial_{u_1}\left(B_{s+\gamma_o}^{-1}A_{\alpha_1,\alpha_2}B_{\gamma_o}\right) &= \left(\mathrm{ad}\partial_{u_1}\left(B_{s+\gamma_o}^{-1}\right)\right)A_{\alpha_1,\alpha_2}B_{\gamma_o} \\ &\quad + B_{s+\gamma_o}^{-1}\mathrm{ad}\partial_{u_1}\left(A_{\alpha_1,\alpha_2}\right)B_{\gamma_o} \\ &\quad + B_{s+\gamma_o}^{-1}A_{\alpha_1,\alpha_2}B_{\gamma_o}\ B_{\gamma_o}^{-1}\left(\mathrm{ad}\partial_{u_1}B_{\gamma_o}\right); \end{aligned}$$

the first operator of the right-hand side is in $\mathscr{L}(L^2(\mathbb{R}^n), \mathcal{Q}_1(\mathbb{R}^n))$, the second is in $\mathscr{L}(L^2(\mathbb{R}^n), \mathcal{Q}_\rho(\mathbb{R}^n))$ and the third is in $\mathscr{L}(L^2(\mathbb{R}^n))$. Proceeding recursively, we obtain that $B_{s+\gamma_o}^{-1}A_{\alpha_1,\alpha_2}B_{\gamma_o}$ satisfies the hypothesis of Theorem 6.4.19. We then conclude as in the proof of Theorem 6.4.18. □

6.4.4 The λ-Shubin classes $\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$

The Shubin metric depending on a parameter $\lambda \in \mathbb{R}\backslash\{0\}$ is the metric $g^{(\lambda)}$ on $\mathbb{R}^{2n}$ defined via

$$g^{(\rho,\lambda)}_{\xi,u}(d\xi, du) := \left(\frac{|\lambda|}{1+|\lambda|(1+|\xi|^2+|u|^2)}\right)^{\rho}(d\xi^2+du^2).$$

The associated positive function $M^{(\lambda)}$ on $\mathbb{R}^{2n}$ is defined via

$$M^{(\lambda)}(\xi,u) := \big(1+|\lambda|(1+|\xi|^2+|u|^2)\big)^{\frac{1}{2}}.$$

These λ-families of metrics and weights were first introduced in [BFKG12a] in the case $\rho = 1$. The authors of [BFKG12a] realised that, placing λ as above, the structural constants may be chosen independently of λ:

Proposition 6.4.21. *For each $\lambda \in \mathbb{R}\backslash\{0\}$, the metric $g^{(\rho,\lambda)}$ is of Hörmander type (see Definition 6.4.2) and the function $M^{(\lambda)}$ is a $g^{(\rho,\lambda)}$-weight (see Definition 6.4.7). Furthermore, if $\rho \in (0,1]$ is fixed, then the structural constants for $g^{(\rho,\lambda)}$ and for $M^{(\lambda)}$ can be chosen independent of λ.*

The proof of Proposition 6.4.21 follows the proof of the case $\rho = 1$ given in [BFKG12a, Proposition 1.20].

Proof of Proposition 6.4.21. The conjugate of $g^{(\rho,\lambda)}_{\xi,u}$ is $(g^{(\rho,\lambda)}_{\xi,u})^\omega$ given by

$$(g^{(\rho,\lambda)}_{\xi,u})^\omega(d\xi, du) = \left(\frac{1+|\lambda|(1+|\xi|^2+|u|^2)}{|\lambda|}\right)^{\rho}(d\xi^2+du^2).$$

The gain is then

$$\Lambda_{g^{(\rho,\lambda)}_{\xi,u}} = \left(\frac{1+|\lambda|(1+|\xi|^2+|u|^2)}{|\lambda|}\right)^{2\rho}.$$

We have for any ρ, λ, ξ, u:

$$\Lambda_{g^{(\rho,\lambda)}_{\xi,u}} \geq \left(\frac{1+|\lambda|}{|\lambda|}\right)^{2\rho} \geq 1.$$

This proves the uniform uncertain property in Definition 6.4.2.

To show that the metric $g^{\rho,\lambda}$ is slowly varying, we notice that it is of the form $\phi(X)^{-2}|T|^2$ as in Example 6.4.5 with

$$\phi(X) = \left(\frac{1+|\lambda|(1+|X|^2)}{|\lambda|}\right)^{\rho/2}.$$

We compute the gradient of ϕ and obtain

$$\begin{array}{rcl}|\nabla_X \phi| & = & \rho|\lambda|^{1-\frac{\rho}{2}}|X|(1+|\lambda|(1+|X|^2))^{\frac{\rho}{2}-1} \\ & \leq & \begin{cases} \rho\left(\frac{|\lambda|}{1+|\lambda|}\right)^{1-\frac{\rho}{2}} \leq \rho & \text{if } |X| \leq 1, \\ \rho\left(\frac{|\lambda||X|^2}{1+|\lambda||X|^2}\right)^{1-\frac{\rho}{2}} |X|^{1-2(1-\frac{\rho}{2})} \leq \rho & \text{if } |X| > 1. \end{cases}\end{array}$$

So ϕ is ρ-Lipschitz on $\mathbb{R}^{2n}$. Therefore, $g^{\rho,\lambda}$ is slowly varying with a constant $\bar{C}$ independent of λ (see Example 6.4.5 as well as Remarks 6.4.4 and 6.4.6).

Let us prove that $g^{\rho,\lambda}$ is temperate. For any $X, Y \in \mathbb{R}^{2n}$ we have

$$|Y|^2 \leq 2|X|^2 + 2|X-Y|^2;$$

thus

$$\frac{1+|\lambda|(1+|Y|^2)}{1+|\lambda|(1+|X|^2)} \leq 2 + 2\frac{|\lambda|}{1+|\lambda|(1+|X|^2)}|X-Y|^2. \tag{6.52}$$

Now

$$|\lambda| \leq 1+|\lambda|(1+|X|^2) \quad \text{thus} \quad \left(\frac{|\lambda|}{1+|\lambda|(1+|X|^2)}\right)^{1+\rho} \leq 1,$$

and

$$\frac{|\lambda|}{1+|\lambda|(1+|X|^2)} \leq \left(\frac{1+|\lambda|(1+|X|^2)}{|\lambda|}\right)^{\rho}.$$

Plugging this into (6.52), we obtain

$$\frac{1+|\lambda|(1+|Y|^2)}{1+|\lambda|(1+|X|^2)} \leq 2 + 2\left(\frac{1+|\lambda|(1+|X|^2)}{|\lambda|}\right)^{\rho} |X-Y|^2.$$

Taking the ρth power yields

$$\begin{array}{rcl}\dfrac{g_X^{(\rho,\lambda)}(T)}{g_Y^{(\rho,\lambda)}(T)} & = & \left(\dfrac{1+|\lambda|(1+|Y|^2)}{1+|\lambda|(1+|X|^2)}\right)^{\rho} \\ & \leq & 2^{\rho}\left(1+\left(\dfrac{1+|\lambda|(1+|X|^2)}{|\lambda|}\right)^{\rho}|X-Y|^2\right)^{\rho} \\ & & = 2^{\rho}\left(1+(g_X^{(\rho,\lambda)})^{\omega}(X-Y)\right)^{\rho}.\end{array}$$

This shows that $g^{(\rho,\lambda)}$ is temperate with constant independent of λ.

So far we have shown that $g^{(\rho,\lambda)}$ is a metric of Hörmander type. Following the same computations, it is not difficult to show that $M^{(\lambda)}$ are g-weights with constants independent of λ. This concludes the proof of Proposition 6.4.21. □

Let $\rho \in (0,1]$ be a fixed parameter.

For each parameter $\lambda \in \mathbb{R}\backslash\{0\}$, we define the λ-*Shubin classes* by

$$\Sigma^m_{\rho,\lambda}(\mathbb{R}^n) := S\left(\left(M^{(\lambda)}\right)^m, g^{(\rho,\lambda)}\right),$$

where we have used the Hörmander notation to define a class of symbols in terms of a metric and a weight, see Definition 6.4.8.

Here this means that $\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$ is the class of functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for each $N \in \mathbb{N}_0$, the quantity

$$\|a\|_{\Sigma^m_{\rho,\lambda},N} := \sup_{\substack{(\xi,u)\in\mathbb{R}^n\times\mathbb{R}^n \\ |\alpha|,|\beta|\leq N}} |\lambda|^{-\rho\frac{|\alpha|+|\beta|}{2}} \left(1+|\lambda|(1+|\xi|^2+|u|^2)\right)^{-\frac{m-\rho(|\alpha|+|\beta|)}{2}} |\partial^\alpha_\xi \partial^\beta_u a(\xi,u)|,$$

is finite. This also means that a symbol $a = \{a(\xi,u)\}$ is in $\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$ if and only if it satisfies

$$\begin{aligned} &\forall \alpha,\beta \in \mathbb{N}^n_0 \qquad \exists C = C_{\alpha,\beta} > 0 \qquad \forall (\xi,u) \in \mathbb{R}^n \times \mathbb{R}^n \\ &|\partial^\alpha_\xi \partial^\beta_u a(\xi,u)| \leq C|\lambda|^{\rho\frac{|\alpha|+|\beta|}{2}} \left(1+|\lambda|(1+|\xi|^2+|u|^2)\right)^{\frac{m-\rho(|\alpha|+|\beta|)}{2}}. \end{aligned} \tag{6.53}$$

The class of symbols $\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$ is a vector subspace of $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which becomes a Fréchet space when endowed with the family of seminorms $\|\cdot\|_{\Sigma^m_{\rho,\lambda},N}$, $N \in \mathbb{N}_0$. We denote by

$$\Psi\Sigma^m_{\rho,\lambda}(\mathbb{R}^n) := \mathrm{Op}^W(\Sigma^m_{\rho,\lambda}(\mathbb{R}^n))$$

the corresponding class of operators, and by $\|\cdot\|_{\Psi\Sigma^m_{\rho,\lambda},N}$ the corresponding seminorms on the Fréchet space $\Psi\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$.

It is clear that all the spaces of the same order m and parameter ρ coincide in the sense that

$$\forall \lambda \neq 0 \qquad \Sigma^m_{\rho,\lambda}(\mathbb{R}^n) = \Sigma^m_{\rho,1}(\mathbb{R}^n) = \Sigma^m_\rho(\mathbb{R}^n), \tag{6.54}$$

and the same is true for $\Psi\Sigma^m_{\rho,\lambda}(\mathbb{R}^n) = \Psi\Sigma^m_\rho(\mathbb{R}^n)$. However, the seminorms

$$\|\cdot\|_{\Sigma^m_{\rho,\lambda},N} \quad \text{and} \quad \|\cdot\|_{\Psi\Sigma^m_{\rho,\lambda},N}$$

carry the dependence on λ. This dependence on λ will be crucial for our purposes. From the general properties of metrics of Hörmander type (see Theorem 6.4.9 and Proposition 6.4.21), we readily obtain the following 'λ-uniform' calculus.

Proposition 6.4.22. *1. If, for each $\lambda \in \mathbb{R}\backslash\{0\}$, we are given a symbol $a_\lambda = \{a_\lambda(\xi,u)\}$ in $\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$ such that*

$$\forall N \in \mathbb{N}_0 \qquad \sup_{\lambda\neq 0} \|a_\lambda\|_{\Sigma^m_{\rho,\lambda},N} < \infty, \tag{6.55}$$

then each symbol b_λ defined by

$$\mathrm{Op}^W b_\lambda = \left(\mathrm{Op}^W a_\lambda\right)^*$$

is in $\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$ as well. Furthermore, for any $\ell \in \mathbb{N}_0$ there exist a constant $C > 0$ and a integer $\ell' \in \mathbb{N}_0$ such that for any $\lambda \neq 0$

$$\|b_\lambda\|_{\Sigma^m_{\rho,\lambda},\ell} \leq C \|a_\lambda\|_{\Sigma^m_{\rho,\lambda},\ell'}.$$

The constant C and the integer ℓ' may be chosen to depend on ℓ, m, n and to be independent of λ and a.

2. *If, for each $\lambda \in \mathbb{R}\backslash\{0\}$, we are given two symbols $a_{1,\lambda} = \{a_{1,\lambda}(\xi, u)\}$ in $\Sigma^{m_1}_{\rho,\lambda}(\mathbb{R}^n)$ and $a_{2,\lambda} = \{a_{2,\lambda}(\xi, u)\}$ in $\Sigma^{m_2}_{\rho,\lambda}(\mathbb{R}^n)$ such that*

$$\forall N \in \mathbb{N}_0 \qquad \sup_{\lambda\neq 0} \|a_{1,\lambda}\|_{\Sigma^{m_1}_{\rho,\lambda},N} < \infty \quad \text{and} \quad \sup_{\lambda\neq 0} \|a_{2,\lambda}\|_{\Sigma^{m_2}_{\rho,\lambda},N} < \infty,$$

then each symbol b_λ defined by

$$\mathrm{Op}^W b_\lambda = \left(\mathrm{Op}^W a_{1,\lambda}\right) \left(\mathrm{Op}^W a_{2,\lambda}\right),$$

is in $\Sigma^{m_1+m_2}_{\rho,\lambda}(\mathbb{R}^n)$. Furthermore, for any $\ell \in \mathbb{N}_0$ there exist a constant $C > 0$ and two integers $\ell_1, \ell_2 \in \mathbb{N}_0$ such that

$$\|b_\lambda\|_{\Sigma^{m_1+m_2}_\lambda,\ell} \leq C \|a_{1,\lambda}\|_{\Sigma^{m_1}_{\rho,\lambda},\ell_1} \|a_{2,\lambda}\|_{\Sigma^{m_2}_{\rho,\lambda},\ell_2}.$$

The constant C and the integers ℓ_1, ℓ_2 may be chosen to depend on ℓ, m_1, m_2, n and to be independent of λ and $a_{1,\lambda}, a_{2,\lambda}$.

We will say that a family of symbols $a_\lambda = \{a_\lambda(\xi, u)\}$, $\lambda \in \mathbb{R}\backslash\{0\}$, which satisfies Property (6.55) is λ-*uniform* in $\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$. The corresponding family of operators via the Weyl quantization is said to be λ-uniform in $\Psi\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$.

Let us give some useful examples of such families of operators.

Example 6.4.23. The families of symbols given by

$$\pi_\lambda(X_j) = i\sqrt{|\lambda|}\xi_j, \quad \pi_\lambda(Y_j) = i\sqrt{\lambda}u_j \quad \text{and} \quad \pi_\lambda(T) = i\lambda$$

are λ-uniform in $\Sigma^1_{1,\lambda}(\mathbb{R}^n)$, $\Sigma^1_{1,\lambda}(\mathbb{R}^n)$, and $\Sigma^2_{1,\lambda}(\mathbb{R}^n)$, respectively.

In particular, the constant operator $\pi_\lambda(T) = i\lambda$ has to be considered as being of order 2 because of the dependence on λ.

Proof. We want to estimate the supremum over $\lambda \neq 0$ of each of the seminorms

$$\|\pi_\lambda(X_j)\|_{\Psi\Sigma^1_{1,\lambda},N} = \|i\sqrt{|\lambda|}\xi_j\|_{\Sigma^1_{1,\lambda},N} \quad \text{and} \quad \|\pi_\lambda(Y_j)\|_{\Psi\Sigma^1_{1,\lambda},N} = \|i\sqrt{\lambda}u_j\|_{\Sigma^1_{1,\lambda},N}.$$

We compute directly for $N = 0$:

$$\sup_{\lambda\neq 0} \|i\sqrt{|\lambda|}\xi_j\|_{\Sigma^1_{1,\lambda},0} = \sup_{\lambda\neq 0,(\xi,u)\in\mathbb{R}^n\times\mathbb{R}^n} \frac{\sqrt{|\lambda|}|\xi_j|}{\sqrt{1+|\lambda|(1+|\xi|^2+|u|^2)}} < \infty,$$

$$\sup_{\lambda\neq 0} \|i\sqrt{\lambda}u_j\|_{\Sigma^1_{1,\lambda},0} = \sup_{\lambda\neq 0,(\xi,u)\in\mathbb{R}^n\times\mathbb{R}^n} \frac{\sqrt{|\lambda|}|u_j|}{\sqrt{1+|\lambda|(1+|\xi|^2+|u|^2)}} < \infty,$$

and

$$\sup_{\substack{|\alpha|+|\beta|=1\\(\xi,u)\in\mathbb{R}^n\times\mathbb{R}^n}} |\partial_\xi^\alpha\partial_u^\beta\{\sqrt{|\lambda|}\xi_j\}| = \sup_{\substack{|\alpha|+|\beta|=1\\(\xi,u)\in\mathbb{R}^n\times\mathbb{R}^n}} |\partial_\xi^\alpha\partial_u^\beta\{\sqrt{\lambda}u_j\}| = \sqrt{|\lambda|},$$

therefore

$$\sup_{\lambda\neq 0} \|i\sqrt{|\lambda|}\xi_j\|_{\Sigma^1_{1,\lambda},1} < \infty \quad \text{and} \quad \sup_{\lambda\neq 0} \|i\sqrt{\lambda}u_j\|_{\Sigma^1_{1,\lambda},1} < \infty.$$

Since all the higher derivatives $\partial_\xi^\alpha\partial_u^\beta$ with $|\alpha|+|\beta|>1$ of the symbols $i\sqrt{|\lambda|}\xi_j$ and $i\sqrt{\lambda}u_j$ are zero, we obtain that the families of symbols given by $\pi_\lambda(X_j)$, $\pi_\lambda(Y_j)$, are λ-uniform in $\Sigma^1_{1,\lambda}(\mathbb{R}^n)$.

For $\pi_\lambda(T) = \mathrm{Op}^W(i\lambda)$, we see that

$$\|i\lambda\|_{\Sigma^2_{1,\lambda},0} = \sup_{(\xi,u)\in\mathbb{R}^n\times\mathbb{R}^n} \frac{|i\lambda|}{1+|\lambda|(1+|\xi|^2+|u|^2)} < \infty,$$

and since $i\lambda$ is a constant, its derivatives are zero and the family of symbols given by $\pi_\lambda(T)$, is λ-uniform in $\Sigma^2_{1,\lambda}(\mathbb{R}^n)$. □

As a consequence of Example 6.4.23 and Proposition 6.4.22, we also have

Example 6.4.24. The family of operators

$$\pi_\lambda(\mathcal{L}) = \sum_{j=1}^n \left\{\pi_\lambda(X_j)^2 + \pi_\lambda(Y_j)^2\right\} = -|\lambda|Q$$

is λ-uniform in $\Psi\Sigma^2_{1,\lambda}(\mathbb{R}^n)$.

Standard computations also show:

Example 6.4.25. For each $m\in\mathbb{R}$, the family of symbols b_λ^m, $\lambda\in\mathbb{R}\backslash\{0\}$, where

$$b_\lambda(\xi,u) = \sqrt{1+|\lambda|(1+|u|^2+|\xi|^2)},$$

is λ-uniform in $\Psi\Sigma^m_{1,\lambda}(\mathbb{R}^n)$.

6.4.5 Commutator characterisation of λ-Shubin classes

In this section, we characterise the λ-Shubin classes in terms of commutators and continuity on the Shubin Sobolev spaces.

First we need to understand some properties of the Sobolev spaces associated with the λ-dependent metric used to define the λ-Shubin symbols.

Proposition 6.4.26. *1. For each $\lambda \in \mathbb{R}\backslash\{0\}$ and $s \in \mathbb{R}$, the Sobolev space corresponding to $g^{(1,\lambda)}$ and $\left(M^{(\lambda)}\right)^s$ coincides with the Shubin Sobolev space:*

$$H\left(\left(M^{(\lambda)}\right)^s, g^{(1,\lambda)}\right) = \mathcal{Q}_s(\mathbb{R}^n).$$

2. The following define norms on $\mathcal{Q}_s(\mathbb{R}^n)$ equivalent to $\|\cdot\|_{\mathcal{Q}_s}$:

$$\begin{aligned} \|h\|_{\mathcal{Q}_{s,\lambda}} &:= \|(\mathrm{I}+|\lambda|Q)^{s/2}h\|_{L^2(\mathbb{R}^n)}, \\ \|h\|^{(b_\lambda)}_{\mathcal{Q}_{s,\lambda}} &:= \|\mathrm{Op}^W(b^s_\lambda)\,h\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

where b_λ was defined in Example 6.4.25. Moreover, in the case $s \in \mathbb{N}_0$, we also have an equivalent norm

$$\|h\|^{(int)}_{\mathcal{Q}_{s,\lambda}} := \sum_{|\alpha|+|\beta|\le s} |\lambda|^{\frac{|\alpha|+|\beta|}{2}} \|u^\alpha \partial_u^\beta h\|_{L^2(\mathbb{R}^n)}.$$

3. Furthermore, for each $s \in \mathbb{R}$ there exists a constant $C_1 = C_{1,s} > 0$ such that

$$\forall \lambda \in \mathbb{R}\backslash\{0\},\ h \in \mathcal{Q}_s(\mathbb{R}^n) \qquad C_1^{-1}\|h\|_{\mathcal{Q}_{s,\lambda}} \le \|h\|^{(b_\lambda)}_{\mathcal{Q}_{s,\lambda}} \le C_1\|h\|_{\mathcal{Q}_{s,\lambda}},$$

and for each $s \in \mathbb{N}_0$ there exists a constant $C_2 = C_{2,s} > 0$ such that

$$\forall \lambda \in \mathbb{R}\backslash\{0\},\ h \in \mathcal{Q}_s(\mathbb{R}^n) \qquad C_2^{-1}\|h\|_{\mathcal{Q}_{s,\lambda}} \le \|h\|^{(int)}_{\mathcal{Q}_{s,\lambda}} \le C_2\|h\|_{\mathcal{Q}_{s,\lambda}}.$$

Naturally, in Part (2), the constants in the equivalences between each of the norms $\|\cdot\|_{\mathcal{Q}_{s,\lambda}}$, $\|\cdot\|^{(int)}_{\mathcal{Q}_{s,\lambda}}$, $\|\cdot\|^{(b_\lambda)}_{\mathcal{Q}_{s,\lambda}}$, and the norm $\|\cdot\|_{\mathcal{Q}_s}$, depend on λ.

Proof of Proposition 6.4.26. Part (1) follows easily from (6.54), Definition 6.4.10, Theorem 6.4.16 especially Part (5).

Using the Shubin calculus $\cup_m \Psi\Sigma_1^m$, it is not difficult to see that the norms $\|\cdot\|^{(b)}_{\mathcal{Q}_s}$ and $\|\cdot\|^{(b_\lambda)}_{\mathcal{Q}_{s,\lambda}}$ are equivalent.

The fact that the norms $\|\cdot\|_{\mathcal{Q}_{s,\lambda}}$, $\|\cdot\|^{(b_\lambda)}_{\mathcal{Q}_{s,\lambda}}$ and, if $s \in \mathbb{N}_0$, $\|\cdot\|^{(int)}_{\mathcal{Q}_{s,\lambda}}$, are equivalent with λ-uniform constants comes from following the same proof as Theorem 6.4.16 but using the seminorms of $\cup_m \Sigma^m_{1,\lambda}$. This is left to the reader and concludes the proof of Proposition 6.4.26. □

Theorem 6.4.27. *We assume that* $\rho \in (0,1]$. *Let* $A_\lambda : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, $\lambda \in \mathbb{R}\backslash\{0\}$, *be a family of linear continuous operators.*

We assume that for every $\alpha_1, \alpha_2 \in \mathbb{N}_0^n$ *all the operators*

$$|\lambda|^{-\frac{|\alpha_1|+|\alpha_2}{2}} (\mathrm{ad} u)^{\alpha_1} (\mathrm{ad}\partial_u)^{\alpha_2} A_\lambda, \quad \lambda \in \mathbb{R}\backslash\{0\},$$

are λ*-uniformly in* $\mathscr{L}(L^2(\mathbb{R}^n), \mathcal{Q}_{-m+\rho(|\alpha_1|+|\alpha_2|)})$. *This means that*

$$\sup_{\lambda \in \mathbb{R}\backslash\{0\}} |\lambda|^{-\frac{|\alpha_1|+|\alpha_2}{2}} \|(\mathrm{ad} u)^{\alpha_1} (\mathrm{ad}\partial_u)^{\alpha_2} A_\lambda\|_{\mathscr{L}(L^2(\mathbb{R}^n), \mathcal{Q}_{-m+\rho(|\alpha_1|+|\alpha_2|)})} < \infty. \quad (6.56)$$

Then $A_\lambda \in \Psi\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$. *Moreover, for any* $\ell \in \mathbb{N}$, *there exist a constant* C *and an integer* ℓ', *both independent of* $\{A_{\lambda'}\}$ *and* λ, *such that*

$$\|A_\lambda\|_{\Psi\Sigma^m_{\rho,\lambda},\ell} \leq C \sum_{|\alpha_1|+|\alpha_2| \leq \ell'} |\lambda|^{-\frac{|\alpha_1|+|\alpha_2}{2}} \|(\mathrm{ad} u)^{\alpha_1} (\mathrm{ad}\partial_u)^{\alpha_2} A_\lambda\|_{\mathscr{L}(L^2(\mathbb{R}^n), \mathcal{Q}_{-m+\rho(|\alpha_1|+|\alpha_2|)})}.$$

Proof. The proof follows exactly the same steps as the proof of Theorem 6.4.18 using the calculi $\cup_m \Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$ to give the uniformity in λ. This is left to the reader. □

The converse is true from the λ-Shubin calculus: if $A_\lambda : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, $\lambda \in \mathbb{R}\backslash\{0\}$, is uniformly in $\Psi\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$ in the sense that

$$\forall N \in \mathbb{N}_0 \qquad \sup_{\lambda \in \mathbb{R}\backslash\{0\}} \|A_\lambda\|_{\Psi\Sigma^m_{\rho,\lambda},N} < \infty, \quad (6.57)$$

then (6.56) holds for every $\alpha_1, \alpha_2 \in \mathbb{N}_0^n$.

Proceeding as for Corollary 6.4.20, we obtain

Corollary 6.4.28. *We assume that* $\rho \in (0,1]$. *Let* $A_\lambda : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, $\lambda \in \mathbb{R}\backslash\{0\}$, *be a family of linear continuous operators.*

The family of operators $\{A_\lambda, \lambda \in \mathbb{R}\backslash\{0\}\}$ *is uniformly in* $\Psi\Sigma^m_{\rho,\lambda}(\mathbb{R}^n)$ *in the sense of (6.57) if and only if there exists* $\gamma_o \in \mathbb{R}$ *such that for each* $\alpha_1, \alpha_2 \in \mathbb{N}_0^n$,

$$\sup_{\lambda \in \mathbb{R}\backslash\{0\}} |\lambda|^{-\frac{|\alpha_1|+|\alpha_2}{2}} \|(\mathrm{ad} u)^{\alpha_1} (\mathrm{ad}\partial_u)^{\alpha_2} A_\lambda\|_{\mathscr{L}(\mathcal{Q}_{\gamma_o}(\mathbb{R}^n), \mathcal{Q}_{-m+\rho(|\alpha_1|+|\alpha_2|)+\gamma_o})} < \infty.$$

In this case this property is also true for every $\gamma \in \mathbb{R}$. *Moreover, for any* $\gamma \in \mathbb{R}$ *and* $\ell \in \mathbb{N}$, *there exist a constant* C *and an integer* ℓ', *both independent of* $\{A_{\lambda'}\}$ *and* λ, *such that*

$$\|A_\lambda\|_{\Psi\Sigma^m_{\rho,\lambda},\ell} \leq C \sum_{|\alpha|+|\alpha_2| \leq \ell'} |\lambda|^{-\frac{|\alpha_1|+|\alpha_2}{2}} \|(\mathrm{ad} u)^{\alpha_1} (\mathrm{ad}\partial_u)^{\alpha_2} A_\lambda\|_{\mathscr{L}(\mathcal{Q}_{\gamma}(\mathbb{R}^n), \mathcal{Q}_{-m+\rho(|\alpha_1|+|\alpha_2|)+\gamma})}.$$

6.5 Quantization and symbol classes $S^m_{\rho,\delta}$ on the Heisenberg group

We recall that in Section 5.2.2 we have introduced symbol classes $S^m_{\rho,\delta}(G)$ for general graded Lie groups G. In particular, this yields symbol classes $S^m_{\rho,\delta}(\mathbb{H}_n)$ for the particular case of $G = \mathbb{H}_n$. In this section, working with Schrödinger representations π_λ, we obtain a characterisation of these symbol classes $S^m_{\rho,\delta}(\mathbb{H}_n)$ in terms of scalar-valued symbols which will depend on the parameter $\lambda \in \mathbb{R}\backslash\{0\}$; these symbols will be called λ-symbols. The dependence on λ will be of crucial importance here.

We start by adapting the notation of the general construction described in Chapter 5 to the case of the Heisenberg group $\mathbb{H}_n$. It will be convenient to change slightly the notation with respect to the general case. Firstly we want to keep the letter x for denoting part of the coordinates of the Heisenberg group and we choose to denote the general element of the Heisenberg group by, e.g.,

$$g = (x, y, t) \in \mathbb{H}_n.$$

Secondly we may define a symbol as parametrised by

$$\sigma(g, \lambda) := \sigma(g, \pi_\lambda), \quad (g, \lambda) \in \mathbb{H}_n \times \mathbb{R}\backslash\{0\}.$$

Thirdly we modify the indices $\alpha \in \mathbb{N}_0^{2n+1}$ in order to write them as

$$\alpha = (\alpha_1, \alpha_2, \alpha_3),$$

with

$$\alpha_1 = (\alpha_{1,1}, \ldots, \alpha_{1,n}) \in \mathbb{N}_0^n, \quad \alpha_2 = (\alpha_{2,1}, \ldots, \alpha_{2,n}) \in \mathbb{N}_0^n, \quad \alpha_3 \in \mathbb{N}_0.$$

The homogeneous degree of α is then

$$[\alpha] = |\alpha_1| + |\alpha_2| + 2\alpha_3.$$

6.5.1 Quantization on the Heisenberg group

Here we summarise the quantization formula of Section 5.1.3 and its consequences in the particular setting of the Heisenberg group $\mathbb{H}_n$.

As introduced in Definition 5.1.33, a symbol is given by a field of operators

$$\sigma = \{\sigma(g, \lambda) : \mathcal{S}(\mathbb{R}^n) \to L^2(\mathbb{R}^n), (g, \lambda) \in \mathbb{H}_n \times (\mathbb{R}\backslash\{0\})\},$$

satisfying (quite weak) properties so that the quantization makes sense. More rigorously, we require that, for each $\beta \in \mathbb{N}_0^{2n+1}$, the map $g \longmapsto \partial_g^\beta \sigma(g, \lambda)$ is continuous from $\mathbb{H}_n$ to some $L^\infty_{a,b}(\widehat{\mathbb{H}}_n)$.

Recall now, that on the Heisenberg group $\mathbb{H}_n$, the Plancherel measure is given by $c_n|\lambda|^n d\lambda$ (see Proposition 6.2.7). By Theorem 5.1.39, the quantization of a symbol σ as above is the operator

$$A = \mathrm{Op}(\sigma)$$

given by

$$A\phi(g) = c_n \int_{\mathbb{R}\setminus\{0\}} \mathrm{Tr}\left(\pi_\lambda(g)\ \sigma(g,\lambda)\ \widehat{\phi}(\pi_\lambda)\right) |\lambda|^n d\lambda, \tag{6.58}$$

for any $\phi \in \mathcal{S}(\mathbb{H}_n)$ and $g = (x, y, t) \in \mathbb{H}_n$.

Note that, by (1.5), we have

$$\widehat{\varphi}(\pi_\lambda)\,\pi_\lambda(g) = \mathcal{F}_{\mathbb{H}_n}(\varphi(g\,\cdot))(\pi_\lambda),$$

thus the properties of the trace imply that

$$\mathrm{Tr}\left(\pi_\lambda(g)\sigma(g,\lambda)\widehat{\phi}(\pi_\lambda)\right) = \mathrm{Tr}\left(\sigma(g,\lambda)\ \mathcal{F}_{\mathbb{H}_n}(\varphi(g\,\cdot))(\pi_\lambda)\right). \tag{6.59}$$

Furthermore, by (6.20), we have

$$\mathcal{F}_{\mathbb{H}_n}(\varphi(g\,\cdot))(\pi_\lambda) = (2\pi)^{\frac{2n+1}{2}} \mathrm{Op}^W \left[\mathcal{F}_{\mathbb{R}^{2n+1}}(\varphi(g\,\cdot))(\sqrt{|\lambda|}\,\cdot, \sqrt{\lambda}\,\cdot, \lambda)\right]. \tag{6.60}$$

This formula shows that the Weyl quantization is playing an important role in the quantization (6.58) due to its close relation to the group Fourier transform on the Heiseneberg group.

Now, for each $(g, \lambda) \in \mathbb{H}_n \times (\mathbb{R}\setminus\{0\})$, each operator $\sigma(g, \lambda) : \mathcal{S}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ in the symbol σ can also be written as the Weyl quantization of some symbol on the Euclidean space $\mathbb{R}^n$, depending on (g, λ). In other words, we can think of the symbol σ as

$$\sigma(g,\lambda) = \mathrm{Op}^W(a_{g,\lambda}), \tag{6.61}$$

where $a = \{a(g, \lambda, \xi, u) = a_{g,\lambda}(\xi, u)\}$ is a function on $\mathbb{H}_n \times \mathbb{R}\setminus\{0\} \times \mathbb{R}^n \times \mathbb{R}^n$. This scalar-valued symbol a will be called the λ-*symbol* of the operator A in (6.58).

In other words, the symbol of the operator A acting on the Heisenberg group is σ, related to A by the quantization formula (6.58). For each (g, λ), the symbol $\sigma_{g,\lambda}$ is itself an operator mapping the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. So, the λ-symbol a of the operator A is given by the collection of the Weyl symbols $a_{g,\lambda}$ of $\sigma(g, \lambda)$.

Note that if $A \in \Psi^m_{\rho,\delta}$, then its symbol acts on smooth vectors so $\sigma_{g,\lambda}$ is itself an operator mapping the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ to itself, for each (g, λ).