# Chapter 4

# Rockland operators and Sobolev spaces

In this chapter, we study a special type of operators: the (homogeneous) Rockland operators. These operators can be viewed as a generalisation of sub-Laplacians to the non-stratified but still homogeneous (graded) setting. The terminology comes from a property conjectured by Rockland and eventually proved by Helffer and Nourrigat in [HN79], see Section 4.1.3.

First, we discuss these operators in general. Subsequently, we concentrate on positive Rockland operators and study the heat semi-group, the Bessel and Riesz potentials and the Sobolev spaces naturally associated with a positive Rockland operator. Most results concerning the heat semi-group are known [FS82, ch.3.B]. To the authors' knowledge, however, this chapter is the first systematic presentation of the fractional powers and the homogeneous and inhomogeneous Sobolev spaces associated with a positive Rockland operator on a graded Lie group.

In fact, this appears to be the greatest generality for such constructions, since the existence of a Rockland (differential) operator on a homogeneous Lie group implies that the group must admit a graded structure, see Proposition 4.1.3. In the case of stratified Lie groups, Sobolev spaces have been developed by Folland [Fol75] for $1 < p < \infty$, for the Rockland operator being a sub-Laplacian (see also [Sak79]). Since sub-Laplacians are not always available on graded Lie groups, our constructions are based on general positive Rockland operators. In particular, this allows one to still cover the case of stratified Lie groups, but permitting taking Rockland operators other than a canonical sub-Laplacian.

Although we define Sobolev spaces using a fixed Rockland operator, Theorem 4.4.20 shows that these spaces are actually independent of the choice of a homogeneous positive Rockland operator.

 
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## 4.1 Rockland operators

We start with the discussion of general Rockland operators, giving definitions, examples, and then relating them to the hypoellipticity questions.

### 4.1.1 Definition of Rockland operators

The first definition of a Rockland operator uses the representations of the group. We use the notation which has become quite conventional nowadays in this part of the theory of group representations and which is explained in Section 1.7. In particular, $\widehat{G}$ denotes the unitary dual of $G$ and $\mathcal{H}_\pi^\infty$ the smooth vectors of a representation $\pi \in \widehat{G}$, see Definition 1.7.2. For a left-invariant differential operator $T$ we will denote $\pi(T) := d\pi(T)$, see Definition 1.7.4.

**Definition 4.1.1.** Let $T$ be a left-invariant differential operator on a Lie group $G$. Then $T$ satisfies the *Rockland condition* when

(R) for each representation $\pi \in \widehat{G}$, except for the trivial representation, the operator $\pi(T)$ is injective on $\mathcal{H}_\pi^\infty$, that is,

$$\forall v \in \mathcal{H}_\pi^\infty \qquad \pi(T)v = 0 \implies v = 0.$$

There is a similar definition of the Rockland condition for right-invariant differential operators, and also for left or right-invariant $L^2(G)$-bounded operators (for the latter, see Głowacki [Gło89, Gło91]). See also Section 4.4.8.

**Definition 4.1.2.** Let $G$ be a homogeneous Lie group. A *Rockland operator* $\mathcal{R}$ on $G$ is a left-invariant differential operator which is homogeneous of positive degree and satisfies the Rockland condition.

Some other authors may define non-homogeneous Rockland operators as operators of the form $\mathcal{R} = \sum_{[\alpha]\leq\nu} c_\alpha X^\alpha$ with the 'main' term $\sum_{[\alpha]=\nu} c_\alpha X^\alpha$ satisfying the Rockland property given in (R). Here we have chosen to assume that a Rockland operator is homogeneous to study directly the main term.

We will give examples of Rockland operators in Section 4.1.2. Before this, we show that their existence on a homogeneous Lie group implies that the group is graded and that the weights could be chosen in $\mathbb{N}$. This property influences the examples we can produce, and the subsequent development of the theory of pseudo-differential operators.

**Proposition 4.1.3.** *Let $G$ be a homogeneous Lie group. If there exists a Rockland operator on $G$ then the group $G$ is graded.*

*Furthermore, the dilations' weights $\upsilon_1, \ldots, \upsilon_n$ satisfy*

$$a_1\upsilon_1 = \ldots = a_n\upsilon_n$$

*for some integers $a_1, \ldots, a_n$.*

This property was shown by Miller in [Mil80], with a small gap in the proof later corrected by ter Elst and Robinson (see [tER97]).

*Proof of Proposition 4.1.3.* Let $G$ be a homogeneous Lie group. Its Lie algebra $\mathfrak{g}$ is endowed with the dilations $D_r = \mathrm{Exp}(\ln r A)$. Let the number $n'$ and $\{X_1, \ldots, X_n\}$ be the basis described in Lemma 3.1.14. We assume that there exists a $\nu$-homogeneous Rockland operator $\mathcal{R}$ which we can write as

$$\mathcal{R} = \sum_{[\alpha]=\nu} c_\alpha X^\alpha.$$

We fix an integer $j \leq n'$. Let $\phi : \mathfrak{g} \to \mathbb{R}$ be the linear functional such that $\phi(X_k) = \delta_{j,k}$, that is, $\phi(X_j) = 1$ while $\phi(X_k) = 0$ for any $k \neq j$. Since $X_j \notin [\mathfrak{g}, \mathfrak{g}]$, $\phi$ is identically zero on $[\mathfrak{g}, \mathfrak{g}]$. We set for any $X \in \mathfrak{g}$:

$$\pi(\exp_G X) := \exp\left(i\phi(X)\right).$$

This defines a one-dimensional representation $\pi$ of $G$. Indeed, if $x, y \in G$, we can write $x = \exp_G X$ and $y = \exp_G Y$ and we have

$$xy = \exp_G X \exp_G Y = \exp_G(X + Y + Z)$$

with $Z \in [\mathfrak{g}, \mathfrak{g}]$ by the Baker-Campbell-Hausdorff formula (see Theorem 1.3.2). Thus, $\phi(Z) = 0$ and we obtain

$$\begin{aligned} \pi(xy) &= \exp\left(i\phi(X + Y + Z)\right) = \exp\left(i\phi(X) + i\phi(Y)\right) \\ &= \exp\left(i\phi(X)\right)\exp\left(i\phi(Y)\right) = \pi(x)\pi(y). \end{aligned}$$

So $\pi$ is a one-dimensional representation of $G$ and we see that

$$\pi(X_k) = \partial_{t=0}\pi(e^{tX_k}) = \partial_{t=0}\exp\left(i\phi(tX_k)\right) = \partial_{t=0}\exp\left(it\phi(X_k)\right) = i\delta_{j,k}.$$

As $\pi$ is a non-trivial one-dimensional representation of $G$ and $\mathcal{R}$ satisfies the Rockland condition,

$$\pi(\mathcal{R}) = \sum_{[\alpha]=\nu} c_\alpha \pi(X^\alpha)$$

must be non-zero. We see that $\pi(X^\alpha)$ is always zero unless $\alpha$ is of the form $ae_j$ for $a \in \mathbb{N}$ where $e_j$ is the multi-index with 1 in the $j$-th place and zeros elsewhere; in this case $[\alpha] = \upsilon_j a$. So $\nu$ must be of the form $\nu = \upsilon_j a$ for some integer $a = a_j \in \mathbb{N}$ which may depend on $j$. And this is true for any $j = 1, \ldots, n'$.

Since $X_1, \ldots, X_{n'}$ generate the Lie algebra $\mathfrak{g}$, the other weights are linear combinations with coefficients in $\mathbb{N}_0$ of the $\upsilon_j$'s, $j \leq n'$. This shows that the operators $D'_r = \mathrm{Exp}(\frac{\ln r}{\nu} A)$ are dilations over $\mathfrak{g}$ with rational weights. By Lemma 3.1.9, the group $G$ is graded. □

*Remark* 4.1.4. Proposition 4.1.3 and Remark 3.1.8 imply that the natural context for the study of Rockland operators is a graded Lie group endowed with a family of dilations with integer weights.

One may further assume that the weights have no common divisor other than 1 but we do not assume so unless we specify it.

From the proof of Proposition 4.1.3, we see:

**Corollary 4.1.5.** *Let $G$ be a graded Lie group and let $\{X_1, \ldots, X_n\}$ be the basis described in Lemma 3.1.14. We keep the notation of the lemma.*

*The homogeneous degree of any Rockland operator is a multiple of $\upsilon_1, \ldots, \upsilon_{n'}$.*

*If $\mathcal{R}$ is a Rockland operator satisfying $\mathcal{R}^t = \mathcal{R}$ then its homogeneous degree is even.*

### 4.1.2 Examples of Rockland operators

On $(\mathbb{R}^n, +)$, it is easy to see that Rockland differential operators are exactly the operators $P(-i\partial_1, \ldots, -i\partial_n)$ where $P$ is a polynomial which is homogeneous (for the standard dilations) and does not vanish except at zero. For instance homogeneous elliptic operators on $\mathbb{R}^n$ with constant coefficients are Rockland operators. More generally, let us prove that sub-Laplacians on a stratified Lie group are Rockland operators. First let us recall their definition.

**Definition 4.1.6.** If $G$ is a stratified Lie group with a given basis $Z_1, \ldots, Z_p$ for the first stratum of its Lie algebra, then the left-invariant differential operator on $G$ given by

$$Z_1^2 + \ldots + Z_p^2$$

is called a *sub-Laplacian.*

For example, the canonical sub-Laplacian of the Heisenberg group $\mathbb{H}_{n_o}$ is

$$X_1^2 + Y_1^2 + \ldots + X_{n_o}^2 + Y_{n_o}^2,$$

see Examples 1.6.4, 3.1.2 and 3.1.3 for our notation regarding the Heisenberg group.

**Lemma 4.1.7.** *Any sub-Laplacian on a stratified Lie group is a Rockland operator of homogeneous degree 2.*

This could be seen as a consequence of famous powerful theorems, namely from combining Hörmander's sums of squares and Helffer-Nourrigat (see Theorems A.1.2 and 4.1.12 in the sequel) but we prefer to give a direct and easy proof.

*Proof.* Let

$$\mathcal{R} = Z_1^2 + \ldots + Z_p^2$$

be a sub-Laplacian on the stratified Lie group $G$, where $Z_1, \ldots, Z_p$ is a given basis for the first stratum $V_1$ of the Lie algebra of $G$.

Clearly $\mathcal{R}$ is a homogeneous left-invariant differential operator of degree 2. Let $\pi \in \widehat{G}\backslash\{1\}$ and $v \in \mathcal{H}_\pi^\infty$ be such that $\pi(\mathcal{R})v = 0$. Then

$$\begin{aligned} 0 &= (\pi(\mathcal{R})v, v)_{\mathcal{H}_\pi} = (\pi(Z_1)^2 v, v)_{\mathcal{H}_\pi} + \ldots + (\pi(Z_p)^2 v, v)_{\mathcal{H}_\pi} \\ &= -(\pi(Z_1)v, \pi(Z_1)v)_{\mathcal{H}_\pi} - \ldots - (\pi(Z_p)v, \pi(Z_p)v)_{\mathcal{H}_\pi} \\ &= -\|\pi(Z_1)v\|_{\mathcal{H}_\pi}^2 - \ldots - \|\pi(Z_p)v\|_{\mathcal{H}_\pi}^2, \end{aligned}$$

and hence

$$\pi(Z_1)v = \ldots = \pi(Z_p)v = 0.$$

Since $\{Z_1, \ldots, Z_p\}$ generates linearly the first stratum $V_1$ of $\mathfrak{g}$ and $V_1$ generates $\mathfrak{g}$ as a Lie algebra, we see that $\pi(X)v = 0$ for any vector $X \in \mathfrak{g}$. But since $\pi$ is non-trivial and irreducible, this forces $v$ to be zero. □

Looking at the proof of Lemma 4.1.7, it is not difficult to construct the 'classical' Rockland differential operators on graded Lie groups $G$:

**Lemma 4.1.8.** *Let $G$ be a graded Lie group of dimension $n$, i.e. $G \sim \mathbb{R}^n$. We denote by $\{D_r\}_{r>0}$ the natural family of dilations on its Lie algebra $\mathfrak{g}$, and by $\upsilon_1, \ldots, \upsilon_n$ its weights. We fix a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ satisfying*

$$D_r X_j = r^{\upsilon_j} X_j, \quad j = 1, \ldots, n, \quad r > 0.$$

*If $\nu_o$ is any common multiple of $\upsilon_1, \ldots, \upsilon_n$, the operator*

$$\sum_{j=1}^{n} (-1)^{\frac{\nu_o}{\upsilon_j}} c_j X_j^{2\frac{\nu_o}{\upsilon_j}} \quad \textit{with} \quad c_j > 0, \tag{4.1}$$

*is a Rockland operator of homogeneous degree $2\nu_o$.*

*Proof.* The operator $\mathcal{R}$ given in (4.1) is clearly a homogeneous left-invariant differential operator of homogeneous degree $2\nu_o$. Let $\pi \in \widehat{G}\backslash\{1\}$ and $v \in \mathcal{H}_\pi^\infty$ be such that $\pi(\mathcal{R})v = 0$. Then

$$\begin{aligned} 0 &= (\pi(\mathcal{R})v, v)_{\mathcal{H}_\pi} = \sum_{j=1}^{n} (-1)^{\frac{\nu_o}{\upsilon_j}} c_j (\pi(X_j)^{2\frac{\nu_o}{\upsilon_j}} v, v)_{\mathcal{H}_\pi} \\ &= \sum_{j=1}^{n} c_j \|\pi(X_j)^{\frac{\nu_o}{\upsilon_j}} v\|_{\mathcal{H}_\pi}, \end{aligned}$$

and hence $\pi(X_j)^{\frac{\nu_o}{\upsilon_j}} v = 0$ for $j = 1, \ldots, n$.

Let us observe the following simple fact regarding any positive integer $p$ and any $Z \in \mathfrak{U}(\mathfrak{g})$: the hypothesis $\pi(Z)^p v = 0$ implies that

- if $p$ is odd then $\pi(Z)^{p+1} v = \pi(Z)\pi(Z)^p v = 0$,

- whereas if $p$ is even then

$$0 = (\pi(Z)^p v, v)_{\mathcal{H}_\pi} = (-1)^{p/2} (\pi(Z)^{\frac{p}{2}} v, \pi(Z)^{\frac{p}{2}} v)_{\mathcal{H}_\pi} = (-1)^{p/2} \|\pi(Z)^{\frac{p}{2}} v\|^2_{\mathcal{H}_\pi},$$

and hence $\pi(Z)^{\frac{p}{2}} v = 0$.

Applying this argument inductively on $Z = X_j$ and $p = \nu_o/\upsilon_j$, $\nu_o/2\upsilon_j, \ldots$, we obtain that $\pi(X_j)v = 0$ for each $j$. Hence $v = 0$. □

*Remark* 4.1.9. By Proposition 4.1.3 and its proof, if a homogeneous Lie group $G$ admits a Rockland operator, then, up to rescaling the dilations (cf. Remark 3.1.8), we may assume that the group $G$ is graded and endowed with its natural family of dilations $\{D_r\}_{r>0}$. Lemma 4.1.8 gives the converse: on such a group, we can always find a Rockland operator.

The proof of Lemma 4.1.8 can easily be modified using an adapted basis constructed in Lemma 3.1.14 to obtain

**Corollary 4.1.10.** *Let $G$ be a graded Lie group endowed with a family of dilations $\{D_r\}_{r>0}$. Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$ as in Lemma 3.1.14. In particular, the vectors $X_1, \ldots, X_{n'}$ generate the Lie algebra $\mathfrak{g}$.*

*If $\nu_o$ is any common multiple of $\upsilon_1, \ldots, \upsilon_{n'}$, the operator*

$$\sum_{j=1}^{n'} (-1)^{\frac{\nu_o}{\upsilon_j}} X_j^{2\frac{\nu_o}{\upsilon_j}}, \tag{4.2}$$

*is a Rockland operator of homogeneous degree $2\nu_o$.*

If the group $G$ is stratified, the vectors $X_1, \ldots, X_{n'}$ span linearly the first stratum and we obtain the sub-Laplacian if we choose $\nu_o = \upsilon_1$.

From one Rockland operator, we can construct many since powers of a Rockland operator or its complex conjugate operator are Rockland:

**Lemma 4.1.11.** *Let $\mathcal{R}$ be a Rockland operator on a graded Lie group $G$ endowed with a family of dilations with integer weights. Then the operators $\mathcal{R}^k$ for any $k \in \mathbb{N}$ and $\bar{\mathcal{R}}$ are also Rockland operators.*

The operator $\bar{\mathcal{R}}$ as an element of $\mathfrak{U}(\mathfrak{g})$ was defined in (1.8).

*Proof.* It is clear that $\bar{\mathcal{R}}$ and $\mathcal{R}^k$ are left-invariant homogeneous differential operators on $G$.

Let $\pi \in \widehat{G}\backslash\{1\}$. We have

$$\pi(\bar{\mathcal{R}}) = \overline{\pi(\mathcal{R})}.$$

This holds in fact for any left-invariant differential operator viewed as an element of $\mathfrak{U}(\mathfrak{g})$. Therefore, $\bar{\mathcal{R}}$ is Rockland. For the case of $\mathcal{R}^k$, let $v \in \mathcal{H}_\pi^\infty$ be such that $\pi(\mathcal{R}^k)v = 0$. Applying recursively the simple fact explained in the proof of Lemma 4.1.8, we obtain $\pi(\mathcal{R})v = 0$ and this implies $v = 0$ because $\mathcal{R}$ is Rockland. Therefore, $\mathcal{R}^k$ is also Rockland. □

### 4.1.3 Hypoellipticity and functional calculus

The analysis of left-invariant homogeneous operators on a nilpotent graded Lie group has played a very important role in the understanding of hypoellipticity. We refer the interested reader on this subject to the lecture notes by Helffer and Nier [HN05]. For the definition of hypoellipticity, see Section A.1.

In [Roc78], Rockland showed that if $T$ is a homogeneous left-invariant differential operators on the Heisenberg group $\mathbb{H}_{n_o}$, then the hypoellipticity of $T$ and $T^t$ is equivalent to the Rockland condition (see Definition 4.1.1). He also asked whether this equivalence would be true for more general homogeneous Lie groups. Just afterwards, Beals showed [Bea77b] that the hypoellipticity of a homogeneous left-invariant differential operator on any homogeneous Lie group implies the Rockland condition. At the same time he also showed that the converse holds in some step-two cases. Eventually in [HN79], Helffer and Nourrigat settled what has become Rockland's conjecture by proving the following equivalence:

**Theorem 4.1.12.** *Let $\mathcal{R}$ be a left-invariant and homogeneous differential operator on a homogeneous Lie group $G$. The hypoellipticity of $\mathcal{R}$ is equivalent to $\mathcal{R}$ satisfying the Rockland condition.*

*In this case, any operator of the form*

$$\mathcal{R} + \sum_{[\alpha]<\nu} c_\alpha X^\alpha,$$

*where $\nu$ is the degree of homogeneity of $\mathcal{R}$ and $c_\alpha$ any complex number, is also hypoelliptic.*

The proof of Theorem 4.1.12 relies on the description of $\widehat{G}$ via Kirillov's orbit method.

*Remark* 4.1.13.

1. The hypotheses of Theorem 4.1.12 with the existence of a Rockland operator imply that the family of dilations of the group may be rescaled to have integer weights and consequently that the group may be viewed as graded, see Proposition 4.1.3. When describing properties of a Rockland operator $\mathcal{R}$ on a homogeneous Lie group $G$, unless stated otherwise, we will always assume that the group $G$ is graded in such a way that the operator $\mathcal{R}$ is homogeneous for the natural family of dilations (with integer weights).
2. Combining the theorems of Hellfer-Nourrigat and of Hörmander (see Theorems 4.1.12 and A.1.2) gives another proof that the sub-Laplacians are Rockland operators, see Lemma 4.1.7.
3. If $\mathcal{R}$ is a Rockland operator formally self-adjoint, i.e. $\mathcal{R}^* = \mathcal{R}$ as elements of $\mathfrak{U}(\mathfrak{g})$, then $\mathcal{R}^t = \bar{\mathcal{R}}$ must also be Rockland by Lemma 4.1.11. Hence Theorem 4.1.12 implies that any formally self-adjoint Rockland operator satisfies the hypothesis of Theorem 3.2.40 and thus admits fundamental solutions. It also satisfies the hypothesis of the Liouville theorem as in Theorem 3.2.45.

4. Let us also mention an alternative reformulation of the Hellfer-Nourrigat theorem given by Rothschild [Rot83]: a left-invariant homogeneous operator $\mathcal{R}$ on a graded Lie group $G$ is hypoelliptic if and only if there is no non-constant bounded function $f$ on $G$ such that $\mathcal{R}f = 0$ on $G$. The proof of this relies on the Liouville theorem from Section 3.2.8. Essentially, in one direction this is Beals' result as above, while in the other it will follow from Corollary 4.3.4.

Along the proof of Theorem 4.1.12 (see [HN79, Estimate (6.1)]), Helffer and Nourrigat also showed the following property which will be used in the sequel.

**Corollary 4.1.14.** *Let $G$ be a graded Lie group endowed with a family of dilations with integer weights. Let $\mathcal{R}$ be a Rockland operator $G$ of homogeneous degree $\nu$. Then there exists $C > 0$ such that*

$$\forall \phi \in \mathcal{S}(G) \qquad \sum_{[\alpha]=\nu} \|X^\alpha \phi\|^2_{L^2(G)} \leq C \left( \|\mathcal{R}\phi\|^2_{L^2(G)} + \|\phi\|^2_{L^2(G)} \right).$$

After developing the Sobolev spaces on $G$, we will be actually able to prove its $L^p$-version, see Lemma 4.4.19.

The following property of Rockland differential operators is technically important and relies on hypoellipticity.

**Proposition 4.1.15.** *Let $\mathcal{R}$ be a Rockland operator on a graded Lie group $G$. We assume that $\mathcal{R}$ is formally self-adjoint. Let $\pi$ be a strongly continuous unitary representation of $G$.*

*Then the operators $\mathcal{R}$ and $\pi(\mathcal{R})$ densely defined on $\mathcal{D}(G) \subset L^2(G)$ and $\mathcal{H}_\pi^\infty \subset \mathcal{H}_\pi$, respectively, are essentially self-adjoint.*

That $\mathcal{R}$ is formally self-adjoint means that $\mathcal{R}^* = \mathcal{R}$ as elements of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, see (1.9).

Before we prove it, let us point out its consequences:

**Corollary 4.1.16** (Functional calculus of Rockland operators and their Fourier transform)**.** *Let $\mathcal{R}$ be a Rockland operator on a graded Lie group $G$. We assume that $\mathcal{R}$ is formally self-adjoint as an element of $\mathfrak{U}(\mathfrak{g})$. Then $\mathcal{R}$ is essentially self-adjoint on $L^2(G)$ and we denote by $\mathcal{R}_2$ its self-adjoint extension on $L^2(G)$. Moreover, for each strongly continuous unitary representation $\pi$ of $G$, $\pi(\mathcal{R})$ is essentially self-adjoint on $\mathcal{H}_\pi$ and we keep the same notation for its self-adjoint extension. Let $E$, $E_\pi$ be the spectral measures of $\mathcal{R}_2$ and $\pi(\mathcal{R})$:*

$$\mathcal{R}_2 = \int_{\mathbb{R}} \lambda dE(\lambda) \quad \textit{and} \quad \pi(\mathcal{R}) = \int_{\mathbb{R}} \lambda dE_\pi(\lambda).$$

*For any Borel subset $B \subset \mathbb{R}$, the orthogonal projection $E(B)$ is left-invariant hence $E(B) \in \mathscr{L}_L(L^2(G))$. The group Fourier transform of its convolution kernel $E(B)\delta_0 \in \mathcal{K}(G)$ is*

$$\mathcal{F}_G(E(B)\delta_0)(\pi) = E_\pi(B).$$

*If $\phi$ is a measurable function on $\mathbb{R}$, the spectral multiplier operator $\phi(\mathcal{R}_2)$ is defined by*

$$\phi(\mathcal{R}_2) := \int_{\mathbb{R}} \phi(\lambda) dE(\lambda),$$

*and its domain $\mathrm{Dom}(\phi(\mathcal{R}_2))$ is the space of function $f \in L^2(G)$ such that the integral $\int_{\mathbb{R}} |\phi(\lambda)|^2 d(E(\lambda)f, f)$ is finite. It satisfies for all $f \in \mathrm{Dom}(\phi(\mathcal{R}_2))$ and $r > 0$:*

$$f(r\,\cdot) \in \mathrm{Dom}(\phi(r^{-\nu}\mathcal{R}_2)) \quad \textit{and} \quad \phi(\mathcal{R}_2)f = \phi(r^{-\nu}\mathcal{R}_2)\left(f(r\,\cdot)\right)(r^{-1}\,\cdot). \tag{4.3}$$

*If $\pi_1$ is another strongly continuous representation such that $\pi_1 \sim_T \pi$, that is, $T$ is a unitary operator satisfying $T\pi_1 = \pi T$, then $TE_{\pi_1} = E_\pi T$ and we have for any measurable function $\phi$ the equality*

$$T\phi(\pi_1(\mathcal{R})) = \phi(\pi(\mathcal{R}))T. \tag{4.4}$$

*Let $\phi \in L^\infty(\mathbb{R})$ be any measurable bounded function. Then the spectral multiplier operator $\phi(\mathcal{R}_2)$ is in $\mathscr{L}_L(L^2(G))$, that is, it is bounded on $L^2(G)$ and left-invariant. Its convolution kernel denoted by $\phi(\mathcal{R}_2)\delta_o$ is the unique tempered distribution $\phi(\mathcal{R}_2)\delta_o \in \mathcal{S}'(G)$ such that*

$$\forall f \in \mathcal{S}(G) \quad \phi(\mathcal{R}_2)f = f * \phi(\mathcal{R}_2)\delta_o.$$

*In fact $\phi(\mathcal{R}_2)\delta_o \in \mathcal{K}(G)$ and its group Fourier transform is*

$$\mathcal{F}\{\phi(\mathcal{R}_2)\delta_o\}(\pi) = \phi(\pi(\mathcal{R})) = \int_{\mathbb{R}} \phi(\lambda) dE_\pi(\lambda). \tag{4.5}$$

*Consequently, for any $f \in L^2(G)$,*

$$\mathcal{F}\{\phi(\mathcal{R}_2)f\}(\pi) = \phi(\pi(\mathcal{R}))\widehat{f}(\pi). \tag{4.6}$$

*We have for any $r > 0$ and $x \in G$:*

$$\phi(r^\nu \mathcal{R}_2)\delta_o(x) = r^{-Q}\phi(\mathcal{R}_2)\delta_o(r^{-1}x). \tag{4.7}$$

*For any $\phi \in L^\infty(\mathbb{R})$,*

$$\{\phi(\mathcal{R}_2)\delta_0\}^* = \bar{\phi}(\mathcal{R})\delta_0, \quad \textit{where} \quad \{\phi(\mathcal{R}_2)\delta_0\}^*(x) = \overline{\phi(\mathcal{R}_2)\delta_0}(x). \tag{4.8}$$

*If $\phi$ is also real-valued, then $\phi(\mathcal{R}_2)$ is a self-adjoint operator and its kernel satisfies $\phi(\mathcal{R}_2)\delta_o = (\phi(\mathcal{R}_2)\delta_o)^*$, that is, in the sense of distributions,*

$$\phi(\mathcal{R}_2)\delta_o(x) = \overline{\phi(\mathcal{R}_2)\delta_o}(x^{-1}).$$

*If $\phi$ is real-valued and furthermore if $\mathcal{R}^t = \mathcal{R}$, then $\phi(\mathcal{R}_2)\delta_o$ is real-valued (as a distribution).*

*Remark* 4.1.17. For any measurable function $\phi : \mathbb{R} \to \mathbb{C}$ such that for every $\pi_1 \in \operatorname{Rep} G$, the domain of $\phi(\pi_1(\mathcal{R}))$ contains $\mathcal{H}_{\pi_1}^\infty$, the corresponding $\widehat{G}$-field of operators $\{\phi(\pi(\mathcal{R})) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi\}$ is well defined in the sense of Definition 1.8.13 because of (4.4). This is the case for instance if $\phi$ is bounded since in this case $\phi(\pi_1(\mathcal{R}))$ is a bounded and therefore defined on the whole space $\mathcal{H}_{\pi_1}$.

The rest of this section is devoted to the proof of Proposition 4.1.15 and Corollary 4.1.16; it may be skipped at first reading. Proposition 4.1.15 follows from a Theorem by Nelson and Stinespring [NS59, Theorem 2.2] regarding elliptic operators on Lie groups as well as the adaptation of its proof due to Folland and Stein [FS82, ch.3.B] to our case. Let us sketch briefly the ideas for the sake of completeness. Nelson and Stinespring's Theorem can be reformulated here as the following:

**Proposition 4.1.18.** *Let $\mathcal{R}$ be a Rockland operator on a graded Lie group $G$. We assume that $\mathcal{R}$ is formally self-adjoint as an element of $\mathfrak{U}(\mathfrak{g})$.*

*If $\pi$ is a strongly continuous unitary representation of $G$, then the closure of $\pi(\mathcal{R}^*)$ is the adjoint of $\pi(\mathcal{R})$.*

*Proof of Proposition 4.1.18.* Let $v \in \mathcal{H}_\pi$ be orthogonal to the range of $\pi(\mathcal{R}) + \mathrm{I}$. Then for all $\phi \in \mathcal{D}(G)$,

$$0 = ((\pi(\mathcal{R}) + \mathrm{I})\pi(\phi)v, v)_{\mathcal{H}_\pi} = \int_G (\mathcal{R} + \mathrm{I})\phi(x) \ (\pi(x)^* v, v)_{\mathcal{H}_\pi} \, dx.$$

In other words, the continuous function $f_\pi$ defined by

$$f_\pi(x) := (\pi(x)^* v, v)_{\mathcal{H}_\pi} = (v, \pi(x)v)_{\mathcal{H}_\pi}, \quad x \in G,$$

is a solution in the sense of distributions of the partial differential equation $(\mathcal{R} + \mathrm{I})f = 0$. By Theorem 4.1.12, the operator $\mathcal{R}+\mathrm{I}$ is hypoelliptic. Hence $f_\pi$ is smooth on $G$ and the equation $(\mathcal{R} + \mathrm{I})f_\pi = 0$ holds in the ordinary pointwise sense. We observe that for any $X \in \mathfrak{U}(\mathfrak{g})$ identified with a left-invariant vector field we have

$$X f_\pi(x) = \partial_{t=0} \left\{ \left(v, \pi(xe^{tX})v\right)_{\mathcal{H}_\pi} \right\} = (v, \pi(x)\pi(X)v)_{\mathcal{H}_\pi}.$$

Thus,

$$(\mathcal{R} + \mathrm{I})f_\pi(x) = (v, \pi(x)\pi(\mathcal{R})v)_{\mathcal{H}_\pi} + (v, \pi(x)v)_{\mathcal{H}_\pi}.$$

Therefore, $(\mathcal{R} + \mathrm{I})f_\pi(0) = 0$ implies

$$(v, \pi(\mathcal{R})v)_{\mathcal{H}_\pi} = -(v, v)_{\mathcal{H}_\pi} = -\|v\|_{\mathcal{H}_\pi}^2.$$

If $\mathcal{R}$ can be written as $S^*S$ for some non-constant $S \in \mathfrak{U}(\mathfrak{g})$, then the left-hand side is equal to $\|\pi(S)v\|^2$ so $v = 0$. In the general case, we apply the argument above to $\mathcal{R}^*\mathcal{R} = \mathcal{R}^2$ which is also a Rockland operator by Lemma 4.1.11, and we obtain the desired conclusion thanks to the following lemma applied to $T = \pi(\mathcal{R})$, $T' = \pi(\mathcal{R}^*)$ and $\mathcal{D} = \mathcal{H}_\pi^\infty$. □

**Lemma 4.1.19.** *Let $\mathcal{D}$ be a dense vector subspace of a Hilbert space $\mathcal{H}$. Let $T$ and $T'$ be two linear operators on $\mathcal{H}$, whose domains are $\mathcal{D}$ and whose ranges are contained in $\mathcal{D}$ such that $T'$ is contained in the adjoint of $T$. If $T'T$ is essentially self-adjoint then the closure of $T'$ is the adjoint of $T$.*

*Proof of Lemma 4.1.19.* We denote by $T_*$ the adjoint of $T$. Let $(u, v)$ be an element of the graph of $T_*$ which is orthogonal to the graph of $T'$. This means

$$v = T_* u \quad \text{and} \quad \forall w \in \mathcal{D} \qquad (u, w)_{\mathcal{H}} + (v, T'w)_{\mathcal{H}} = 0.$$

In particular, for $w = Tx$ with $x \in \mathcal{D}$, we obtain

$$0 = (u, Tx)_{\mathcal{H}} + (v, T'Tx)_{\mathcal{H}} = (v, x)_{\mathcal{H}} + (v, T'Tx)_{\mathcal{H}}, \quad x \in \mathcal{D}.$$

But it is not difficult to see that $\mathrm{I} + T'T$ has a dense range. Consequently $v = 0$. So $(u, w)_{\mathcal{H}} = 0$ for all $w \in \mathcal{D}$ and therefore $u = 0$. This shows that the graph of $T_*$ contains no non-zero element orthogonal to the graph of $T'$; hence the closure of $T'$ is $T_*$. □

*Proof of Proposition 4.1.15.* We apply Proposition 4.1.18 to the left regular action on $L^2(G)$ and the strongly continuous unitary representation $\pi$ of $G$. □

*Proof of Corollary 4.1.16.* Applying the spectral theorem to the self-adjoint operators $\mathcal{R}_2$ and $\pi(\mathcal{R})$ (see, e.g., Rudin [Rud91, Part III]) we obtain the spectral measures $E$ and $E_\pi$ together with the definition of the spectral multipliers.

For each $x_o \in G$ and $r > 0$ we set for any Borel set $B \subset \mathbb{R}$ and any function $f \in L^2(G)$,

$$\begin{aligned} E^{(x_o)}(B)f &:= (E(B))\,(f(x_o\,\cdot))\,(x_o^{-1}\,\cdot), \\ E^{(r)}(B)f &:= \big(E(r^{-\nu}B)\big)\,(f(r\,\cdot))\,(r^{-1}\,\cdot), \end{aligned}$$

where the dilation of a subset of $\mathbb{R}$ is defined in the usual sense. It is not difficult to check that this defines new spectral measures $E^{(x_o)}$ and $E^{(r)}$ and, that for any function $f \in \mathcal{S}(G)$,

$$\begin{aligned} \int_{\mathbb{R}} \lambda dE^{(x_o)}(\lambda) f &= \int_{\mathbb{R}} \lambda d\,(E(\lambda))\,(f(x_o\,\cdot))\,(x_o^{-1}\,\cdot) = \mathcal{R}_2\,(f(x_o\,\cdot))\,(x_o^{-1}\,\cdot) \\ &= \mathcal{R}\,(f(x_o\,\cdot))\,(x_o^{-1}\,\cdot) = \mathcal{R}f = \mathcal{R}_2 f, \\ \int_{\mathbb{R}} \lambda dE^{(r)}(\lambda) f &= \int_{\mathbb{R}} (r^{-\nu}\lambda) d\,(E(\lambda))\,(f(r\,\cdot))\,(r^{-1}\,\cdot) = r^{-\nu}\mathcal{R}_2\,(f(r\,\cdot))\,(r^{-1}\,\cdot) \\ &= r^{-\nu}\mathcal{R}\,(f(r\,\cdot))\,(r^{-1}\,\cdot) = \mathcal{R}f = \mathcal{R}_2 f, \end{aligned}$$

since $\mathcal{R}$ is left-invariant and $\nu$-homogeneous. By density of $\mathcal{S}(G)$ in $L^2(G)$, we have obtained for any $f \in L^2(G)$ that

$$\int_{\mathbb{R}} \lambda dE^{(x_o)}(\lambda) f = \mathcal{R}_2 f \quad \text{and} \quad \int_{\mathbb{R}} \lambda dE^{(r)}(\lambda) f = \mathcal{R}_2 f.$$

By uniqueness of the spectral measure of $\mathcal{R}_2$, the spectral measures $E^{(x_o)}$, $E^{(r)}$ and $E$ coincide. For $E^{(r)}$ this implies (4.3).

For $E^{(x_o)}$ this means that for each Borel subset $B \subset \mathbb{R}$, the projection $E(B)$ is a left-invariant operator on $L^2(G)$. By the Plancherel theorem (see Section 1.8.2) the group Fourier transform of its convolution kernel $E(B)\delta_0 \in \mathcal{K}(G)$ satisfies

$$\forall f \in L^2(G) \qquad \pi(E(B)f) = \pi(E(B)\delta_0)\pi(f). \tag{4.9}$$

It is not difficult, using the uniqueness of the group Fourier transform, to check that

$$F : B \longmapsto \pi(E(B)\delta_0) =: F(B),$$

is a spectral measure on $\mathcal{H}_\pi$. Equality (4.9) can be rewritten for any $f \in L^2(G)$ as

$$\mathcal{F}_G\left(\int_{\mathbb{R}} \phi(\lambda) dE(\lambda) f\right)(\pi) = \left(\int_{\mathbb{R}} \phi(\lambda) dF(\lambda)\right) \widehat{f}(\pi), \tag{4.10}$$

with $\phi = 1_B$, that is, the characteristic function of a Borel subset $B \subset \mathbb{R}$. Hence Equality (4.10) also holds for a finite linear combination of characteristic functions, and then, passing through the limit carefully, for any $\phi \in L^\infty(\mathbb{R})$ with $f \in L^2(G)$ and $\phi(\lambda) = \lambda$ for $f \in \mathcal{S}(G)$. The latter yields

$$\begin{aligned}\left(\int_{\mathbb{R}} \lambda dF(\lambda)\right) \widehat{f}(\pi) &= \mathcal{F}_G\left(\int_{\mathbb{R}} \lambda dE(\lambda) f\right)(\pi) \\ &= \mathcal{F}_G(\mathcal{R}_2 f)(\pi) = \pi(\mathcal{R})\widehat{f}(\pi).\end{aligned}$$

Since the space $\mathcal{H}_\pi^\infty$ of smooth vectors is linearly spanned by elements of the form $\widehat{f}(\pi)v$, $f \in \mathcal{S}(G)$, $v \in \mathcal{H}_\pi$ (see Theorem 1.7.8), we have on $\mathcal{H}_\pi^\infty$

$$\int_{\mathbb{R}} \lambda dF(\lambda) = \pi(\mathcal{R}).$$

The uniqueness of the spectral measure $E_\pi$ shows that

$$E_\pi(B) = F(B) = \pi(E(B)\delta_0).$$

Equality (4.5) follows from (4.10) for $\phi \in L^\infty(\mathbb{R})$.

If $\pi_1 \sim_T \pi$, then we set $E_\pi^{(T)} := TE_{\pi_1}T^{-1}$, where $E_{\pi_1}$ denotes the spectral measure of $\pi_1(\mathcal{R})$. We check easily that $E_\pi^{(T)}$ is a spectral measure on $\mathcal{H}_\pi$ and that

$$\int_{\mathbb{R}} \lambda dE_\pi^{(T)} = T \int_{\mathbb{R}} \lambda dE_{\pi_1} T^{-1} = T\pi_1(\mathcal{R})T^{-1} = T\pi_1 T^{-1}(\mathcal{R}) = \pi(\mathcal{R}).$$

The property of the spectral measure $E_\pi$, that is, its uniqueness and the functional calculus, shows that $E_\pi^{(T)} = E_\pi$ and that (4.4) holds.

The rest of the statement follows from the Schwartz kernel theorem (see Corollary 3.2.1) and basic properties of the convolution. □

## 4.2 Positive Rockland operators

In this section we concentrate on positive Rockland operators, i.e. Rockland operators which are positive in the operator sense. Positive Rockland operators always exist on a graded Lie group, see Remark 4.2.4 below. Among Rockland operators, positive ones enjoy a number of additional useful properties. In particular, in this section, we analyse the heat semi-group associated to a positive Rockland operator and the corresponding heat kernel.

### 4.2.1 First properties

We shall be interested in Rockland differential operators which are positive in the sense of operators:

**Definition 4.2.1.** An operator $T$ on a Hilbert space $\mathcal{H}$ is *positive* when for any vectors $v, v_1, v_2 \in \mathcal{H}$ in the domain of $T$, we have

$$(Tv_1, v_2)_{\mathcal{H}} = (v_1, Tv_2)_{\mathcal{H}} \quad \text{and} \quad (Tv, v)_{\mathcal{H}} \geq 0.$$

In the case of left-invariant differential operator, this is easily equivalent to

**Proposition 4.2.2.** *Let $T$ be a left-invariant differential operator on a Lie group $G$. Then $T$ is positive on $L^2(G)$ when $T$ is formally self-adjoint, that is, $T^* = T$ in $\mathfrak{U}(\mathfrak{g})$, and satisfies*

$$\forall f \in \mathcal{D}(G) \qquad \int_G Tf(x)\overline{f(x)}\, dx \geq 0.$$

For the definition of $T^*$, see (1.9).

The following properties of positive operators are easy to prove:

**Lemma 4.2.3.** *1. A linear combination with non-negative coefficients of positive operators is a positive operator.*

*2. If $X$ is a left-invariant vector field and $p \in 2\mathbb{N}_0$, then the operator $(-1)^{\frac{p}{2}} X^p$ is positive on $G$.*

*3. If $T$ is a positive differential operator on $G$ then for any $k \in \mathbb{N}$ the differential operator $T^k$ is also positive.*

*Proof.* The first property is clear.

The second is true since each invariant vector field is essentially skew-symmetric, see Section 1.3.

Let us prove the third property. Let $T$ be a positive differential operator and $k \in \mathbb{N}$. Clearly $T^k$ is also formally self-adjoint and we obtain recursively if $k = 2\ell$:

$$\int_G T^k f(x)\overline{f(x)} dx = \int_G T^\ell f(x)\overline{T^\ell f(x)} dx = \int_G \left|T^\ell f(x)\right|^2 dx,$$

which is necessarily non-negative, whereas if $k = 2\ell + 1$,

$$\int_G T^k f(x)\overline{f(x)}dx = \int_G T(T^\ell f(x))\ \overline{T^\ell f(x)}dx,$$

which is non-negative since $T$ is positive. $\square$

We observe that the signs of the coefficients of a positive differential operator can not be guessed, as the example $-(\partial_1 \pm \partial_2)^2$ on $\mathbb{R}^2$ shows.

*Remark* 4.2.4. By Lemma 4.2.3, Parts 1 and 2, we see that the examples in Section 4.1.2 yield positive Rockland operators. For instance, on stratified Lie groups, the sub-Laplacians give operators $-\mathcal{R}$ with $\mathcal{R}$ positive and Rockland. Also, the operators in (4.1) and (4.2) give positive Rockland operators. In particular, this shows that any graded Lie group admits a positive Rockland operator.

We may obtain other positive Rockland operators as powers of those since a direct consequence of Lemma 4.1.11 and Lemma 4.2.3, Part 3, is the following

**Lemma 4.2.5.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. Then $\mathcal{R}^k$ for every $k \in \mathbb{N}$ and $\bar{\mathcal{R}} = \mathcal{R}^t$ are also positive Rockland operators.*

We fix a positive Rockland operator $\mathcal{R}$. By Proposition 4.2.2, $\mathcal{R}$ is essentially self-adjoint and we may adopt the same notation as in Corollary 4.1.16. Since $\mathcal{R}$ is positive, the spectrum of $\mathcal{R}_2$ is included in $[0, \infty)$ and we have

$$\mathcal{R}_2 = \int_0^\infty \lambda dE(\lambda).$$

**Proposition 4.2.6.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. If $\pi \in \widehat{G}$, then the operator $\pi(\mathcal{R})$ is positive. Furthermore, if $\pi$ is non-trivial and*

$$(\pi(\mathcal{R})v, v)_{\mathcal{H}_\pi} = 0$$

*then $v = 0$.*

*Proof.* By Proposition 4.1.15, $\pi(E(B)) = E_\pi(B)$. Since $E$ is supported in $[0, \infty)$ then so is $E_\pi$ and the operator $\pi(\mathcal{R})$ is positive:

$$\forall v \in \mathcal{H}_\pi^\infty \qquad (\pi(\mathcal{R})v, v)_{\mathcal{H}_\pi} = \int_0^\infty \lambda d(E_\pi(\lambda)v, v)_{\mathcal{H}_\pi} \geq 0.$$

If $(\pi(\mathcal{R})v, v)_{\mathcal{H}_\pi} = 0$ then the (real non-negative) measure $(E_\pi(\lambda)v, v)_{\mathcal{H}_\pi}$ is concentrated on $\{\lambda = 0\}$ and this means that $v = E_\pi(0)v$ is in the nullspace of $\pi(\mathcal{R})$. Thus $v = 0$ since $\mathcal{R}$ satisfies the Rockland condition and $\pi$ is non-trivial. $\square$

### 4.2.2 The heat semi-group and the heat kernel

In this section, we fix a positive Rockland operator $\mathcal{R}$ which is homogeneous of degree $\nu \in \mathbb{N}$.

By the functional calculus (see Corollary 4.1.16), we define the multipliers

$$e^{-t\mathcal{R}_2} := \int_0^\infty e^{-t\lambda} dE(\lambda), \quad t > 0.$$

We then have

$$\|e^{-t\mathcal{R}_2}\|_{\mathscr{L}(L^2(G))} \leq \sup_{\lambda \geq 0} |e^{-t\lambda}| = 1 \quad \text{and} \quad e^{-t\mathcal{R}_2} e^{-s\mathcal{R}_2} = e^{-(t+s)\mathcal{R}_2},$$

since $e^{-s\lambda} e^{-t\lambda} = e^{-(t+s)\lambda}$. Thus $\{e^{-t\mathcal{R}_2}\}_{t>0}$ is a contraction semi-group of operators on $L^2(G)$ (see Section A.2). This semi-group is often called the *heat semi-group*. The corresponding convolution kernels $h_t \in \mathcal{S}'(G)$, $t > 0$, are called *heat kernels*. We summarise its main properties in the following theorem:

**Theorem 4.2.7.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. Then the heat kernels $h_t$ associated with $\mathcal{R}$ satisfy the following properties. Each function $h_t$ is Schwartz and we have*

$$\begin{aligned}
\forall s, t > 0 \qquad h_t * h_s &= h_{t+s}, &(4.11)\\
\forall x \in G,\ t, r > 0 \qquad h_{r^\nu t}(rx) &= r^{-Q} h_t(x), &(4.12)\\
\forall x \in G \qquad h_t(x) &= \overline{h_t(x^{-1})}, &(4.13)\\
\int_G h_t(x) dx &= 1. &(4.14)
\end{aligned}$$

*The function $h : G \times \mathbb{R} \to \mathbb{C}$ defined by*

$$h(x,t) := \begin{cases} h_t(x) & \text{if } t > 0 \text{ and } x \in G, \\ 0 & \text{if } t \leq 0 \text{ and } x \in G, \end{cases}$$

*is smooth on $(G \times \mathbb{R}) \backslash \{(0,0)\}$ and satisfies*

$$(\mathcal{R} + \partial_t) h = \delta_{0,0},$$

*where $\delta_{0,0}$ is the delta-distribution at $(0,0) \in G \times \mathbb{R}$.*

*Having fixed a homogeneous norm $|\cdot|$ on $G$, we have for any $N \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$ and $\ell \in \mathbb{N}_0$, that*

$$\exists C = C_{\alpha,N,\ell} > 0 \quad \forall t \in (0,1] \quad \sup_{|x|=1} |\partial_t^\ell X^\alpha h_t(x)| \leq C_{\alpha,N} t^N. \qquad (4.15)$$

The proof of Theorem 4.2.7 is given in the next section. We finish this section with some comments and some corollaries of this theorem.

*Remark* 4.2.8. 1. If the group is stratified and $\mathcal{R} = -\mathcal{L}$ where $\mathcal{L}$ is a sub-Laplacian, then $\mathcal{R}$ is of order two and the proof relies on Hunt's theorem [Hun56], cf. [FS82, ch1.G]. In this case, the heat kernel is real-valued and moreover non-negative. The heat semi-group is then a semi-group of contraction which preserves positivity.

2. The behaviour of the heat kernel in the general case is quite well understood. For instance, it can be extended to the complex right-half plane. Then the heat kernel $h_z$ with $z \in \mathbb{C}$, $\operatorname{Re} z > 0$ decays exponentially. See [Dzi93, DHZ94, AtER94].

3. Since $\mathcal{R}_2$ is a positive operator, only the values of $\phi \in L^\infty(\mathbb{R})$ on $[0, \infty)$ are taken into account for the multipliers $\phi(\mathcal{R}_2)$. But in fact, the value at 0 can be neglected too, as a consequence of the property of the heat kernel. Indeed, from $h_t \in \mathcal{S}(G)$ and (4.12), it is not difficult to show

$$\|f * h_t\|_{L^2(G)} \underset{t\to\infty}{\longrightarrow} 0,$$

first for $f \in \mathcal{D}(G)$ and then by density for any $f \in L^2$. This shows

$$\|e^{-t\mathcal{R}_2} f\|_{L^2(G)} \underset{t\to\infty}{\longrightarrow} 0,$$

and therefore we have

$$\| \int_0^\epsilon dE(\lambda)\|_{L^2(G)} \underset{\epsilon\to 0}{\longrightarrow} 0.$$

4. Another consequence of the heat kernel being Schwartz, proved in [HJL85], is that the spectrum of $\pi(\mathcal{R})$ is discrete and lies in $(0, \infty)$ for any $\pi \in \widehat{G}\backslash\{1\}$. Indeed, it is easy to see that $\pi(\mathcal{R})$ is the infinitesimal generator of the semi-group $\{\pi(e^{-t\mathcal{R}})\}_{t>0}$ in $\mathcal{H}_\pi$ and that $\pi(e^{-t\mathcal{R}}) = \pi(h_t)$ is a compact operator since $h_t \in \mathcal{S}(G)$ (for this last property, see [CG90, Theorem 4.2.1]).

   Moreover, strong properties of the eigenvalue distributions of $\pi(\mathcal{R})$ are known, see [tER97].

Theorem 4.2.7 shows that the functions $h_t$ provide a commutative approximation of the identity, see Remark 3.1.60. We already know that $\{e^{-t\mathcal{R}_2}\}_{t>0}$ is a strongly continuous contraction semi-group. Moreover, we have the following properties for any $p$:

**Corollary 4.2.9.** *The operators*

$$f \mapsto f * h_t, \quad t > 0,$$

*form a strongly continuous semi-group on $L^p(G)$ for any $p \in [1, \infty)$ and on $C_o(G)$. Furthermore, for any $f \in \mathcal{D}(G)$ and any $p \in [1, \infty]$ (finite or infinite), we have the convergence*

$$\left\| \frac{1}{t} (f * h_t - f) - \mathcal{R} f \right\|_p \longrightarrow_{t\to 0} 0. \tag{4.16}$$

Finally, we formulate a simple but useful corollary of Theorem 4.2.7.

**Corollary 4.2.10.** *Setting $r = t^{-\frac{1}{\nu}}$ in (4.12), we get*

$$\forall x \in G,\ t > 0 \qquad h_t(x) = t^{-\frac{Q}{\nu}} h_1(t^{-\frac{1}{\nu}} x) \tag{4.17}$$

*and*

$$\text{for } x \in G \backslash \{0\} \text{ fixed, } X_x^\alpha h(x,t) = \begin{cases} O(t^{-\frac{Q+[\alpha]}{\nu}}) \text{ as } t \to \infty, \\ O(t^N) \text{ for all } N \in \mathbb{N}_0 \text{ as } t \to 0. \end{cases} \tag{4.18}$$

*Inequalities (4.18) are also valid for any $x$ in a fixed compact subset of $G\backslash\{0\}$.*

### 4.2.3 Proof of the heat kernel theorem and its corollaries

This section is entirely devoted to the proofs of Theorem 4.2.7 and Corollaries 4.2.9 and 4.2.10. This may be skipped at first reading. The proofs essentially follow the arguments of Folland and Stein [FS82, Ch. 4. B].

Since $h_t$ is the convolution kernel of the $\mathcal{R}_2$-multiplier operator, Corollary 4.1.16 yield that $h_t \in \mathcal{S}'(G)$ is a distribution which satisfies Properties (4.12) and (4.13) for each $t > 0$ fixed. Note that (4.12) easily yields (4.17).

By the Schwartz kernel theorem (see Corollary 3.2.1), since $(0,\infty) \ni t \mapsto e^{-t\mathcal{R}_2} \in \mathscr{L}(L^2(G))$ is a strongly continuous mapping, the function $(0,\infty) \ni t \mapsto h_t \in \mathcal{S}'(G)$ is continuous. Consequently the mapping $(t,x) \mapsto h_t(x)$ is a distribution on $(0,\infty) \times G$.

By the properties of semi-groups (cf. Proposition A.2.3 (4)), we have

$$\forall \phi \in \mathcal{D}(G),\ t > 0, \qquad \partial_t(e^{-t\mathcal{R}_2}\phi) = -\mathcal{R}_2(e^{-t\mathcal{R}_2}\phi) = -\mathcal{R}(e^{-t\mathcal{R}_2}\phi).$$

Taking this equation at $0_G$ shows that $(t,x) \mapsto h_t(x)$ is a solution in the sense of distributions of the equation $(\partial_t + \mathcal{R})f = 0$ on $(0,\infty) \times G$.

The next lemma is independent of the rest of the proof and shows that $\partial_t + \mathcal{R}$ can be turned into a Rockland operator:

**Lemma 4.2.11.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. We equip the group $H := G \times \mathbb{R}$ (which is the direct product of the groups $G$ and $(\mathbb{R},+)$) with the dilations*

$$D_r(x,t) := (rx, r^\nu t), \quad x \in G,\ t \in \mathbb{R}.$$

*The group $H$ has become a homogeneous Lie group and the operators $\mathcal{R} + \partial_t$ and $\mathcal{R} - \partial_t$ are Rockland operators on $H$.*

*Proof of Lemma 4.2.11.* The dual of $H$ is easily seen to be isomorphic to $\widehat{G} \times \mathbb{R}$:

- if $\pi \in \widehat{G}$ and $\lambda \in \mathbb{R}$, we can construct the representation $\rho = \rho_{\pi,\lambda}$ of $H$ on $\mathcal{H}_\rho = \mathcal{H}_\pi$ by $\rho(x,t) := e^{i\lambda t}\pi(x)$;

- conversely, any representation $\rho \in \widehat{H}$ can be realised into a representation of the form $\rho_{\pi,\lambda}$.

Let $\rho = \rho_{\pi,\lambda} \in \widehat{H}$. We observe that $\mathcal{H}_\rho^\infty = \mathcal{H}_\pi^\infty$, $\rho(\mathcal{R}) = \pi(\mathcal{R})$, and $\rho(\partial_t) = i\lambda$. If $v \in \mathcal{H}_\rho^\infty$ is such that $\rho(\mathcal{R} + \partial_t)v = 0$ then

$$0 = (\rho(\mathcal{R} \pm \partial_t)v, v)_{\mathcal{H}_\rho} = (\pi(\mathcal{R})v, v)_{\mathcal{H}_\pi} \pm i\lambda(v, v)_{\mathcal{H}_\pi} = (\pi(\mathcal{R})v, v)_{\mathcal{H}_\pi} \pm i\lambda \|v\|^2_{\mathcal{H}_\pi}.$$

Since, by Proposition 4.2.6, $(\pi(\mathcal{R})v, v)_{\mathcal{H}_\pi} \geq 0$, the real part of the previous equalities is $(\pi(\mathcal{R})v, v)_{\mathcal{H}_\pi} = 0$. Again by Proposition 4.2.6, necessarily $v = 0$. □

*Remark* 4.2.12. A similar proof implies that $\mathcal{R} \pm \partial_t^k$ for $k \in \mathbb{N}$ odd is a Rockland operator on the group $G \times \mathbb{R}$ endowed with the dilations $D_r(x, t) = (rx, r^{\nu/k}t)$.

**Corollary 4.2.13.** *The distribution $(t, x) \mapsto h_t(x)$ is smooth on $(0, \infty) \times G$ and satisfies the equation*

$$(\partial_t + \mathcal{R})f = 0.$$

*Furthermore, for any $t > 0$, $h_t \in L^2(G)$ and*

$$\int_G |h_t(x)|^2 dx = t^{-\frac{Q}{\nu}} \int_G |h_1(x)|^2 dx < \infty. \tag{4.19}$$

*Proof.* The operator $\partial_t + \mathcal{R}$ is Rockland on $G \times \mathbb{R}$ by Lemma 4.2.11, therefore hypoelliptic by the Hellfer-Nourrigat theorem (see Theorem 4.1.12). Since the distribution $(t, x) \mapsto h_t(x)$ is a solution of the equation $(\partial_t + \mathcal{R})f = 0$ on $(0, \infty) \times G$, it is in fact smooth.

Since $\mathcal{R}$ is a positive Rockland operator, $\mathcal{R}^t$ is also a positive Rockland operator (see Lemma 4.2.5) and we can apply Lemma 4.2.11 to both. Therefore, $\mathcal{R} + \partial_t$ and its transpose are Rockland and thus hypoelliptic on $G \times \mathbb{R}$. By the Schwartz-Treves theorem (see Theorem A.1.6), the distribution topology on $G \times (0, \infty)$ and the $C^\infty$-topology agree on the the nullspace of $\mathcal{R} + \partial_t$

$$\mathcal{N} = \{f \in \mathcal{D}'(G \times (0, \infty)) \ : \ (\mathcal{R} + \partial_t)f = 0\}.$$

Since $(0, \infty) \ni t \mapsto h_t \in \mathcal{S}'(G)$ is continuous and $(t, x) \mapsto h_t(x)$ is smooth on $(0, \infty) \times G$, the mapping $T$ defined via

$$T\phi(x, t) = (e^{-t\mathcal{R}_2}\phi)(x) = \int_G h_t(x)\phi(x)dx, \quad \phi \in L^2(G),\ x \in G,\ t > 0,$$

is continuous from $L^2(G)$ to $\mathcal{D}'(G \times (0, \infty))$. Furthermore, the semi-group properties imply that the range of $T$ lies in $\mathcal{N}$. Therefore, the mapping

$$L^2(G) \ni \phi \longmapsto T\phi(0, 1) = \int_G \phi(x)h_1(x)dx,$$

is a continuous functional. Hence $h_1$ must be square integrable.

By homogeneity (see (4.17)), for any $t > 0$, we see that $h_t \in L^2(G)$ as a consequence of Corollary 4.2.10 and (4.19) must hold. □

We now define the function $h : G \times \mathbb{R} \to \mathbb{C}$ as in the statement of Theorem 4.2.7 by

$$h(x,t) := \begin{cases} h_t(x) & \text{if } t > 0 \text{ and } x \in G, \\ 0 & \text{if } t \leq 0 \text{ and } x \in G. \end{cases}$$

By Corollary 4.2.13, the function $h$ is smooth on $G \times (\mathbb{R}\backslash\{0\})$ and satisfies the equation $(\mathcal{R} + \partial_t)h = 0$ on $G \times (\mathbb{R}\backslash\{0\})$. However, it is not obvious that it is a distribution on $G \times \mathbb{R}$. Our next goal is to prove that it is indeed a distribution and that it satisfies the equation $(\mathcal{R} + \partial_t)h = 0$ on $G \times \mathbb{R}$.

It is easy to prove that $h$ is a distribution under the assumption $\nu > Q/2$ since it is then locally integrable:

**Lemma 4.2.14.** *If $\nu > Q/2$, then $h$ is locally integrable on $G \times \mathbb{R}$.*

*Proof of Lemma 4.2.14.* We assume $\nu > Q/2$. We see that for any $\epsilon > 0$ and $R > 0$, using the homogeneity property given in (4.19),

$$\begin{aligned} \int_0^\epsilon \int_{|x|<R} |h(x,t)| dx dt &\leq \int_0^\epsilon \left( \int_{|x|<R} |h_t(x)|^2 dx \right)^{\frac{1}{2}} \left( \int_{|x|<R} 1 dx \right)^{\frac{1}{2}} dt \\ &\leq |B(0,1)|^{\frac{1}{2}} R^{Q/2} \left( \int_G |h_1(x)|^2 dx \right)^{\frac{1}{2}} \int_0^\epsilon t^{-\frac{Q}{2\nu}} dt \\ &= C R^{Q/2} \epsilon^{1-\frac{Q}{2\nu}}, \end{aligned}$$

since we assumed $\nu > Q/2$. This shows that $h$ is locally integrable on $G \times \mathbb{R}$ and hence defines a distribution. □

If we know that $h$ is a distribution, being a solution of $(\mathcal{R} + \partial_t)h = \delta_{0,0}$ is almost granted:

**Lemma 4.2.15.** *Let us assume that $h \in \mathcal{D}'(G \times \mathbb{R})$ is a distribution and that*

- *either $h_1 \in L^2(G)$ and $\nu > Q/2$,*
- *or $h_1 \in L^1(G)$ (without restriction on $\nu > Q/2$).*

*Then $h$ satisfies the equation*

$$(\mathcal{R} + \partial_t)h = \delta_{0,0}$$

*as a distribution.*

The proof of Lemma 4.2.15 will require the following technical property which is independent of the rest of the proof:

**Lemma 4.2.16.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G \sim \mathbb{R}^n$ with homogeneous degree $\nu$. If $m\nu \geq \lceil \frac{n}{2} \rceil$, the functions in the domain of $\mathcal{R}^m$ are continuous on $\Omega$, i.e.*

$$\text{Dom}(\mathcal{R}^m) \subset C(\Omega),$$

*where $C(\Omega)$ denotes the space of continuous functions on $\Omega$. Furthermore, for any compact subset $\Omega$ of $G$, there exists a constant $C = C_{\Omega,\mathcal{R},G,m}$ such that*

$$\forall \phi \in \mathrm{Dom}(\mathcal{R}^m) \qquad \sup_{x\in\Omega} |\phi(x)| \leq C \left(\|\phi\|_{L^2} + \|\mathcal{R}^m \phi\|_{L^2}\right).$$

This is a (very) weak form of Sobolev embeddings. We will later on obtain stronger results in Theorem 4.4.25. The proof below uses Corollary 4.1.14 showed by Helffer and Nourrigat during their proof of Theorem 4.1.12.

*Proof of Lemma 4.2.16.* By the classical Sobolev embedding theorem on $\mathbb{R}^n$, see e.g. [Ste70a, p.124], if $\phi \in L^2(\mathbb{R}^n)$ together with $\partial_x^\alpha \phi \in L^2(\mathbb{R}^n)$ for any multi-index $\alpha$ satisfying $|\alpha| \leq \lceil \frac{n}{2} \rceil$, then $\phi$ may be modified on a set of zero measure so that the resulting function, still denoted by $\phi$, is continuous.

Furthermore, for any compact subset $\Omega$ of $G$, we may choose a closed ball $B(0,R)$ strictly containing $\Omega$, and there exists a constant $C = C_{\Omega,R}$ independent of $\phi$ such that

$$\sup_\Omega |\phi| \leq C \sum_{|\alpha| \leq \lceil \frac{n}{2} \rceil} \|\partial_x^\alpha \phi\|_{L^2(B(0,R))}.$$

As the abelian derivatives may be expressed as linear combination of left-invariant ones, see Section 3.1.5, there exists another constant $C = C_R$ such that

$$\sum_{|\alpha| \leq \lceil \frac{n}{2} \rceil} \|\partial_x^\alpha \psi\|_{L^2(B(0,R))} \leq C \sum_{|\alpha| \leq \lceil \frac{n}{2} \rceil} \|X^\alpha \psi\|_{L^2(B(0,R))}$$

for any $\psi$ such that the right-hand side makes sense. By the corollary of the Helffer-Nourrigat theorem applied to $\mathcal{R}^m$ (see Corollary 4.1.14, see also Lemma 4.2.5), there exists $C = C_{\mathcal{R},m} > 0$ such that

$$\forall \psi \in \mathcal{S}(G) \qquad \sum_{[\alpha] \leq m\nu} \|X^\alpha \psi\|_{L^2(G)} \leq C \left(\|\mathcal{R}^m \psi\|_{L^2(G)} + \|\psi\|_{L^2(G)}\right).$$

The last two properties yield easily

$$\sum_{|\alpha| \leq \lceil \frac{n}{2} \rceil} \|\partial_x^\alpha \psi\|_{L^2(B(0,R))} \leq C \left(\|\mathcal{R}^m \psi\|_{L^2(G)} + \|\psi\|_{L^2(G)}\right),$$

for any function $\psi \in L^2(G)$ for which the right-hand side makes sense, for some constant $C = C_{R,\mathcal{R},m}$ independent of $\psi$, as long as $m\nu \geq \lceil \frac{n}{2} \rceil$. Together with the embedding property recalled at the beginning of the proof, this shows Lemma 4.2.16. □

We can now go back to the proof of the heat kernel theorem, and more precisely, the proof of Lemma 4.2.15.

*Proof of Lemma 4.2.15.* If we set for each $\epsilon > 0$ and $(x,t) \in G \times \mathbb{R}$,

$$h^{(\epsilon)}(x,t) := \begin{cases} h(x,t) & \text{if } t > \epsilon, \\ 0 & \text{if } t \leq \epsilon, \end{cases}$$

it is clear that this defines a distribution $h^{(\epsilon)} \in \mathcal{D}'(G \times \mathbb{R})$ and that $\{h^{(\epsilon)}\}$ converges to $h$ in $\mathcal{D}'(G \times \mathbb{R})$ as $\epsilon$ tends to 0. To prove that

$$(\mathcal{R} + \partial_t)h = \delta_{0,0},$$

it suffices to show that $(\mathcal{R} + \partial_t)h^{(\epsilon)}$ converges to $\delta_{0,0}$ in $\mathcal{D}'(G \times \mathbb{R})$ as $\epsilon$ tends to 0; this means:

$$\forall \phi \in \mathcal{D}(G \times \mathbb{R}) \qquad \langle h^{(\epsilon)}, (\mathcal{R}^t - \partial_t)\phi \rangle = \langle (\mathcal{R} + \partial_t)h^{(\epsilon)}, \phi \rangle \xrightarrow[\epsilon \to 0]{\mathcal{D}'} \phi(0).$$

Using the translation of the group $H = G \times \mathbb{R}$ which is the direct product of the groups $G$ and $(\mathbb{R}, +)$, this is equivalent to the pointwise convergence in $H$:

$$\forall \phi \in \mathcal{D}(H),\ (x,t) \in H \qquad (\mathcal{R} + \partial_t)(\phi * h^{(\epsilon)})(x,t) \underset{\epsilon \to 0}{\longrightarrow} \phi(x,t), \tag{4.20}$$

since

$$(\mathcal{R} + \partial_t)(\phi * h^{(\epsilon)})(x,t) = \phi * ((\mathcal{R} + \partial_t)h^{(\epsilon)})(x,t) = \langle (\mathcal{R} + \partial_t)h^{(\epsilon)}, \phi((x,t)\,\cdot^{-1}) \rangle.$$

The above convolution is in $H$, given by

$$\begin{aligned} \left(\phi * h^{(\epsilon)}\right)(x,t) &= \int_G \int_{\mathbb{R}} \phi(y,u)\, h^{(\epsilon)}((y,u)^{-1}(x,t))dydu \\ &= \int_G \int_{u=-\infty}^{t-\epsilon} \phi(y,u)\, h(y^{-1}x, -u+t)dydu. \end{aligned}$$

We see that

$$\begin{aligned} (\mathcal{R} + \partial_t)(\phi * h^{(\epsilon)})(x,t) &= \int_G \int_{u=-\infty}^{t-\epsilon} \phi(y,u)\, (\mathcal{R}_x + \partial_t)h(y^{-1}x, -u+t)dydu \\ &\quad + \int_G \phi(y, t-\epsilon)\, h(y^{-1}x, \epsilon)dy, \end{aligned}$$

and the first term of the right hand side is zero since $(\mathcal{R} + \partial_t)h = 0$ on $G \times (0, \infty)$ and $\mathcal{R} + \partial_t$ is left-invariant on $H$. Hence

$$(\mathcal{R} + \partial_t)(\phi * h^{(\epsilon)})(x,t) = \phi(\cdot, t-\epsilon) * h_\epsilon(x), \tag{4.21}$$

using the convolution in $H$ and $G$ for the left and right hand sides respectively.

We now fix $t$ and set $\phi_\epsilon(y) := \phi(y, t-\epsilon)$. Then

$$\phi(\cdot, t-\epsilon) * h_\epsilon = \phi_\epsilon * h_\epsilon,$$

and we can write

$$\phi_\epsilon * h_\epsilon - \phi_0 = (\phi_\epsilon - \phi_0) * h_\epsilon - (\phi_0 * h_\epsilon - \phi_0). \tag{4.22}$$

For the first term in the right-hand side of (4.22), we need to separate the case $h_1 \in L^2(G)$ with $\nu > Q/2$ from the case $h_1 \in L^1(G)$. Indeed if $h_1 \in L^2(G)$ with $\nu > Q/2$, then by (4.19),

$$\|h_\epsilon\|_2 = \epsilon^{-\frac{Q}{2\nu}} \|h_1\|_2$$

and the Cauchy-Schwartz inequality yields

$$\|(\phi_\epsilon - \phi_0) * h_\epsilon\|_\infty \leq \|\phi_\epsilon - \phi_0\|_2 \|h_\epsilon\|_2.$$

We easily obtain $\|\phi_\epsilon - \phi_0\|_2 \leq C\epsilon$ as $\phi \in \mathcal{D}(G \times \mathbb{R})$. Thus

$$\|(\phi_\epsilon - \phi_0) * h_\epsilon\|_\infty \leq C' \epsilon^{1-\frac{Q}{2\nu}} \longrightarrow_{\epsilon \to 0} 0,$$

since we assumed $\nu > Q/2$. If $h_1 \in L^1(G)$, then by (4.19), $\|h_\epsilon\|_1 = \|h_1\|_1$ and the Hölder inequality yields

$$\|(\phi_\epsilon - \phi_0) * h_\epsilon\|_\infty \leq \|\phi_\epsilon - \phi_0\|_\infty \|h_\epsilon\|_1 = \|h_1\|_1 \|\phi_\epsilon - \phi_0\|_\infty.$$

Again $\|\phi_\epsilon - \phi_0\|_2 \leq C\epsilon$ as $\phi \in \mathcal{D}(G \times \mathbb{R})$ thus

$$\|(\phi_\epsilon - \phi_0) * h_\epsilon\|_\infty \leq C'\epsilon \longrightarrow_{\epsilon \to 0} 0.$$

For the second term in the right-hand side of (4.22), the functional calculus of $\mathcal{R}_2$ yields the convergence in $L^2(G)$

$$\phi_0 * h_\epsilon = e^{-\epsilon \mathcal{R}_2} \phi_0 \longrightarrow_{\epsilon \to 0} \phi_0.$$

As $\mathcal{R}_2$ commutes with the $\mathcal{R}_2$-multiplier $e^{-\epsilon \mathcal{R}_2}$ and since $\phi_0 \in \mathcal{D}(G)$, $\mathcal{R}_2 \phi_0 = \mathcal{R}\phi_0$, we know that $\phi_0 * h_\epsilon = e^{-\epsilon \mathcal{R}_2} \phi_0 \in \text{Dom}(\mathcal{R}_2)$ and moreover

$$(\mathcal{R}\phi_0) * h_\epsilon = (\mathcal{R}_2 \phi_0) * h_\epsilon = e^{-\epsilon \mathcal{R}_2} \mathcal{R}_2 \phi_0 = \mathcal{R}_2 e^{-\epsilon \mathcal{R}_2} \phi_0 \overset{L^2(G)}{\longrightarrow}_{\epsilon \to 0} \mathcal{R}_2 \phi_0.$$

More generally, for any $m \in \mathbb{N}$, $\phi_0 * h_\epsilon = e^{-\epsilon \mathcal{R}_2} \phi_0 \in \text{Dom}(\mathcal{R}_2^m)$ and

$$\mathcal{R}_2^m e^{-\epsilon \mathcal{R}_2} \phi_0 \overset{L^2(G)}{\longrightarrow}_{\epsilon \to 0} \mathcal{R}_2^m \phi_0.$$

By Lemma 4.2.16, this implies that $\phi_0 * h_\epsilon - \phi_0$ is continuous on $G$. Furthermore, for any compact subset $\Omega$ of $G \sim \mathbb{R}^n$ and any $m \in \mathbb{N}$ with $m\nu > \lfloor \frac{n}{2} \rfloor$, we have

$$\sup_\Omega |\phi_0 * h_\epsilon - \phi_0| \leq C \left( \|\phi_0 * h_\epsilon - \phi_0\|_2 + \|\mathcal{R}^m (\phi_0 * h_\epsilon - \phi_0)\|_2 \right) \longrightarrow_{\epsilon \to 0} 0.$$

Hence we have obtained that both terms on the right-hand side of (4.22) go to zero for the supremum norm on any compact subset of $G$. Therefore, the expression in (4.21) tends to

$$(\mathcal{R} + \partial_t)(\phi * h^{(\epsilon)})(x, t) \longrightarrow_{\epsilon \to 0} \phi(\cdot, t - \epsilon) * h_\epsilon(x),$$

for $t$ fixed, locally in $x$. This is even stronger than the pointwise convergence in $H$ we wanted in (4.20) and concludes the proof of Lemma 4.2.15. $\square$

**Corollary 4.2.17.** *Under the hypothesis of Lemma 4.2.15, $h$ is smooth on $(G \times \mathbb{R})\backslash\{(0,0)\}$ and satisfies (4.15) and (4.18). Moreover, each function $h_t$ is Schwartz on $G$ and*

$$\int_G h_t(x)dx = 1.$$

*Proof of Corollary 4.2.17.* By Lemma 4.2.15, the distribution $h$ annihilates the hypoelliptic operator $\mathcal{R}+\partial_t$ on $(G\times\mathbb{R})\backslash\{0\}$, and thus $h$ is smooth on $(G\times\mathbb{R})\backslash\{0\}$. Since $h(x,t)=0$ for $t\leq 0$, this implies that $h(x,t)$ vanish to infinite order as $t\to 0$:

$$\forall x \in G\backslash\{0\},\ N\in\mathbb{N}_0 \quad \exists\epsilon>0,\ C>0 \quad \forall t\in(0,\epsilon) \quad |h(x,t)|\leq Ct^N.$$

We can choose $\epsilon=1$ since $h$ is smooth on $G\times(0,\infty)$. In fact this estimate remains true for any $x$-derivatives $(\frac{\partial}{\partial x})^\alpha h(x,t)$. It is also uniform in $x$ when $x$ runs over a fixed compact set which does not contain 0. Choosing this compact set to be the unit sphere of a given quasi-norm $|\cdot|$, we have

$$\forall N\in\mathbb{N}_0 \quad \exists C>0 \quad \forall t\in(0,1] \quad \sup_{|x|=1}\left|\left(\frac{\partial}{\partial x}\right)^\alpha h(x,t)\right|\leq Ct^N.$$

We may replace the abelian derivatives $(\frac{\partial}{\partial x})^\alpha$ by the left-invariant ones, see Section 3.1.5. This implies (4.15).

Using the homogeneity of $h$ (see Property (4.12) which was already proven and Proposition 3.1.23), we have

$$\forall x\in G,\ r>0 \qquad X^\alpha h(x,t)=r^{Q-[\alpha]}X^\alpha h_{r^\nu t}(rx),$$

and so, in particular, if $|x|\geq 1$ then we obtain, because of (4.15), that

$$|X^\alpha h_1(x)|=|x|^{-Q+[\alpha]}|X^\alpha h_{|x|^{-\nu}}(|x|^{-1}x)|\leq C_{\alpha,N}|x|^{-Q+[\alpha]-\nu N}.$$

Since $h_1$ is smooth on $G$, this shows that $h_1$ is Schwartz. This is also the case for $h_t$ by homogeneity, see (4.17). Note that the same homogeneity property together with (4.15) implies (4.18).

Since each function $h_t$ satisfies the homogeneity property given in (4.17) and is integrable, the functions $h_t$ form a commutative approximation of the identity, see Remark 3.1.60. In particular,

$$\phi * h_t \longrightarrow_{t\to 0} c\phi \quad \text{in } L^2(G),$$

with $c=\int_G h_1(x)dx$. Since we know

$$\phi * h_t = e^{-t\mathcal{R}_2}\phi \longrightarrow_{\epsilon\to 0}\phi \quad \text{in } L^2(G),$$

this constant $c$ must be equal to 1. By homogeneity,

$$\forall t>0 \qquad \int_G h_t(x)dx=\int_G h_1(x)dx=c=1.$$

□

Lemmata 4.2.14 and 4.2.15 imply Theorem 4.2.7 and Corollary 4.2.10 under the assumption $\nu > Q/2$. We now need to remove this assumption. For this, we will use the following formula which is a consequence of the principle of subordination:

**Lemma 4.2.18.** *For any $\gamma > 0$, we have*

$$e^{-\gamma} = \int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}} e^{-\frac{\gamma^2}{4s}} ds. \tag{4.23}$$

*Sketch of the proof of Lemma 4.2.18.* We follow [Ste70a, p.61]. We start from the well known identity

$$\pi e^{-\gamma} = \int_{-\infty}^\infty \frac{e^{i\gamma x}}{1+x^2} dx, \tag{4.24}$$

which is an application of the Residue theorem to the function

$$z \mapsto \frac{e^{i\gamma z}}{z^2+1}.$$

In (4.24) we replace $1+x^2$ using

$$\frac{1}{1+x^2} = \int_0^\infty e^{-(1+x^2)u} du,$$

and we obtain the double integral

$$\pi e^{-\gamma} = \int_{-\infty}^\infty e^{i\gamma x} \int_0^\infty e^{-(1+x^2)u} du\ dx.$$

One can show that it is possible to invert the order of integration:

$$\pi e^{-\gamma} = \int_0^\infty e^{-u} \int_{-\infty}^\infty e^{i\gamma x} e^{-x^2 u} dx\ du.$$

It is well known that the inner integral in $dx$ is equal to

$$\frac{e^{-\frac{\gamma^2}{4u}}}{\sqrt{\pi u}}.$$

And this shows (4.23). □

We can now finish the proofs of Theorem 4.2.7 and Corollary 4.2.10.

*End of the proofs of Theorem 4.2.7 and Corollary 4.2.10.* Since the case $\nu > Q/2$ is already proven, we may assume $\nu \leq Q/2$.

For any $m \in \mathbb{N}_0$, $\mathcal{R}^{2^m}$ is a positive Rockland operator (see Lemma 4.2.5), with homogeneous degree $2^m \nu$. We denote by $K_m$ the function on $G \times \mathbb{R}$ giving its heat kernel in the sense that if $t > 0$, $K_m(\cdot, t) \in \mathcal{S}'(G)$ is the kernel of $e^{-t\mathcal{R}_2^{2^m}}$

and if $t \leq 0$ then $K_m(x,t) = 0$ for any $x \in G$. This is possible since, by Corollary 4.2.13, $K_m$ is smooth on $G \times (0,\infty)$. By homogeneity, it will always satisfy

$$\forall x \in G,\ t > 0 \qquad K_m(x,t) = t^{-\frac{Q}{\nu 2^m}} K_m(t^{-\frac{1}{\nu 2^m}} x, 1). \tag{4.25}$$

In (4.23), replacing $\gamma$ by $t\lambda^{2^{m-1}}$, one finds that

$$e^{-t\lambda^{2^{m-1}}} = \int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}} e^{-\frac{t^2\lambda^{2^m}}{4s}} ds.$$

Using the functional calculus on $\mathcal{R}$, that is, integrating against the spectral measure $dE(\lambda)$ of $\mathcal{R}_2$, we obtain formally that for any non-negative integer $m \in \mathbb{N}_0$ and $t > 0$,

$$e^{-t\mathcal{R}_2^{2^{m-1}}} = \int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}} e^{-\frac{t^2}{4s}\mathcal{R}_2^{2^m}} ds, \tag{4.26}$$

and for the kernels of these operators,

$$K_{m-1}(x,t) = \int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}} K_m(x, \frac{t^2}{4s}) ds. \tag{4.27}$$

It is not difficult to see that Formulae (4.26) and (4.27) hold as operators and continuous integrable functions respectively when, for instance, $K_m(\cdot,t)$ is integrable on $G$ for each $t > 0$ and

$$\int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}} \|K_m(\cdot, \frac{t^2}{4s})\|_{L^1(G)} ds < \infty.$$

Indeed under this hypothesis, $K_{m-1}(\cdot,t)$ is integrable on $G$ for any fixed $t > 0$ and

$$\|K_{m-1}(\cdot,t)\|_{L^1(G)} \leq \int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}} \|K_m(\cdot, \frac{t^2}{4s})\|_{L^1(G)} ds < \infty. \tag{4.28}$$

It is then a standard procedure to make sense of (4.26) by first integrating $\lambda$ over $[0,N]$ and then letting $N$ tend to infinity.

We first assume that $2^m\nu > Q/2$, so that the conclusion of Theorem 4.2.7 holds for $K_m$. In particular, $K_m(\cdot,1) \in \mathcal{S}(G)$ and by homogeneity, the $L^1$-norm of $K_m(\cdot,t)$ is

$$\int_G |K_m(x,t)| dx = \int_G |K_m(x,1)| dx,$$

is finite and independent of $t$. Therefore

$$\int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}} \int_G |K_m(x, \frac{t^2}{4s})| dx ds = \int_G |K_m(x,1)| dx \int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}} ds,$$

is finite. Consequently Formula (4.27) holds and by (4.28),

$$\|K_{m-1}(t,\cdot)\|_{L^1(G)} \leq \int_G |K_m(x,1)| dx \int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}} ds < \infty.$$

By homogeneity, $\int_G |K_{m-1}(x,t)|dx$ must also be independent of $t>0$, while it is identically zero if $t \leq 0$. This implies that $K_{m-1}$ is locally integrable on $G \times \mathbb{R}$ and that $K_{m-1}(\cdot, 1) \in L^1(G)$. By Lemmata 4.2.14 and 4.2.15, $K_{m-1}$ satisfy the properties of the heat kernel described in Theorem 4.2.7 and Corollary 4.2.10.

Now we can repeat the same reasoning with $m$ replaced successively by $m-1, m-2, \ldots, 2, 1$. Since $K_0 = h$, this concludes the proofs of Theorem 4.2.7 and Corollary 4.2.10. □

We still have to show Corollary 4.2.9.

*Proof of Corollary 4.2.9.* Since the heat kernels $h_t$, $t>0$, form a commutative approximation of the identity (see Theorem 4.2.7 and Remark 3.1.60 in Section 3.1.10), the operators $f \mapsto f * h_t$, $t > 0$, form a strongly continuous semi-group on $L^p(G)$ for any $p \in [1,\infty)$ and on $C_o(G)$, see Lemma 3.1.58. It is naturally equibounded by $\|h_1\|$ since

$$\|f * h_t\|_p \leq \|f\|_p \|h_t\|_1 \quad \text{and} \quad \|h_t\|_1 = \|h_1\|.$$

Let us prove the convergence in (4.16) for $p=\infty$. Let $f \in \mathcal{D}(G)$. By Lemma 4.2.16, for any compact subset $\Omega \subset G$,

$$\begin{aligned}
&\sup_\Omega \left| \frac{1}{t}\left(f * h_t - f\right) - \mathcal{R}f \right| \\
&\leq C \left( \left\| \frac{1}{t}\left(f * h_t - f\right) - \mathcal{R}f \right\|_2 + \left\| \frac{1}{t}\mathcal{R}^m \left(f * h_t - f\right) - \mathcal{R}^{m+1} f \right\|_2 \right),
\end{aligned}$$

where $m$ is an integer such that $m\nu \geq \lceil \frac{n}{2} \rceil$. Since $\mathcal{D}(G) \subset \mathrm{Dom}(\mathcal{R})$ and

$$e^{-t\mathcal{R}_2} f = f * h_t,$$

we have for any integer $m' \in \mathbb{N}_0$ that

$$\begin{aligned}
&\frac{1}{t}\mathcal{R}^{m'}\left(f * h_t - f\right) - \mathcal{R}^{m'+1} f = \frac{1}{t}\mathcal{R}_2^{m'}\left(e^{-t\mathcal{R}_2} f - f\right) - \mathcal{R}_2^{m'+1} f \\
&\quad = \frac{1}{t}\left(e^{-t\mathcal{R}_2}\mathcal{R}_2^{m'} f - \mathcal{R}_2^{m'} f\right) - \mathcal{R}_2^{m'+1} f = \frac{1}{t}\left((\mathcal{R}^{m'} f) * h_t - \mathcal{R}^{m'} f\right) - \mathcal{R}^{m'+1} f \\
&\quad \longrightarrow_{t\to 0} 0 \quad \text{in } L^2(G).
\end{aligned}$$

Therefore,

$$\sup_\Omega \left| \frac{1}{t}\left(f * h_t - f\right) - \mathcal{R}f \right| \longrightarrow_{t \to 0} 0.$$

We fix a quasi-norm $|\cdot|$. By Part 2 of Remark 3.2.16 and the existence of a homogeneous norm (Theorem 3.1.39), without loss of generality, we may assume $|\cdot|$ to be also a norm, that is, the triangular inequality is satisfied with constant 1; although we could give a proof without this hypothesis, it simplifies the constants below. Let $\bar{B}_R$ be a closed ball about 0 of radius $R$ which contains the support of $f$. We choose $\Omega = \bar{B}_{2R}$ the closed ball about 0 and with radius $2R$. If $x \notin \Omega$, then since $f$ is supported in $\bar{B}_R \subset \Omega$,

$$\left(\frac{1}{t}\left(f * h_t - f\right) - \mathcal{R}f\right)(x) = \frac{1}{t} f * h_t(x) = \frac{1}{t}\int_{|y|\leq R} f(y) h_t(y^{-1}x)dy,$$

hence

$$\left|\frac{1}{t} f * h_t(x)\right| \leq \frac{\|f\|_\infty}{t}\int_{|y|\leq R} |h_t(y^{-1}x)|dy = \frac{\|f\|_\infty}{t}\int_{|xt^{\frac{1}{\nu}}z^{-1}|\leq R} |h_1(z)|dz,$$

as $h_t$ satisfies (4.17). Note that $\{z : |xt^{\frac{1}{\nu}}z^{-1}| \leq R\} \subset \{z : |t^{\frac{1}{\nu}}z| > R/2\}$ since

$$|t^{\frac{1}{\nu}}z| \leq R/2 \implies |xt^{\frac{1}{\nu}}z^{-1}| \geq |x| - |t^{\frac{1}{\nu}}z^{-1}| \geq \frac{3}{2}R.$$

Therefore

$$\int_{|xt^{\frac{1}{\nu}}z^{-1}|\leq R} |h_1(z)|dz \leq \int_{|z|>t^{-\frac{1}{\nu}}R/2} |h_1(z)|dz.$$

Since $h_1$ is Schwartz, we must have

$$\exists C \quad \forall z \in G\backslash\{0\} \quad |h_1(z)| \leq C|z|^{-a},$$

for $a = Q + 2\nu$ for instance. This together with the polar change of variable (cf. Proposition 3.1.42) yield

$$\int_{|z|>t^{-\frac{1}{\nu}}R/2} |h_1(z)|dz \leq C\int_{r=t^{-\frac{1}{\nu}}R/2}^{\infty} r^{-a-Q-1}dr = C't^2.$$

Consequently, denoting by $\Omega^c$ the complement of $\Omega$ in $G$, we have

$$\sup_{\Omega^c}\left|\frac{1}{t}\left(f * h_t - f\right) - \mathcal{R}f\right| \leq C't \longrightarrow_{t\to 0} 0.$$

This shows the convergence in (4.16) for $p = \infty$.

We proceed in a similar way to prove the convergence in (4.16) for $p$ finite. As above we fix $f \in \mathcal{D}(G)$ supported in $\bar{B}_R$. We decompose

$$\begin{aligned}&\|\frac{1}{t}\left(f * h_t - f\right) - \mathcal{R}f\|_p \\ &\quad \leq \|\frac{1}{t}(f * h_t - f) - \mathcal{R}f\|_{L^p(\bar{B}_{2R})} + \|\frac{1}{t}(f * h_t - f) - \mathcal{R}f\|_{L^p(B_{2R}^c)}.\end{aligned}$$

For the first term,

$$\|\frac{1}{t}(f * h_t - f) - \mathcal{R}f\|_{L^p(\bar{B}_{2R})} \leq |\bar{B}_{2R}|^{\frac{1}{p}}\|\frac{1}{t}(f * h_t - f) - \mathcal{R}f\|_\infty \underset{t\to 0}{\longrightarrow} 0,$$

as we have already proved the convergence in (4.16) for $p = \infty$. For the second term, we obtain for the reasons explained in the case $p = \infty$ that

$$\begin{aligned}
\|\frac{1}{t}(f * h_t - f) - \mathcal{R}f\|_{L^p(B_{2R}^c)} &= \frac{1}{t}\|f * h_t\|_{L^p(B_{2R}^c)} \\
&= \frac{1}{t}\left(\int_{|x|>2R}\left|\int_{|y|<R} f(y)\, h_t(y^{-1}x)dy\right|^p dx\right)^{\frac{1}{p}} \\
&\leq \frac{C}{t}\left(\int_{|x|>2R}\left(\int_{|y|<R} |f(y)|\, t^{-\frac{Q}{\nu}}|t^{-\frac{1}{\nu}}(y^{-1}x)|^{-a}dy\right)^p dx\right)^{\frac{1}{p}} \\
&\leq Ct^{-1+p(-\frac{Q}{\nu}+\frac{a}{\nu})}\|f\|_{L^1}\left(\int_{|x|>2R}(|x|-R)^{-ap}dx\right)^{\frac{1}{p}},
\end{aligned}$$

where we have used that the reverse triangle inequality

$$|y^{-1}x| \geq |x| - |y| \geq |x| - R.$$

Consequently we obtain the convergence in (4.16) for $p$ finite if we choose $a$ large enough. □

## 4.3 Fractional powers of positive Rockland operators

In this section we aim at defining fractional powers of positive Rockland operators. We will carry out the construction on the scale of $L^p$-spaces for $1 \leq p \leq \infty$, with $L^\infty(G)$ substituted by the space $C_o(G)$ of continuous functions vanishing at infinity. The extension of a positive Rockland operator $\mathcal{R}$ to $L^p(G)$ will be denoted by $\mathcal{R}_p$, and first we discuss the essential properties of such an extension. Then we define its complex powers. Before studying the corresponding Riesz and Bessel potentials, we will show that imaginary powers are continuous operators on $L^p$, $p \in (1, \infty)$.

### 4.3.1 Positive Rockland operators on $L^p$

We start by defining the analogue $\mathcal{R}_p$ of the operator $\mathcal{R}$ on $L^p(G)$.

**Definition 4.3.1.** Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$.

For $p \in [1, \infty)$, we denote by $\mathcal{R}_p$ the operator such that $-\mathcal{R}_p$ is the infinitesimal generator of the semi-group of operators $f \mapsto f * h_t$, $t > 0$, on $L^p(G)$.

We also denote by $\mathcal{R}_{\infty_o}$ the operator such that $-\mathcal{R}_{\infty_o}$ is the infinitesimal generator of the semi-group of operators $f \mapsto f * h_t$, $t > 0$, on $C_o(G)$.

For the moment it seems that $\mathcal{R}_2$ denotes the self-adjoint extension of $\mathcal{R}$ on $L^2(G)$ and minus the generator of $f \mapsto f * h_t$, $t > 0$, on $L^2(G)$. In the sequel, in fact in Theorem 4.3.3 below, we show that the two operators coincide and there is no conflict of notation.

The case $p = \infty$ is somewhat irrelevant and will be often replaced by $p = \infty_o$, especially when using duality. The next lemma aims at clarifying this point.

**Lemma 4.3.2.**

- *If $p \in (1,\infty)$, any bounded linear functional on $L^p(G)$ can be realised by integration against a function in $L^{p'}(G)$, where $p'$ is the conjugate exponent of $p$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Consequently, the dual $L^p(G)'$ of $L^p(G)$ may be identified with $L^{p'}(G)$ and the corresponding norms coincide.*
- *If $p = 1$, any bounded linear functional on $L^1(G)$ can be realised by integration against a bounded function on $G$. Consequently, the dual $L^1(G)'$ of $L^1(G)$ may be identified with $L^\infty(G)$ and the corresponding norms coincide. In particular, $L^1(G)'$ contains $C_o(G)$.*
- *If $p = \infty_o$, any bounded linear functional on $C_o(G)$ can be realised by integration against a regular complex measure. Consequently, the dual $C_o(G)'$ of $C_o(G)$ may be identified with the Banach space $M(G)$ of regular complex measures endowed with the total mass $\|\cdot\|_{M(G)}$ as its norm, and the corresponding norms coincide. With this identification, $C_o(G)'$ contains $L^1(G)$ and the corresponding norms coincide.*

*Proof.* See, e.g., Rudin [Rud87, ch.6]. □

We can now describe the properties of $\mathcal{R}_p$.

**Theorem 4.3.3.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. In this statement, $p \in [1,\infty) \cup \{\infty_o\}$.*

*(i) The semi-group $\{f \mapsto f * h_t\}_{t>0}$ is strongly continuous and equicontinuous on $L^p(G)$ if $p \in [1,\infty)$ or on $C_o(G)$ if $p = \infty_o$:*

$$\forall t > 0,\ \forall f \in L^p(G) \text{ or } C_o(G) \qquad \|f * h_t\|_p \le \|h_1\|_1 \|f\|_p.$$

*Consequently, the operator $\mathcal{R}_p$ is closed. The domain of $\mathcal{R}_p$ contains $\mathcal{D}(G)$, and for $f \in \mathcal{D}(G)$ we have $\mathcal{R}_p f = \mathcal{R} f$.*

*(ii) The operator $\bar{\mathcal{R}}_p$ is the infinitesimal generator of the strongly continuous semi-group $\{f \mapsto f * \bar{h}_t\}_{t>0}$ on $L^p(G)$.*

*(iii) We use the identifications of Lemma 4.3.2. If $p \in (1,\infty)$ then the dual of $\mathcal{R}_p$ is $\bar{\mathcal{R}}_{p'}$. The dual of $\mathcal{R}_{\infty_o}$ restricted to $L^1(G)$ is $\bar{\mathcal{R}}_1$. The dual of $\mathcal{R}_1$ restricted to $C_o(G) \subset L^\infty(G)$ is $\bar{\mathcal{R}}_{\infty_o}$.*

*(iv) If $p \in [1,\infty)$, the operator $\mathcal{R}_p$ is the maximal restriction of $\mathcal{R}$ to $L^p(G)$, that is, the domain of $\mathcal{R}_p$ consists of all the functions $f \in L^p(G)$ such that the distributional derivative $\mathcal{R} f$ is in $L^p(G)$ and $\mathcal{R}_p f = \mathcal{R} f$.*

*The operator $\mathcal{R}_{\infty_o}$ is the maximal restriction of $\mathcal{R}$ to $C_o(G)$, that is, the domain of $\mathcal{R}_{\infty_o}$ consists of all the functions $f \in C_o(G)$ such that the distributional derivative $\mathcal{R} f$ is in $C_o(G)$ and $\mathcal{R}_p f = \mathcal{R} f$.*

(v) *If $p \in [1,\infty)$, the operator $\mathcal{R}_p$ is the smallest closed extension of $\mathcal{R}|_{\mathcal{D}(G)}$ on $L^p(G)$. For $p=2$, $\mathcal{R}_2$ is the self-adjoint extension of $\mathcal{R}$ on $L^2(G)$.*

*Proof.* Part (i) is a consequence of Corollary 4.2.9, see also Section A.2.

Part (i) implies, intertwining with the complex conjugate, that $\{f \mapsto f * \bar{h}_t\}_{t>0}$ is also a strongly continuous semi-group on $L^p(G)$. On $\mathcal{D}(G)$, its infinitesimal operator coincide with $\bar{\mathcal{R}} = \mathcal{R}^t$ which is a positive Rockland operator (see Lemma 4.2.5) and it is easy to see that

$$\forall \phi \in \mathcal{D}(G),\ t>0 \qquad e^{-t\bar{\mathcal{R}}_2}\phi = \overline{e^{-t\mathcal{R}_2}\bar{\phi}} = \overline{\bar{\phi} * h_t} = \phi * \bar{h}_t.$$

This shows Part (ii).

For Part (iii), we observe that using (1.14) and (4.13), we have

$$\forall f_1, f_2 \in \mathcal{D}(G) \qquad \langle f_1 * h_t, f_2\rangle = \langle f_1, f_2 * \bar{h}_t\rangle. \tag{4.29}$$

Thus we have for any $f,g \in \mathcal{D}(G)$ and $p \in [1,\infty) \cup \{\infty_o\}$

$$\langle \frac{1}{t}(e^{-t\mathcal{R}_p} f - f), g\rangle = \frac{1}{t}\langle f * h_t - f, g\rangle = \frac{1}{t}\langle f, g * \bar{h}_t - g\rangle = \frac{1}{t}\langle f, e^{-t\bar{\mathcal{R}}_{p'}} g - g\rangle.$$

Here the brackets refer to the duality in the sense of distributions or, equivalently, to the duality explained in Lemma 4.3.2. Taking the limit as $t \to 0$ of the first and last expressions proves Part (iii).

We now prove Part (iv) for any $p \in [1,\infty) \cup \{\infty_o\}$. Let $f \in \text{Dom}(\mathcal{R}_p)$ and $\phi \in \mathcal{D}(G)$. Since $\mathcal{R}$ is formally self-adjoint, we know that $\mathcal{R}^t = \bar{\mathcal{R}}$, and by Part (i), we have $\mathcal{R}_q\phi = \mathcal{R}\phi$ for any $q \in [1,\infty) \cup \{\infty_o\}$. Thus by Part (iii) we have

$$\langle \mathcal{R}_p f, \phi\rangle = \langle f, \bar{\mathcal{R}}_{p'}\phi\rangle = \langle f, \mathcal{R}^t\phi\rangle = \langle \mathcal{R} f, \phi\rangle,$$

and $\mathcal{R}_p f = \mathcal{R} f$ in the sense of distributions. Thus

$$\text{Dom}(\mathcal{R}_p) \subset \{f \in L^p(G) \ : \ \mathcal{R} f \in L^p(G)\}.$$

We now prove the reverse inclusion. Let $f \in L^p(G)$ such that $\mathcal{R} f \in L^p(G)$. Let also $\phi \in \mathcal{D}(G)$. The following computations are justified by the properties of $\mathcal{R}$ and $h_t$ (see Theorem 4.2.7), Fubini's Theorem, and (4.29):

$$\begin{aligned}
\langle f * h_t - f, \phi\rangle &= \langle f, \phi * \bar{h}_t - \phi\rangle = \langle f, \int_0^t \partial_s(\phi * \bar{h}_s) ds\rangle \\
&= \langle f, \int_0^t -\bar{\mathcal{R}}(\phi * \bar{h}_s) ds\rangle = -\langle f, \bar{\mathcal{R}} \int_0^t (\phi * \bar{h}_s) ds\rangle \\
&= -\langle \mathcal{R} f, \int_0^t \phi * \bar{h}_s ds\rangle = -\int_0^t \langle \mathcal{R} f, \phi * \bar{h}_s\rangle ds \\
&= -\int_0^t \langle (\mathcal{R} f) * h_s, \phi\rangle ds = -\langle \int_0^t (\mathcal{R} f) * h_s ds, \phi\rangle.
\end{aligned}$$

Therefore,

$$f * h_t - f = -\int_0^t (\mathcal{R}f) * h_s ds.$$

Let us recall the following general property: if $t \mapsto x_t$ is a continuous mapping from $[0, \infty)$ to a Banach space $\mathcal{X}$, then $\frac{1}{t}\int_0^t x_s ds$ converges to $x_0$ in the strong topology of $\mathcal{X}$ as $t \to 0$. We apply this property to $\mathcal{X} = L^p(G)$ and $t \mapsto (\mathcal{R}f) * h_t$; the hypotheses are indeed satisfied because of the properties of the heat kernel, see Theorem 4.2.7. Hence we have the following convergence in $L^p(G)$:

$$\frac{1}{t}(f * h_t - f) = -\frac{1}{t}\int_0^t (\mathcal{R}f) * h_s ds \underset{t\to 0}{\longrightarrow} -\mathcal{R}f.$$

This shows $f \in \text{Dom}(\mathcal{R}_p)$ and concludes the proof of (iv).

Part (v) follows from (iv). This also shows that the self-adjoint extension of $\mathcal{R}$ coincides with $\mathcal{R}_2$ as defined in Definition 4.3.1 and concludes the proof of Theorem 4.3.3. □

Theorem 4.3.3 has the following couple of corollaries which will enable us to define the fractional powers of $\mathcal{R}_p$.

**Corollary 4.3.4.** *We keep the same setting and notation as in Theorem 4.3.3.*

*(i) The operator $\mathcal{R}_p$ is injective on $L^p(G)$ for $p \in [1,\infty)$ and $\mathcal{R}_{\infty_o}$ is injective on $C_o(G)$, namely,*

$$\text{for } p \in [1,\infty) \cup \{\infty_o\} \ : \qquad \forall f \in \text{Dom}(\mathcal{R}_p) \qquad \mathcal{R}_p f = 0 \Longrightarrow f = 0.$$

*(ii) If $p \in (1,\infty)$ then the operator $\mathcal{R}_p$ has dense range in $L^p(G)$. The operator $\mathcal{R}_{\infty_o}$ has dense range in $C_o(G)$. The closure of the range of $\mathcal{R}_1$ is the closed subspace $\{\phi \in L^1(G) : \int_G \phi = 0\}$ of $L^1(G)$.*

*Proof.* Let $f \in \text{Dom}(\mathcal{R}_p)$ be such that $\mathcal{R}_p f = 0$ for $p \in [1,\infty) \cup \{\infty_o\}$. By Theorem 4.3.3 (iv), $f \in \mathcal{S}'(G)$ and $\mathcal{R}f = 0$. In Remark 4.1.13 (3), we noticed that any positive Rockland operator satisfies the hypotheses of Liouville's Theorem for homogeneous Lie groups, that is, Theorem 3.2.45. Consequently $f$ is a polynomial. Since $f$ is also in $L^p(G)$ for $p \in [1,\infty)$ or in $C_o(G)$ for $p = \infty_o$, $f$ must be identically zero. This proves (i).

For (ii), let $\Psi$ be a bounded linear functional on $L^p(G)$ if $p \in [1,\infty)$ or on $C_o(G)$ if $p = \infty_o$ such that $\Psi$ vanishes identically on Range($\mathcal{R}_p$). Then $\Psi$ can be realised as the integration against a function $f \in L^{p'}(G)$ if $p \in [1,\infty)$ or a measure also denoted by $f \in M(G)$ if $p = \infty_o$, see Lemma 4.3.2. Using the distributional notation, we have

$$\Psi(\phi) = \langle f, \phi \rangle \qquad \forall \phi \in L^p(G) \quad \text{or} \quad \forall \phi \in C_o(G).$$

Then for any $\phi \in \mathcal{D}(G)$, we know that $\phi \in \text{Dom}(\mathcal{R}_p)$ and $\mathcal{R}_p\phi = \mathcal{R}\phi$ by Theorem 4.3.3 (i) thus

$$0 = \Psi(\mathcal{R}_p(\phi)) = \langle f, \mathcal{R}(\phi)\rangle = \langle \bar{\mathcal{R}} f, \phi\rangle,$$

since $\mathcal{R}^t = \bar{\mathcal{R}}$. This shows that $\bar{\mathcal{R}} f = 0$. Applying again Liouville's Theorem, this time to the positive Rockland operator $\bar{\mathcal{R}}$ (see Lemma 4.2.5), this shows that $f$ is a polynomial. For $p \in (1,\infty)$, $f$ being also a function in $L^{p'}(G)$, this implies that $f \equiv 0$. For $p = \infty_o$, $f \in M(G)$, this shows that $f$ is an integrable polynomial on $G$ hence $f \equiv 0$. For $p = 1$, $f$ being a measurable bounded function and a polynomial, $f$ must be constant, i.e. $f \equiv c$ for some $c \in \mathbb{C}$. This shows that if $p \in (1,\infty)\cup\{\infty_o\}$ then $\Psi = 0$ and $\text{Range}(\mathcal{R}_p)$ is dense in $L^p(G)$ or $C_o(G)$, whereas if $p = 1$ then $\Psi : L^1(G) \ni \phi \mapsto c\int_G \phi$. This shows (ii) for $p \in (1,\infty) \cup \{\infty_o\}$.

Let us study more precisely the case $p = 1$. It is easy to see that

$$\int_G X\phi(x)dx = -\int_G \phi(x)\ (X1)(x)dx = 0$$

holds for any $\phi \in L^1(G)$ such that $X\phi \in L^1(G)$. Consequently, for any $\phi \in \text{Dom}(\mathcal{R}_1)$, we know that $\phi$ and $\mathcal{R}\phi$ are in $L^1(G)$ thus $\int_G \mathcal{R}_1\phi = 0$. So the range of $\mathcal{R}_1$ is included in

$$S := \left\{\phi \in L^1(G)\ :\ \int_G \phi = 0\right\} \supset \text{Range}(\mathcal{R}_1).$$

Moreover, if $\Psi_1$ a bounded linear functional on $S$ such that $\Psi_1$ is identically 0 on $\text{Range}(\mathcal{R}_1)$, by the Hahn-Banach Theorem (see, e.g. [Rud87, Theorem 5.16]), it can be extended into a bounded linear function $\Psi$ on $L^1(G)$. As $\Psi$ vanishes identically on $\text{Range}(\mathcal{R}_1) \subset S$, we have already proven that $\Psi$ must be of the form

$$\Psi : L^1(G) \ni \phi \mapsto c\int_G \phi$$

for some constant $c \in \mathbb{C}$ and its restriction to $S$ is $\Psi_1 \equiv 0$. This concludes the proof of Part (ii). □

Eventually, let us prove that the operator $\mathcal{R}_p$ is Komatsu-non-negative, see hypothesis (iii) in Section A.3:

**Corollary 4.3.5.** *For $p \in [1,\infty) \cup \{\infty_o\}$, and any $\mu > 0$, the operator $\mu\mathrm{I} + \mathcal{R}_p$ is invertible on $L^p(G)$, $p \in [1,\infty)$, and $C_o(G)$ for $p = \infty_o$, and the operator norm of $(\mu\mathrm{I} + \mathcal{R}_p)^{-1}$ is*

$$\|(\mu\mathrm{I} + \mathcal{R}_p)^{-1}\| \leq \|h_1\|\mu^{-1}.$$

*Proof.* Integrating the formula

$$(\mu + \lambda)^{-1} = \int_0^\infty e^{-t(\mu+\lambda)}dt,$$

against the spectral measure $dE(\lambda)$ of $\mathcal{R}_2$, we have formally

$$(\mu \mathrm{I} + \mathcal{R}_2)^{-1} = \int_0^\infty e^{-t(\mu\mathrm{I}+\mathcal{R}_2)} dt, \tag{4.30}$$

and the convolution kernel of the operator on the right-hand side is (still formally) given by

$$\kappa_\mu(x) := \int_0^\infty e^{-t\mu} h_t(x) dt.$$

From the properties of the heat kernel $h_t$ (see Theorem 4.2.7 and Corollary 4.2.10), we see that the function $\kappa_\mu$ defined just above is continuous on $G$ and that

$$\|\kappa_\mu\|_1 \le \int_0^\infty e^{-t\mu} \|h_t\|_1 dt = \|h_1\| \int_0^\infty e^{-t\mu} dt = \frac{\|h_1\|}{\mu} < \infty.$$

As $\kappa_\mu \in L^1(G)$, it is a routine exercise to show that the operator

$$\int_0^\infty e^{-t(\mu\mathrm{I}+\mathcal{R}_2)} dt$$

is bounded on $L^2(G)$ with convolution kernel $\kappa_\mu$ (it suffices to consider integration over $[0, N]$ with $N \to \infty$). Moreover, Formula (4.30) holds in $\mathscr{L}(L^2(G))$.

For any $\phi \in \mathcal{D}(G)$ and $p \in [1, \infty) \cup \{\infty_o\}$, Theorem 4.3.3 (iv) implies

$$(\mu\mathrm{I} + \mathcal{R}_p)\phi = (\mu\mathrm{I} + \mathcal{R})\phi = (\mu\mathrm{I} + \mathcal{R}_2)\phi \in \mathcal{D}(G),$$

thus

$$((\mu\mathrm{I} + \mathcal{R}_p)\phi) * \kappa_\mu = ((\mu\mathrm{I} + \mathcal{R}_2)\phi) * \kappa_\mu = \phi.$$

This yields that the operator $(\mu\mathrm{I} + \mathcal{R}_p)^{-1} : \phi \mapsto \phi * \kappa_\mu$ is bounded on $L^p(G)$ if $p \in [1, \infty)$ and on $C_o(G)$ if $p = \infty_o$. Furthermore, its operator norm is

$$\|(\mu\mathrm{I} + \mathcal{R}_p)^{-1}\| \le \|\kappa_\mu\|_1 \le \|h_1\|\mu^{-1},$$

completing the proof. □

### 4.3.2 Fractional powers of operators $\mathcal{R}_p$

We now apply the general theory of fractional powers outlined in Section A.3 to the operators $\mathcal{R}_p$ and $\mathrm{I} + \mathcal{R}_p$.

**Theorem 4.3.6.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. We consider the operators $\mathcal{R}_p$ defined in Definition 4.3.1. Let $p \in [1, \infty) \cup \{\infty_o\}$.*

1. *Let $\mathcal{A}$ denote either $\mathcal{R}$ or $\mathrm{I} + \mathcal{R}$.*

(a) *For every* $a \in \mathbb{C}$, *the operator* $\mathcal{A}_p^a$ *is closed and injective with* $(\mathcal{A}_p^a)^{-1} = \mathcal{A}_p^{-a}$. *We have* $\mathcal{A}_p^0 = \mathrm{I}$, *and for any* $N \in \mathbb{N}$, $\mathcal{A}_p^N$ *coincides with the usual powers of differential operators on* $\mathcal{S}(G)$ *and* $\mathrm{Dom}(\mathcal{A}^N) \cap \mathrm{Range}(\mathcal{A}^N)$ *is dense in* $\mathrm{Range}(\mathcal{A}_p)$.

(b) *For any* $a, b \in \mathbb{C}$, *in the sense of operator graph, we have* $\mathcal{A}_p^a \mathcal{A}_p^b \subset \mathcal{A}_p^{a+b}$. *If* $\mathrm{Range}(\mathcal{A}_p)$ *is dense then the closure of* $\mathcal{A}_p^a \mathcal{A}_p^b$ *is* $\mathcal{A}_p^{a+b}$.

(c) *Let* $a_o \in \mathbb{C}_+$.
  - *If* $\phi \in \mathrm{Range}(\mathcal{A}_p^{a_o})$ *then* $\phi \in \mathrm{Dom}(\mathcal{A}_p^a)$ *for all* $a \in \mathbb{C}$ *with* $0 < -\mathrm{Re}\, a < \mathrm{Re}\, a_o$ *and the function* $a \mapsto \mathcal{A}_p^a \phi$ *is holomorphic in* $\{a \in \mathbb{C} :\ -\mathrm{Re}\, a_o < \mathrm{Re}\, a < 0\}$.
  - *If* $\phi \in \mathrm{Dom}(\mathcal{A}_p^{a_o})$ *then* $\phi \in \mathrm{Dom}(\mathcal{A}_p^a)$ *for all* $a \in \mathbb{C}$ *with* $0 < \mathrm{Re}\, a < \mathrm{Re}\, a_o$ *and the function* $a \mapsto \mathcal{A}_p^a \phi$ *is holomorphic in* $\{a \in \mathbb{C} :\ 0 < \mathrm{Re}\, a < \mathrm{Re}\, a_o\}$.

(d) *For every* $a \in \mathbb{C}$, *the operator* $\mathcal{A}_p^a$ *is invariant under left translations.*

(e) *If* $p \in (1, \infty)$ *then the dual of* $\mathcal{A}_p$ *is* $\bar{\mathcal{A}}_{p'}$. *The dual of* $\mathcal{A}_{\infty_o}$ *restricted to* $L^1(G)$ *is* $\bar{\mathcal{A}}_1$. *The dual of* $\mathcal{A}_1$ *restricted to* $C_o(G) \subset L^\infty(G)$ *is* $\bar{\mathcal{A}}_{\infty_o}$.

(f) *If* $a, b \in \mathbb{C}_+$ *with* $\mathrm{Re}\, b > \mathrm{Re}\, a$, *then*

$$\exists C = C_{a,b} > 0 \quad \forall \phi \in \mathrm{Dom}(\mathcal{A}_p^b) \quad \|\mathcal{A}_p^a \phi\| \leq C \|\phi\|^{1 - \frac{\mathrm{Re}\, a}{\mathrm{Re}\, b}} \|\mathcal{A}_p^b \phi\|^{\frac{\mathrm{Re}\, a}{\mathrm{Re}\, b}}.$$

(g) *For any* $a \in \mathbb{C}_+$, $\mathrm{Dom}(\mathcal{A}_p^a)$ *contains* $\mathcal{S}(G)$.

(h) *If* $f \in \mathrm{Dom}(\mathcal{A}_p^a) \cap L^q(G)$ *for some* $q \in [1, \infty) \cup \{\infty_o\}$, *then* $f \in \mathrm{Dom}(\mathcal{A}_q^a)$ *if and only if* $\mathcal{A}_p^a f \in L^q(G)$, *in which case* $\mathcal{A}_p^a f = \mathcal{A}_q^a f$.

2. *For each* $a \in \mathbb{C}_+$, *the operators* $(\mathrm{I} + \mathcal{R}_p)^a$ *and* $\mathcal{R}_p^a$ *are unbounded and their domains satisfy for all* $\epsilon > 0$,

$$\mathrm{Dom}\left[(\mathrm{I} + \mathcal{R}_p)^a\right] = \mathrm{Dom}(\mathcal{R}_p^a) = \mathrm{Dom}\left[(\mathcal{R}_p + \epsilon \mathrm{I})^a\right].$$

3. *If* $0 < \mathrm{Re}\, a < 1$ *and* $\phi \in \mathrm{Range}(\mathcal{R}_p)$ *then*

$$\mathcal{R}_p^{-a} \phi = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{-t\mathcal{R}_p} \phi \, dt,$$

*in the sense that* $\lim_{N \to \infty} \int_0^N$ *converges in the norm of* $L^p(G)$ *or* $C_o(G)$.

4. *If* $a \in \mathbb{C}_+$, *then the operator* $(\mathrm{I} + \mathcal{R}_p)^{-a}$ *is bounded and for any* $\phi \in \mathcal{X}$ *with* $\mathcal{X} = L^p(G)$ *or* $C_o(G)$, *we have*

$$(\mathrm{I} + \mathcal{R}_p)^{-a} \phi = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{-t(\mathrm{I} + \mathcal{R}_p)} \phi \, dt,$$

*in the sense of absolute convergence:*

$$\int_0^\infty t^{a-1} \|e^{-t(\mathrm{I} + \mathcal{R}_p)} \phi\|_{\mathcal{X}} dt < \infty.$$

5. *For any* $a, b \in \mathbb{C}$, *the two (possibly unbounded) operators* $\mathcal{R}_p^a$ *and* $(\mathrm{I} + \mathcal{R}_p)^b$ *commute.*

6. *For any* $a \in \mathbb{C}$, *the operator* $\mathcal{R}_p^a$ *is homogeneous of degree* $\nu a$.

Recall (see Definition A.3.2) that the two (possibly unbounded) operators $A$ and $B$ commute when

$$x \in \mathrm{Dom}(AB) \cap \mathrm{Dom}(BA) \Longrightarrow ABx = BAx,$$

and that the domain of the product $AB$ of two (possibly unbounded) operators $A$ and $B$ on the same Banach space $\mathcal{X}$ is formed by the elements $x \in \mathcal{X}$ such that $x \in \mathrm{Dom}(B)$ and $Bx \in \mathrm{Dom}(A)$.

*Proof.* The operator $\mathcal{R}_p$ is closed and densely defined by Theorem 4.3.3 (i), it is injective by Corollary 4.3.4 and Komatsu-non-negative in the sense of Section A.3 (iii) by Corollary 4.3.5. Therefore, $\mathcal{R}_p$ satisfies the hypotheses of Theorem A.3.4. Moreover, $\mathrm{I} + \mathcal{R}_p$ also satisfies these hypotheses by Remark A.3.3, and $-(\mathrm{I} + \mathcal{R}_p)$ generates an exponentially stable semi-group:

$$\|e^{-t(\mathrm{I}+\mathcal{R}_p)}\| \leq e^{-t}\|e^{-t\mathcal{R}_p}\| \leq \|h_1\|_1 e^{-t}.$$

Most of the statements then follow from the general properties of fractional powers constructed via the Balakrishnan formulae recalled in Section A.3. More precisely, from the Balakrishnan formula, for any $N \in \mathbb{N}$, $\mathcal{A}_p^N$ coincides with the usual powers of differential operators on $\mathcal{S}(G)$ and Part (1a) follows from Theorem A.3.4 (1) and (2) and Remark A.3.1.

The duality properties explained in Part (1e) for $p \in (1, \infty)$ hold for the Balakrishnan operators hence they hold for their maximal closure. The cases of $p = 1, \infty_o$ are similar and this proves Part (1e). The properties in Parts (1d), (5) and (6) hold for the Balakrishnan operators hence they hold for their maximal closure and these parts are proved.

Part (1b) follows from Theorem A.3.4 (4).

Part (1c) follows from Theorem A.3.4 (5).

Part (1f) follows from Theorem A.3.4 (6).

Part (1g) follows from Parts (1a) and (1c).

Part (1h) is certainly true for any $f \in \mathcal{S}(G)$ and $\mathrm{Re}\, a > 0$ via the Balakrishnan formulae. By analyticity (see Part (1c)) it is true for any $a \in \mathbb{C}$. The density of $\mathcal{D}(G)$ in $L^p(G)$ (or $C_o(G)$ if $p = \infty_o$) together with the maximality of $\mathcal{A}_p^a$ and the uniqueness of distributional convergence imply the result.

Part (2) follow from Theorem A.3.4 (8).

Parts (3) and (4) follows from Theorem A.3.4 (10).

This concludes the proof of Theorem 4.3.6. □

### 4.3.3 Imaginary powers of $\mathcal{R}_p$ and $\mathrm{I} + \mathcal{R}_p$

In this section, we show that imaginary powers of a positive Rockland operator $\mathcal{R}$ as well as $\mathrm{I} + \mathcal{R}$ are bounded operators on $L^p(G)$, $p \in (1, \infty)$. We prove this as a consequence of the theorem of singular integrals on homogeneous groups, see Section 3.2.3.

We start by showing that if $\mathcal{R}$ is a positive Rockland operator, then the imaginary powers of $\mathrm{I} + \mathcal{R}_p$ are bounded on $L^p(G)$:

**Proposition 4.3.7.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. For any $\tau \in \mathbb{R}$ and $p \in (1, \infty)$, the operator $(\mathrm{I} + \mathcal{R}_p)^{i\tau}$ is bounded on $L^p(G)$. For any $p \in (1, \infty)$, there exists $C = C_{p,\mathcal{R}} > 0$ and $\theta > 0$ such that*

$$\forall \tau \in \mathbb{R} \qquad \|(\mathrm{I} + \mathcal{R}_p)^{i\tau}\|_{\mathscr{L}(L^p(G))} \leq C e^{\theta|\tau|}.$$

*For any $p \in (1, \infty)$ and $a \in \mathbb{C}$, $\mathrm{Dom}((\mathrm{I} + \mathcal{R}_p)^a) = \mathrm{Dom}((\mathrm{I} + \mathcal{R}_p)^{\mathrm{Re}\, a})$.*

The following technical result will be useful in the proof of Proposition 4.3.7 and in other proofs (see Sections 4.3.4 and 4.4.4).

**Lemma 4.3.8.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. Let $h_t$ be its heat kernel as in Section 4.2.2.*

1. *For any homogeneous quasi-norm $|\cdot|$, any multi-index $\alpha \in \mathbb{N}_0^n$, and any real number $a$ with $0 < a < \frac{Q+[\alpha]}{\nu}$, there exists a constant $C > 0$ such that*

$$\int_0^\infty t^{a-1} |X^\alpha h_t(x)| dt \leq C |x|^{-Q-[\alpha]+\nu a}.$$

   *For any homogeneous quasi-norm $|\cdot|$, any multi-index $\alpha \in \mathbb{N}_0^n$, there exists a constant $C > 0$ such that*

$$\int_0^\infty |X^\alpha h_t(x)| e^{-t} dt \leq C |x|^{-Q-[\alpha]}.$$

2. *For any homogeneous quasi-norm $|\cdot|$, any multi-index $\alpha \in \mathbb{N}_0^n$, and any $t > 0$, we have*

$$\int_{|x| \geq 1/2} |X^\alpha h_t(x)| dx \leq t^{-\frac{[\alpha]}{\nu}} \|X^\alpha h_1\|_{L^1}.$$

3. *For any homogeneous quasi-norm $|\cdot|$, any multi-index $\alpha \in \mathbb{N}_0^n$, any $N \in \mathbb{N}$ and any $t \in (0,1)$, there exists a constant $C > 0$ such that*

$$\int_{|x| \geq 1/2} |X^\alpha h_t(x)| dx \leq C t^N.$$

*Proof of Lemma 4.3.8* . Let us prove Part 1. We write

$$\int_0^\infty t^{a-1}|X^\alpha h_t(x)|dt = \int_0^{|x|^\nu} + \int_{|x|^\nu}^\infty .$$

For the second integral, we use the property of homogeneity of $h_t$ (see (4.12) or (4.17))

$$\begin{aligned} \int_{|x|^\nu}^\infty &= \int_{|x|^\nu}^\infty t^{a-1-\frac{Q+[\alpha]}{\nu}}|X^\alpha h_1(t^{-\frac{1}{\nu}}x)|dt \\ &\leq (\frac{Q+[\alpha]}{\nu}-a)^{-1}\|X^\alpha h_1\|_\infty |x|^{\nu(a-\frac{Q+[\alpha]}{\nu})}. \end{aligned}$$

As $h_1 \in \mathcal{S}(G)$, $\|X^\alpha h_1\|_\infty$ is finite. For the first integral, we use again (4.12) to obtain

$$\begin{aligned} \int_0^{|x|^\nu} &= \int_0^{|x|^\nu} t^{a-1}|x|^{-(Q+[\alpha])}\left|X^\alpha h_{|x|^{-\nu}t}\left(\frac{x}{|x|}\right)\right| dt \\ &\leq C_1 a^{-1}|x|^{\nu(a-\frac{Q+[\alpha]}{\nu})}. \end{aligned}$$

where $C_1 := \sup_{|y|=1, 0\leq t_1\leq 1}|X^\alpha h_{t_1}(y)|$ is finite by (4.15). Combining the two estimates above shows the estimates for the first integral in Part 1. We proceed in the same way for the second one:

$$\int_0^\infty |X^\alpha h_t(x)|e^{-t}dt = \int_0^{|x|^\nu} + \int_{|x|^\nu}^\infty .$$

We have (with $C_1$ as above)

$$\begin{aligned} \int_0^{|x|^\nu} &\leq C_1|x|^{\nu(a-\frac{Q+[\alpha]}{\nu})}\int_0^{|x|^\nu} e^{-t}dt = C_1|x|^{\nu(a-\frac{Q+[\alpha]}{\nu})}(1-e^{-|x|^\nu}) \\ &\leq C_1|x|^{\nu(a-\frac{Q+[\alpha]}{\nu})}, \end{aligned}$$

whereas

$$\begin{aligned} \int_{|x|^\nu}^\infty &\leq \|X^\alpha h_1\|_\infty (|x|^\nu)^{-\frac{Q+[\alpha]}{\nu}}\int_{|x|^\nu}^\infty e^{-t}dt = \|X^\alpha h_1\|_\infty |x|^{-(Q+[\alpha])}e^{-|x|^\nu} \\ &\leq \|X^\alpha h_1\|_\infty |x|^{-(Q+[\alpha])}. \end{aligned}$$

We conclude in the same way as above and Part 1 is proved.

Let us prove Part 2. The property of homogeneity of $h_t$ (see (4.17)) together with $h_1 \in \mathcal{S}(G)$ imply

$$\begin{aligned} \int_{|x|\geq 1/2}|X^\alpha h_t(x)|dx &= \int_{|x|\geq 1/2}|X^\alpha h_1(t^{-\frac{1}{\nu}}x)|t^{-\frac{[\alpha]+Q}{\nu}}dx \\ &= t^{-\frac{[\alpha]}{\nu}}\int_{t^{\frac{1}{\nu}}|x'|\geq 1/2}|X^\alpha h_1(x')|dx' \leq t^{-\frac{[\alpha]}{\nu}}\int_G |X^\alpha h_1|, \end{aligned}$$

having used the change of variable $x' = t^{-\frac{1}{\nu}}x$. This shows Part 2.

Let us prove Part 3. The properties of the heat kernel, especially (4.12) and (4.15), imply

$$|X^\alpha h_t(x)| = |x|^{-[\alpha]-Q}|X^\alpha h_{|x|^{-\nu}t}(|x|^{-1}x)| \le C|x|^{-[\alpha]-Q}(|x|^{-\nu}t)^N,$$

if $|x| \ge 1/2$ and $t \in (0,1)$ where $C = \sup_{|x'|=1, 0<t'<1} t'^{-N}|X^\alpha h_{t'}(x')|$ is finite. Hence

$$\int_{|x|\ge 1/2} |X^\alpha h_t(x)|dx \le Ct^N \int_{|x|\ge 1/2} |x|^{-[\alpha]-Q-\nu N}dx.$$

This shows Part 3 and concludes the proof of Lemma 4.3.8. □

*Proof of Proposition 4.3.7.* By Theorem 4.3.6 (1), to show that $(\mathrm{I}+\mathcal{R}_p)^{i\tau}$ is bounded on $L^p(G)$ for some $p \in (1,\infty)$ and $\tau \in \mathbb{R}$, it suffices to show that $(\mathrm{I}+\mathcal{R}_2)^{i\tau}$ can be extended to an $L^p$-bounded operator. To do this, we will show that Corollary 3.2.21 can be applied to $(\mathrm{I}+\mathcal{R}_2)^{i\tau}$.

By functional calculus, $(\mathrm{I}+\mathcal{R}_2)^{i\tau}$ is bounded on $L^2(G)$. Part 1 of Lemma 4.3.8 together with the formula

$$\forall\lambda > 0 \qquad \lambda^{i\tau} = \frac{\lambda}{\Gamma(1-i\tau)} \int_0^\infty t^{-i\tau} e^{-\lambda t} dt,$$

and the functional calculus of $\mathcal{R}_2$ imply that the right convolution kernel of $(\mathrm{I}+\mathcal{R}_2)^{i\tau}$ is the tempered distribution $\kappa$ which coincides with the smooth function away from 0 given via

$$\kappa(x) = \frac{1}{\Gamma(1-i\tau)} \int_0^\infty t^{-i\tau}(\mathrm{I}+\mathcal{R})h_t(x)e^{-t}dt, \qquad x \ne 0. \tag{4.31}$$

Using this formula, we have

$$\int_{|x|\ge 1/2} |\kappa(x)|dx \le |\Gamma(1-i\tau)|^{-1} \int_{t=0}^\infty \int_{|x|\ge 1/2} (|h_t(x)| + |\mathcal{R}h_t(x)|)e^{-t}dxdt.$$

By Part 2 of Lemma 4.3.8, (and $h_1$ being Schwartz), the integrals

$$\int_{t=0}^\infty \int_{|x|\ge 1/2} |h_t(x)|e^{-t}dxdt \quad \text{and} \quad \int_{t=1}^\infty \int_{|x|\ge 1/2} |\mathcal{R}h_t(x)|e^{-t}dxdt,$$

are finite. By Part 3 of Lemma 4.3.8, the integral

$$\int_{t=0}^1 \int_{|x|\ge 1/2} |\mathcal{R}h_t(x)|e^{-t}dxdt \le C\int_{t=0}^1 t^0 dt = C,$$

is finite. This shows that $\int_{|x|\ge 1/2} |\kappa(x)|dx$ is finite.

Using (4.31), we also obtain easily that

$$\sup_{0<|x|<1} |x|^{Q+[\alpha]} |X^\alpha \kappa(x)| \leq |\Gamma(1-i\tau)|^{-1} \sup_{0<|x|<1} |x|^{Q+[\alpha]} \int_0^\infty |X^\alpha h_t(x)| + |X^\alpha \mathcal{R} h_t(x)| dt,$$

and the right-hand side is finite by Lemma 4.3.8. Note that if we denote by $\kappa = \kappa_{\tau,\mathcal{R}}$ the kernel of $(\mathrm{I}+\mathcal{R}_2)^{i\tau}$, then we have

$$\kappa_{\tau,\mathcal{R}}(x^{-1}) = \overline{\kappa_{-\tau,\bar{\mathcal{R}}}(x)},$$

using the formula in (4.31) and

$$\begin{aligned} \left((\mathrm{I}+\mathcal{R})h_t\right)(x^{-1}) &= \left((\mathrm{I}-\partial_t)h_t\right)(x^{-1}) = \left((\mathrm{I}-\partial_t)\bar{h}_t\right)(x) \\ &= \overline{\left((\mathrm{I}-\partial_t)h_t\right)(x)} = \overline{\left((\mathrm{I}+\mathcal{R})h_t\right)(x)}, \end{aligned}$$

where we have used (4.13). Hence we also have that each quantity

$$\sup_{0<|x|<1} |x|^{Q+[\alpha]} |\tilde{X}^\alpha \kappa(x)| = \sup_{0<|x|<1} |x|^{Q+[\alpha]} |X^\alpha \kappa_{-\tau,\bar{\mathcal{R}}}(x)|$$

is finite.

The estimates above show that $\kappa$ satisfies the hypotheses of Corollary 3.2.21 and therefore the operator $(\mathrm{I}+\mathcal{R}_2)^{i\tau}$ is bounded on $L^p(G)$, $p \in (1,\infty)$. The properties of the semi-group (see Theorem A.3.4 (3)) imply the rest of the statement in Proposition 4.3.7. □

Let us now prove the homogeneous case, that is, that the imaginary powers of a positive Rockland operator are bounded on $L^p(G)$:

**Proposition 4.3.9.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. For any $\tau \in \mathbb{R}$ and $p \in (1,\infty)$, the operator $\mathcal{R}_p^{i\tau}$ is bounded on $L^p(G)$. For any $p \in (1,\infty)$, there exists $C = C_{p,\mathcal{R}} > 0$ and $\theta > 0$ such that*

$$\forall \tau \in \mathbb{R} \qquad \|\mathcal{R}_p^{i\tau}\|_{\mathscr{L}(L^p(G))} \leq C e^{\theta|\tau|}.$$

*For any $p \in (1,\infty)$ and $a \in \mathbb{C}$,* $\mathrm{Dom}(\mathcal{R}_p^a) = \mathrm{Dom}(\mathcal{R}_p^{\mathrm{Re}\, a})$.

*Proof of Proposition 4.3.9.* Let $p \in (1,\infty)$ and $\tau \in \mathbb{R}$. Let us denote by $\mathcal{R}_{p,i\tau}$ the (possibly unbounded) operator given as the strong limit in $L^p(G)$ of $(\epsilon+\mathcal{R}_p)^{i\tau}\phi$ as $\epsilon \to 0$, for $\phi \in \mathrm{Dom}((\epsilon+\mathcal{R}_p)^{i\tau})$ for any $\epsilon \in (0,\epsilon_0)$ for some small $\epsilon_0 > 0$ and such that this strong limit exists. The domain of $\mathcal{R}_{p,i\tau}$ is naturally the space of all those functions $\phi$. Note that the homogeneity of $\mathcal{R}$ implies

$$(\epsilon+\mathcal{R}_p)^{i\tau}\phi = \epsilon^{i\tau}(\mathrm{I}+\epsilon^{-1}\mathcal{R}_p)^{i\tau}\phi = \epsilon^{i\tau}(\mathrm{I}+\mathcal{R}_p)^{i\tau}\{\phi(\epsilon^{-1/\nu}\cdot)\}(\epsilon^{1/\nu}\cdot),$$

for any $\epsilon > 0$ and any $\phi \in L^p(G)$ such that

$$\phi(\epsilon^{-1/\nu}\cdot) \in \mathrm{Dom}((\mathrm{I}+\mathcal{R}_p)^{i\tau}).$$

By Proposition 4.3.7, $\mathrm{Dom}((\mathrm{I}+\mathcal{R}_p)^{i\tau}) = L^p(G)$ and the operator $(\mathrm{I}+\mathcal{R}_p)^{i\tau}$ is bounded. Therefore for all $\phi \in L^p(G)$ and $\epsilon > 0$, $\phi$ is in $\mathrm{Dom}((\epsilon+\mathcal{R}_p)^{i\tau})$ and we have

$$\begin{aligned}\|(\epsilon+\mathcal{R}_p)^{i\tau}\phi\|_{L^p(G)} &= \|(\mathrm{I}+\mathcal{R}_p)^{i\tau}\{\phi(\epsilon^{-1/\nu}\cdot)\}(\epsilon^{1/\nu}\cdot)\|_{L^p(G)}\\ &= \epsilon^{-\frac{Q}{p\nu}}\|(\mathrm{I}+\mathcal{R}_p)^{i\tau}\{\phi(\epsilon^{-1/\nu}\cdot)\}\|_{L^p(G)}\\ &\leq \epsilon^{-\frac{Q}{p\nu}}\|(\mathrm{I}+\mathcal{R}_p)^{i\tau}\|_{\mathscr{L}(L^p(G))}\|\phi(\epsilon^{-1/\nu}\cdot)\|_{L^p(G)}\\ &\quad = \|(\mathrm{I}+\mathcal{R}_p)^{i\tau}\|_{\mathscr{L}(L^p(G))}\|\phi\|_{L^p(G)}.\end{aligned}$$

Consequently, $\mathcal{R}_{p,i\tau}$ extends to a bounded operator on $L^p(G)$. By Theorem A.3.4 (9), this implies that $\mathcal{R}_p^{i\tau}$ is also a bounded operator on $L^p(G)$ as $\mathcal{R}_p$ has dense range and domain by Corollary 4.3.4. As in the inhomogeneous case, the properties of the semi-group (see Theorem A.3.4 (3)) imply the rest of the statement in Proposition 4.3.9. □

Given the proof of Proposition 4.3.7, one would be tempted to study the convolution kernel of the operator $\mathcal{R}_2^{i\tau}$ in order to show the $L^p$-boundedness in the proof of Proposition 4.3.9. Indeed, following the same arguments as in the proof of Proposition 4.3.7, one shows that the kernel of $\mathcal{R}_2^{i\tau}$ coincides away from the origin with the smooth function

$$G\backslash\{0\} \ni x \mapsto \frac{1}{\Gamma(1-i\tau)}\int_0^\infty t^{-i\tau}\mathcal{R}h_t(x)dt.$$

However, this function can not be in general a kernel of type $i\tau$: already for the usual Laplacian on $(\mathbb{R}^n,+)$ it is not the case. Indeed, in the Euclidean case, this function is radial and non-zero and its average on the sphere can therefore not vanish.

In the stratified case, Folland proved the $L^p$-boundedness of imaginary powers of the sub-Laplacian $-\mathcal{L}$ and $\mathrm{I}+(-\mathcal{L})$ using general properties of semigroups preserving positivity together with the Laplace transform see [Fol75, Proposition 3.14 and Lemma 3.13]. More precisely, the boundedness follows from the Littlewood-Paley theory and the study of square functions associated with the semi-group. Note that in the case of a sub-Laplacian, the proof in [Fol75] yields a bound of the operator norm by $|\Gamma(1-i\tau)|^{-1}$ up to a constant of $p$.

In our case, we applied a consequence of the theorem of Singular Integrals via Corollary 3.2.20 to obtain the $L^p$-boundedness of the imaginary powers of $\mathrm{I}+\mathcal{R}$ and we have shown

$$\|\mathcal{R}_p^{i\tau}\|_{\mathscr{L}(L^p(G))} \leq \|(\mathrm{I}+\mathcal{R}_p)^{i\tau}\|_{\mathscr{L}(L^p(G))}, \quad p \in (1,\infty),$$

in the proof of Proposition 4.3.9. We can follow the constants in the proof of the theorem of Singular Integrals (see Remark A.4.5 (2)) as well as in our application to show that $\|(\mathrm{I}+\mathcal{R}_p)^{i\tau}\|_{\mathscr{L}(L^p(G))}$ is bounded up to a constant of $p$, by

$$(1+|\Gamma(1-i\tau)|^{-1})^{2|\frac{1}{p}-\frac{1}{2}|}.$$

However, we do not need these precise bounds as the bounds obtained from the general theory of semigroups as stated in Propositions 4.3.7 and 4.3.9 will be sufficient for our purpose in the proofs of interpolation properties for Sobolev spaces in Theorem 4.4.9 and Proposition 4.4.15.

### 4.3.4 Riesz and Bessel potentials

We mimic the usual terminology in the Euclidean setting, to define the Riesz and Bessel potentials associated with a positive Rockland operator.

**Definition 4.3.10.** Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu$. We call the operators $\mathcal{R}^{-a/\nu}$ for $\{a \in \mathbb{C},\ 0 < \operatorname{Re} a < Q\}$ and $(\mathrm{I} + \mathcal{R})^{-a/\nu}$ for $a \in \mathbb{C}_+$, the *Riesz potential* and the *Bessel potential*, respectively.

In the sequel we will denote their kernels by $\mathcal{I}_a$ and $\mathcal{B}_a$, respectively, as defined in the following:

**Corollary 4.3.11.** *We keep the setting and notation of Theorem 4.3.3.*

(i) *Let $a \in \mathbb{C}$ with $0 < \operatorname{Re} a < Q$. The integral*

$$\mathcal{I}_a(x) := \frac{1}{\Gamma(\frac{a}{\nu})} \int_0^\infty t^{\frac{a}{\nu}-1} h_t(x) dt$$

*converges absolutely for every $x \neq 0$. This defines a distribution $\mathcal{I}_a$ which is smooth away from the origin and $(a - Q)$-homogeneous.*

*For any $p \in (1, \infty)$, if $\phi \in \mathcal{S}(G)$ or, more generally, if $\phi \in L^q(G) \cap L^p(G)$ where $q \in [1, \infty)$ is given by $\frac{1}{q} - \frac{1}{p} = \frac{\operatorname{Re} a}{Q}$, then*

$$\phi \in \operatorname{Dom}(\mathcal{R}_p^{-\frac{a}{\nu}}) \quad \text{and} \quad \mathcal{R}_p^{-\frac{a}{\nu}} \phi = \phi * \mathcal{I}_a \in L^p(G).$$

*Consequently,*

$$\forall \phi \in \mathcal{S}(G) \qquad \mathcal{R}_p^{\frac{a}{\nu}} \phi \in L^p(G) \quad \text{and} \quad \phi = (\mathcal{R}_p^{\frac{a}{\nu}} \phi) * \mathcal{I}_a.$$

(ii) *Let $a \in \mathbb{C}_+$. The integral*

$$\mathcal{B}_a(x) := \frac{1}{\Gamma(\frac{a}{\nu})} \int_0^\infty t^{\frac{a}{\nu}-1} e^{-t} h_t(x) dt$$

*converges absolutely for every $x \neq 0$ and defines an integrable function $\mathcal{B}_a$ on $G$. The function $\mathcal{B}_a$ is always smooth away from 0.*

*If $\operatorname{Re} a > Q$, $\mathcal{B}_a$ is also smooth at 0.*

*If $\operatorname{Re} a > Q/2$, then $\mathcal{B}_a$ is square integrable: $\mathcal{B}_a \in L^2(G)$.*

*All the operators* $(\mathrm{I}+\mathcal{R}_p)^{-a/\nu}$, $p \in [1,\infty)\cup\{\infty_o\}$, *are bounded convolution operators with the same (right convolution) kernel* $\mathcal{B}_a$.

*If* $a, b \in \mathbb{C}_+$, *then as integrable functions, we have*

$$\mathcal{B}_a * \mathcal{B}_b = \mathcal{B}_{a+b}.$$

*Remark* 4.3.12. In other words for Part (i), $\mathcal{I}_a$ is a kernel of type $a$ and

$$\mathcal{R}_p^{-a/\nu}\delta_0 = \mathcal{I}_a.$$

This shows that if $\nu < Q$, $\mathcal{I}_1$ is a fundamental solution of $\mathcal{R}$, in fact, the unique homogeneous fundamental solution (cf. Theorem 3.2.40).

Note that we will show in Lemma 4.5.9 that more generally $X^\alpha \mathcal{B}_a \in L^2(G)$ whenever $\mathrm{Re}\, a > [\alpha] + Q/2$, as well as other $L^1$-estimates.

*Proof of Corollary 4.3.11.* The absolute convergence and the smoothness of $\mathcal{I}_a$ and $\mathcal{B}_a$ follow from Lemma 4.3.8.

For the homogeneity of $\mathcal{I}_a$, we use (4.12) and the change of variable $s = r^{-\nu}t$, to get

$$\begin{aligned}\mathcal{I}_a(rx) &= \frac{1}{\Gamma(a/\nu)}\int_0^\infty t^{\frac{a}{\nu}-1} h_t(rx)dt \\ &= \frac{1}{\Gamma(a/\nu)}\int_0^\infty (r^\nu s)^{\frac{a}{\nu}-1} r^{-Q} h_s(x) r^\nu ds = r^{a-Q}\mathcal{I}_a(x).\end{aligned}$$

Hence $\mathcal{I}_a$ is a kernel of type $a$ with $0 < \mathrm{Re}\, a < Q$ (see Definition 3.2.9).

By Lemma 3.2.7, the operator $\mathcal{S}(G) \ni \phi \mapsto \phi * \mathcal{I}_a$ is homogeneous of degree $-a$, and by Proposition 3.2.8, it admits a bounded extension $L^q(G) \to L^p(G)$ when $\frac{1}{p} - \frac{1}{q} = \frac{\mathrm{Re}\,(a)}{Q}$.

Let $\phi \in \mathcal{R}^Q(\mathcal{S}(G))$. By Theorem 4.3.6, the function $a \mapsto \mathcal{R}_p^{-\frac{a}{\nu}}\phi$ is analytic on the strip $\{z \in \mathbb{C}, 0 < \mathrm{Re}\, z < Q\}$ and coincides there with

$$a \mapsto \frac{1}{\Gamma(\frac{a}{\nu})}\int_0^\infty t^{\frac{a}{\nu}-1}\phi * h_t dt.$$

But since the integral defining $\mathcal{I}_a(x)$ is absolutely convergent for all $x \in G\backslash\{0\}$, we have

$$\forall a \in \mathbb{C},\ \mathrm{Re}\, a \in (0,Q), \qquad \frac{1}{\Gamma(\frac{a}{\nu})}\int_0^\infty t^{\frac{a}{\nu}-1}\phi * h_t dt = \phi * \mathcal{I}_a,$$

and $a \mapsto \phi * \mathcal{I}_a$ is analytic on the strip $\{0 < \mathrm{Re}\, a < Q\}$.

Hence we have obtained that

$$\mathcal{R}_p^{-\frac{a}{\nu}}\phi = \phi * \mathcal{I}_a$$

holds for $\operatorname{Re} a \in (0, Q)$ and for any $\phi \in \mathcal{R}^Q(\mathcal{S}(G))$. Note that $\mathcal{R}^Q(\mathcal{S}(G))$ is dense in any $L^r(G)$, $r \in (1, \infty)$ as it suffices to apply Corollary 4.3.4 (ii) to the positive Rockland operator $\mathcal{R}^Q$. Then Corollary 3.2.32 concludes the proof of Part (i).

By Theorem 4.2.7,

$$\int_G |h_t| = \int_G |h_1| < \infty$$

for all $t > 0$, so

$$\int_G |\mathcal{B}_a(x)| dx \leq \frac{1}{|\Gamma(\frac{a}{\nu})|} \int_0^\infty t^{\frac{\operatorname{Re} a}{\nu} - 1} e^{-t} \int_G |h_t(x)| dx\, dt = \frac{\Gamma(\frac{\operatorname{Re} a}{\nu})}{|\Gamma(\frac{a}{\nu})|} \|h_1\|_{L^1}, \quad (4.32)$$

and $\mathcal{B}_a$ is integrable.

By Theorem 4.3.6 Part (4), the integrable function $\mathcal{B}_a$ is the convolution kernel of $(\mathrm{I} + \mathcal{R}_p)^{-a/\nu}$.

Let us show the square integrability of $\mathcal{B}_a$. We compute for any $R > 0$:

$$\begin{aligned} |\Gamma(a/\nu)|^2 \int_{|x|<R} |\mathcal{B}_a(x)|^2 dx &= \int_{|x|<R} \Gamma(a/\nu) \mathcal{B}_a(x) \overline{\Gamma(a/\nu) \mathcal{B}_a(x)} dx \\ &= \int_{|x|<R} \int_0^\infty t^{\frac{a}{\nu}-1} e^{-t} h_t(x) dt \int_0^\infty s^{\frac{\bar{a}}{\nu}-1} e^{-s} \bar{h}_s(x) ds\, dx \\ &= \int_0^\infty \int_0^\infty s^{\frac{a}{\nu}-1} t^{\frac{\bar{a}}{\nu}-1} e^{-(t+s)} \int_{|x|<R} h_t(x) \bar{h}_s(x) dx\, dt ds. \end{aligned}$$

From the properties of the heat kernel (see (4.13) and (4.11)) we see that

$$\int_{|x|<R} h_t(x) \bar{h}_s(x) dx = \int_{|x|<R} h_t(x) h_s(x^{-1}) dx \underset{R\to\infty}{\longrightarrow} h_t * h_s(0),$$

$$\text{and } h_t * h_s(0) = h_{t+s}(0) = (t+s)^{-\frac{Q}{\nu}} h_1(0).$$

Therefore,

$$\begin{aligned} \int_G |\mathcal{B}_a(x)|^2 dx &= \frac{h_1(0)}{|\Gamma(a/\nu)|^2} \int_0^\infty \int_0^\infty s^{\frac{a}{\nu}-1} t^{\frac{\bar{a}}{\nu}-1} e^{-(t+s)} (t+s)^{-\frac{Q}{\nu}} dt ds \\ = \frac{h_1(0)}{|\Gamma(a/\nu)|^2} &\int_{s'=0}^1 s'^{\frac{a}{\nu}-1} (1-s')^{\frac{\bar{a}}{\nu}-1} ds' \int_{u=0}^\infty e^{-u} u^{2(\frac{\operatorname{Re} a}{\nu}-1) - \frac{Q}{\nu}+1} du, \end{aligned} \quad (4.33)$$

after the change of variables $u = s + t$ and $s' = s/u$. The integrals over $s'$ and $u$ converge when $\operatorname{Re} a > Q/2$. Thus $\mathcal{B}_a$ is square integrable under this condition.

The rest of the proof of Corollary 4.3.11 follows easily from the properties of the fractional powers of $\mathrm{I} + \mathcal{R}$. □

The proof of Corollary 4.3.11 implies:

**Corollary 4.3.13.** *We keep the notation of Corollary 4.3.11 and $h_1$ denotes the heat kernel at time $t = 1$ of $\mathcal{R}$.*

1. *For any $a \in \mathbb{C}_+$, the operator norm of $(\mathrm{I}+\mathcal{R}_p)^{-\frac{a}{\nu}}$ on $L^p(G)$ if $p \in [1,\infty)$ or on $C_o(G)$ if $p = \infty_o$ is bounded by $\|\mathcal{B}_a\|_1$ and we have*

$$\|\mathcal{B}_a\|_{L^1(G)} \leq \frac{\Gamma(\frac{\mathrm{Re}\, a}{\nu})}{|\Gamma\left(\frac{a}{\nu}\right)|}\|h_1\|_{L^1(G)}.$$

2. *If $\mathrm{Re}\, a > Q/2$,*

$$\|\mathcal{B}_a\|_{L^2(G)} = \left(h_1(0)\frac{\Gamma(\frac{2\mathrm{Re}\, a - Q}{\nu})}{\Gamma(\frac{2\mathrm{Re}\, a}{\nu})}\right)^{1/2}.$$

3. *If $p \in (1,2)$ and $a > Q(1-\frac{1}{p})$ then $\mathcal{B}_a \in L^p(G)$.*

*Proof.* The first statement follows from (4.32).

For the second part, Estimate (4.33) yields

$$\|\mathcal{B}_a\|_2^2 = h_1(0) C_a,$$

where

$$\begin{aligned} C_a &= |\Gamma(a/\nu)|^{-2}\int_{s'=0}^1 s'^{\frac{a}{\nu}-1}(1-s')^{\frac{\bar{a}}{\nu}-1}ds' \int_{u=0}^\infty e^{-u}u^{2\frac{\mathrm{Re}\, a}{\nu}-\frac{Q}{\nu}-1}du \\ &= |\Gamma(\frac{a}{\nu})|^{-2}\frac{\Gamma(\frac{a}{\nu})\Gamma(\frac{\bar{a}}{\nu})}{\Gamma(\frac{a}{\nu}+\frac{\bar{a}}{\nu})}\Gamma(\frac{2\mathrm{Re}\, a - Q}{\nu}), \end{aligned}$$

thanks to the properties of the Gamma function (see equality (A.4)). We notice that

$$\Gamma(\frac{a}{\nu})\Gamma(\frac{\bar{a}}{\nu}) = \Gamma(\frac{a}{\nu})\overline{\Gamma(\frac{a}{\nu})} = |\Gamma(\frac{a}{\nu})|^2.$$

Thus the constant $C_a$ simplifies into

$$C_a = \frac{\Gamma(\frac{2\mathrm{Re}\, a - Q}{\nu})}{\Gamma(\frac{a}{\nu}+\frac{\bar{a}}{\nu})}.$$

This shows the second part.

The third part is obtained by complex interpolation between Parts 1 and 2. More precisely, we fix $a > 0$ and $b > Q/2$ and we consider the linear functional defined on simple functions in $L^1(G)$ via

$$T_z\phi = \int_G \mathcal{B}_{az+b(1-z)}(x)\phi(x)$$

for any $z \in \mathbb{C}$, $\mathrm{Re}\, z \in [0,1]$. We have

$$|T_z\phi| \leq \|\mathcal{B}_{az+b(1-z)}\|_1\|\phi\|_\infty.$$

Before applying Part 1 to $\|\mathcal{B}_{az+b(1-z)}\|_1$, let us mention that the Stirling formula (A.3) implies that for any $w \in \mathbb{C}_+$,

$$\begin{aligned}\frac{\Gamma(\operatorname{Re} w)}{|\Gamma(w)|} &\lesssim \sqrt{\frac{|w|}{\operatorname{Re} w}}\frac{(\frac{\operatorname{Re} w}{e})^{\operatorname{Re} w}}{|(\frac{w}{e})^w|} \\ &\lesssim \left(\frac{\operatorname{Re} w}{|w|}\right)^{\operatorname{Re} w-\frac{1}{2}}|w^{w-\operatorname{Re} w}| \\ &\lesssim \left(\frac{\operatorname{Re} w}{|w|}\right)^{\operatorname{Re} w-\frac{1}{2}}\exp\big(|\operatorname{Im} w|\ln|w|\big).\end{aligned}$$

This together with Part 1 then yield

$$\ln|T_z\phi| \leq \ln(\|\mathcal{B}_{az+b(1-z)}\|_1\|\phi\|_\infty) \lesssim (1+|\operatorname{Im} z|)\ln(1+|\operatorname{Im} z|),$$

thus $\{T_z\}$ is an admissible family of operator (in the sense of Section A.6). The same arguments also show that

$$|T_{1+iy}\phi| \lesssim (1+|y|)^{-\frac{a}{\nu}+\frac{1}{2}}\exp\left(c|y|\ln(1+|y|)\right)\|\phi\|_\infty,$$

where $c$ is a constant of $a, b, \nu$.

The Cauchy-Schwartz estimate and Part 2 yield

$$|T_{iy}\phi| \leq \|\mathcal{B}_{aiy+b(1-iy)}\|_2\|\phi\|_2,$$

and Part 2 implies that the quantity

$$\|\mathcal{B}_{aiy+b(1-iy)}\|_2 = \left(h_1(0)\frac{\Gamma(\frac{2b-Q}{\nu})}{\Gamma(\frac{2b}{\nu})}\right)^{1/2},$$

is independent of $y$. Hence we can apply Theorem A.6.1 to $\{T_z\}$: $T_t$ extends to an $L^{q_t}$-bounded operator where $t \in (0,1)$ and $\frac{1}{q_t} = \frac{1-t}{2}$. Therefore $\mathcal{B}_{at+b(1-t)} \in L^{q_t'}$ where $q_t'$ is the dual exponent to $q_t$, i.e. $\frac{1}{q_t} + \frac{1}{q_t'} = 1$. This shows Part 3 and concludes the proof of Corollary 4.3.13. □

We finish this section with some technical properties which will be useful in the sequel. The first one is easy to check.

**Lemma 4.3.14.** *If $\mathcal{R}$ is a positive Rockland operator with $\mathcal{B}_a$ being the kernel of the Bessel potential as given in Corollary 4.3.11, then $\bar{\mathcal{R}}$ is also a positive Rockland operator and $\bar{\mathcal{B}}_a$ is the kernel of the Bessel potential associated to $\bar{\mathcal{R}}$.*

**Lemma 4.3.15.** *We keep the notation of Corollary 4.3.11. If $a \in \mathbb{C}_+$, then the function*

$$x \mapsto |x|^N\mathcal{B}_a(x)$$

*is integrable on $G$, where $|\cdot|$ denotes any homogeneous quasi-norm on $G$ and $N$ is any positive integer. Consequently, for any $\phi \in \mathcal{S}(G)$, the function $\phi * \mathcal{B}_a$ is Schwartz and*

$$\phi \mapsto \phi * \mathcal{B}_a$$

*acts continuously from $\mathcal{S}(G)$ to itself.*

Note that we will show in Lemma 4.5.9 that, more generally,

$$|x|^b X^\alpha \mathcal{B}_a \in L^1(G) \quad \text{for } \operatorname{Re} a + b > [\alpha],$$

and that

$$X^\alpha \mathcal{B}_a \in L^2(G) \quad \text{for } \operatorname{Re} a > [\alpha] + Q/2.$$

*Proof of Lemma 4.3.15.* Let $|\cdot|$ be a homogeneous quasi-norm on $G$ and $N \in \mathbb{N}$. We see that

$$\int_G |x|^N |\mathcal{B}_a(x)| dx \leq \frac{1}{|\Gamma(\frac{a}{\nu})|} \int_0^\infty t^{\frac{\operatorname{Re} a}{\nu} - 1} e^{-t} \int_G |x|^N |h_t(x)| dx\, dt,$$

and using the homogeneity of the heat kernel (see (4.17)) and the change of variables $y = t^{-\frac{1}{\nu}} x$, we get

$$\int_G |x|^N |h_t(x)| dx = \int_G |t^{\frac{1}{\nu}} y|^N |h_1(y)| dy = c_N t^{\frac{N}{\nu}},$$

where $c_N = \| |y|^N h_1(y) \|_{L^1(dy)}$ is a finite constant since $h_1 \in \mathcal{S}(G)$. Thus,

$$\int_G |x|^N |\mathcal{B}_a(x)| dx \leq \frac{c_N}{|\Gamma(\frac{a}{\nu})|} \int_0^\infty t^{\frac{\operatorname{Re} a}{\nu} - 1 + \frac{N}{\nu}} e^{-t} dt < \infty,$$

and $x \mapsto |x|^N \mathcal{B}_a(x)$ is integrable.

Let $C_o \geq 1$ denote the constant in the triangle inequality for $|\cdot|$ (see Proposition 3.1.38 and also Inequality (3.43)). Let also $\phi \in \mathcal{S}(G)$. We have for any $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$:

$$\begin{aligned}
(1+|x|)^N \left| \tilde{X}^\alpha \left[\phi * \mathcal{B}_a\right](x) \right| &= (1+|x|)^N \left| \tilde{X}^\alpha \phi * \mathcal{B}_a(x) \right| \\
&\leq (1+|x|)^N \left| \tilde{X}^\alpha \phi \right| * |\mathcal{B}_a| (x) \\
&\leq C_o^N \left| (1+|\cdot|)^N \tilde{X}^\alpha \phi \right| * \left| (1+|\cdot|)^N \mathcal{B}_a(x) \right| (x) \\
&\leq C_o^N \left\| (1+|\cdot|)^N \tilde{X}^\alpha \phi \right\|_\infty \left\| (1+|\cdot|)^N \mathcal{B}_a \right\|_{L^1(G)}.
\end{aligned}$$

This shows that that $\phi * \mathcal{B}_a \in \mathcal{S}(G)$ and that $\phi \mapsto \phi * \mathcal{B}_a$ is continuous as a map of $\mathcal{S}(G)$ to itself (for a description of the Schwartz class, see Section 3.1.9). ☐

**Corollary 4.3.16.** *We keep the notation of Corollary 4.3.11.*

*For any $a \in \mathbb{C}$ and $p \in [1,\infty) \cup \{\infty_o\}$, $\text{Dom}(\text{I}+\mathcal{R}_p)^a \supset \mathcal{S}(G)$ and, moreover,*

$$(\text{I}+\mathcal{R}_p)^a(\mathcal{S}(G)) = \mathcal{S}(G). \tag{4.34}$$

*Furthermore on $\mathcal{S}(G)$, $(\text{I}+\mathcal{R}_p)^a$ does not depend on $p \in [1,\infty) \cup \{\infty_o\}$ and acts continuously on $\mathcal{S}(G)$.*

*If $a \in \mathbb{C}_+$, we have*

$$(\text{I}+\mathcal{R}_p)^a (\phi * \mathcal{B}_{a\nu}) = ((\text{I}+\mathcal{R}_p)^a \phi) * \mathcal{B}_{a\nu} = \phi \qquad (p \in [1,\infty) \cup \{\infty_o\}). \tag{4.35}$$

*Proof.* Formula (4.35) holds for each $p \in [1,\infty) \cup \{\infty_o\}$ by Theorem 4.3.6 and Corollary 4.3.11.

Let us show (4.34) in the case of $a = N \in \mathbb{N}$. By Theorem 4.3.6 (1a), we have the equality $(\text{I}+\mathcal{R}_p)^N \phi = (\text{I}+\mathcal{R})^N \phi$ for any $\phi \in \mathcal{S}(G)$ and $p \in (1,\infty)$. Hence $(\text{I}+\mathcal{R}_p)^N(\mathcal{S}(G)) = (\text{I}+\mathcal{R})^N(\mathcal{S}(G))$. The inclusion $(\text{I}+\mathcal{R})^N(\mathcal{S}(G)) \subset \mathcal{S}(G)$ is immediate. The converse follows easily from Lemma 4.3.15 together with (4.35). This proves (4.34) for $a = N \in \mathbb{N}$. This implies that for any $N \in \mathbb{N}$, $\mathcal{S}(G)$ is included in

$$\text{Dom}\left[(\text{I}+\mathcal{R}_p)^N\right] \cap \text{Range}\left[(\text{I}+\mathcal{R}_p)^N\right]$$

and we can apply the analyticity results (Part (1c)) of Theorem 4.3.6: fixing $\phi \in \mathcal{S}(G)$, the function $a \mapsto (\text{I}+\mathcal{R}_p)^a \phi$ is holomorphic in $\{a \in \mathbb{C} : -N < \text{Re}\, a < N\}$. We observe that by Corollary 4.3.11 (ii), if $-N < \text{Re}\, a < 0$, all the functions $(\text{I}+\mathcal{R}_p)^a \phi$ coincide with $\phi * \mathcal{B}_{a\nu}$ for any $p \in [1,\infty) \cup \{\infty_o\}$. This shows that for each $a \in \mathbb{C}$ fixed, $(\text{I}+\mathcal{R}_p)^a \phi$ is independent of $p$. Furthermore, it is Schwartz. Indeed if $\text{Re}\, a < 0$ this follow from Lemma 4.3.15. If $\text{Re}\, a \geq 0$, we write $a = a_o + a'$ with $a_o \in \mathbb{N}$ and $\text{Re}\, a' < 0$ and we have in the sense of operators

$$(\text{I}+\mathcal{R})^{a'}(\text{I}+\mathcal{R})^{a_o} \subset (\text{I}+\mathcal{R})^a.$$

The operator $(\text{I}+\mathcal{R})^{a_o}$ is a differential operator, hence maps $\mathcal{S}(G)$ to itself, and the operator $(\text{I}+\mathcal{R})^{a'}$ maps $\mathcal{S}(G)$ to itself by Lemma 4.3.15. Thus in any case $(\text{I}+\mathcal{R}_p)^a \phi \in \mathcal{S}(G)$ and is independent of $p$.

We have obtained that $(\text{I}+\mathcal{R}_p)^a(\mathcal{S}(G)) \subset \mathcal{S}(G)$ for any $p \in (1,\infty)$, $a \in \mathbb{C}$. As $\{(\text{I}+\mathcal{R}_p)^a\}^{-1} = (\text{I}+\mathcal{R}_p)^{-a}$ by Theorem 4.3.6 (1a), this proves the equality in (4.34) for any $a \in \mathbb{C}$. Lemma 4.3.15 says that this action is continuous if $\text{Re}\, a < 0$. This is also the case for $\text{Re}\, a \geq 0$ since we can proceed as above and write $a = a_o + a'$ with $a_o \in \mathbb{N}$ and $\text{Re}\, a' < 0$, the action of $(\text{I}+\mathcal{R})^{a_o}$ being continuous on $\mathcal{S}(G)$. This concludes the proof of Corollary 4.3.16. □

Corollary 4.3.16 implies that the following definition makes sense.

**Definition 4.3.17.** Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu$ and let $s \in \mathbb{R}$. For any tempered distribution $f \in \mathcal{S}'(G)$, we denote by $(\text{I}+\mathcal{R})^{s/\nu} f$ the tempered distribution defined by

$$\langle (\text{I}+\mathcal{R})^{s/\nu} f, \phi \rangle = \langle f, (\text{I}+\bar{\mathcal{R}})^{s/\nu} \phi \rangle, \quad \phi \in \mathcal{S}(G).$$

## 4.4 Sobolev spaces on graded Lie groups

In this section we define the (homogeneous and inhomogeneous) Sobolev spaces associated to a positive Rockland operator $\mathcal{R}$ and show that they satisfy similar properties to the Euclidean Sobolev spaces and to the Sobolev spaces defined and studied by Folland [Fol75] on stratified Lie groups. In Section 4.4.5, we show that the constructed spaces are actually independent of the choice of a positive Rockland operator $\mathcal{R}$ on a graded Lie group with which we start our construction. In Section 4.4.7, we list the main properties of our Sobolev spaces.

### 4.4.1 (Inhomogeneous) Sobolev spaces

We first need the following lemma:

**Lemma 4.4.1.** *We keep the notation of Theorem 4.3.6. For any $s \in \mathbb{R}$ and $p \in [1,\infty) \cup \{\infty_o\}$, the domain of the operator $(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}}$ contains $\mathcal{S}(G)$, and the map*

$$f \longmapsto \|(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f\|_{L^p(G)}$$

*defines a norm on $\mathcal{S}(G)$. We denote it by*

$$\|f\|_{L^p_s(G)} := \|(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f\|_{L^p(G)}.$$

*Moreover, any sequence in $\mathcal{S}(G)$ which is Cauchy for $\|\cdot\|_{L^p_s(G)}$ is convergent in $\mathcal{S}'(G)$.*

We have allowed ourselves to write $\|\cdot\|_{L^\infty(G)} = \|\cdot\|_{L^{\infty_o}(G)}$ for the supremum norm. We may also write $\|\cdot\|_\infty$ or $\|\cdot\|_{\infty_o}$.

*Proof.* By Corollary 4.3.16, the domain of $(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}}$ contains $\mathcal{S}(G)$. Since the operator $(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}}$ is linear, it is easy to check that the map $f \mapsto \|(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f\|_p$ is non-negative and satisfies the triangle inequality. Since $(\mathrm{I}+\mathcal{R}_p)^{s/\nu}$ is injective by Theorem 4.3.6, Part (1), we have that $\|f\|_{L^p_s(G)} = 0$ implies $f = 0$.

Clearly $\|\cdot\|_{L^p_0(G)} = \|\cdot\|_p$, so in the case of $s = 0$ a Cauchy sequence of Schwartz functions converges in $L^p$-norm, thus also in $\mathcal{S}'(G)$.

Let us assume $s > 0$. By Corollary 4.3.11 (ii), the operator $(\mathrm{I}+\mathcal{R}_p)^{-\frac{s}{\nu}}$ is bounded on $L^p(G)$. Hence we have

$$\|\cdot\|_{L^p(G)} \leq C\|\cdot\|_{L^p_s(G)}$$

on $\mathcal{S}(G)$. Consequently a $\|\cdot\|_{L^p_s(G)}$-Cauchy sequence of Schwartz functions converge in $L^p$-norm thus in $\mathcal{S}'(G)$.

Now let us assume $s < 0$. Let $\{f_\ell\}_{\ell\in\mathbb{N}}$ be a sequence of Schwartz functions which is Cauchy for the norm $\|\cdot\|_{L^p_s(G)}$. By (4.35) we have

$$f_\ell = \left((\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f_\ell\right) * \mathcal{B}_s.$$

Furthermore, if $\phi \in \mathcal{S}(G)$ then using (1.14) and (4.13), we have

$$\int_G f_\ell(x)\phi(x)dx = \int_G \left((\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f_\ell\right)(x) \ \left(\phi * \mathcal{B}_s\right)(x)\ dx. \tag{4.36}$$

By assumption the sequence $\{(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f_\ell\}_{\ell\in\mathbb{N}}$ is $\|\cdot\|_{L^p(G)}$-Cauchy thus convergent in $L^p(G)$. By Lemma 4.3.15, $\phi * \mathcal{B}_s \in \mathcal{S}(G)$. Therefore, the right-hand side of (4.36) is convergent as $\ell \to \infty$. Hence the scalar sequence $\langle f_\ell, \phi\rangle$ converges for any $\phi \in \mathcal{S}(G)$. This shows that the sequence $\{f_\ell\}$ converges in $\mathcal{S}'(G)$. □

Lemma 4.4.1 allows us to define the (inhomogeneous) Sobolev spaces:

**Definition 4.4.2.** Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. We consider its $L^p$-analogue $\mathcal{R}_p$ and the powers of $(\mathrm{I}+\mathcal{R}_p)^a$ as defined in Theorems 4.3.3 and 4.3.6. Let $s \in \mathbb{R}$.

If $p \in [1,\infty)$, the *Sobolev space* $L^p_{s,\mathcal{R}}(G)$ is the subspace of $\mathcal{S}'(G)$ obtained by completion of $\mathcal{S}(G)$ with respect to the *Sobolev norm*

$$\|f\|_{L^p_{s,\mathcal{R}}(G)} := \|(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f\|_{L^p(G)}, \quad f \in \mathcal{S}(G).$$

If $p = \infty_o$, the *Sobolev space* $L^{\infty_o}_{s,\mathcal{R}}(G)$ is the subspace of $\mathcal{S}'(G)$ obtained by completion of $\mathcal{S}(G)$ with respect to the *Sobolev norm*

$$\|f\|_{L^{\infty_o}_{s,\mathcal{R}}(G)} := \|(\mathrm{I}+\mathcal{R}_{\infty_o})^{\frac{s}{\nu}} f\|_{L^\infty(G)}, \quad f \in \mathcal{S}(G).$$

When the Rockland operator $\mathcal{R}$ is fixed, we may allow ourselves to drop the index $\mathcal{R}$ in $L^p_{s,\mathcal{R}}(G) = L^p_s(G)$ to simplify the notation.

We will see later that the Sobolev spaces actually do not depend on the Rockland operator $\mathcal{R}$, see Theorem 4.4.20.

By construction the Sobolev space $L^p_s(G)$ endowed with the Sobolev norm is a Banach space which contains $\mathcal{S}(G)$ as a dense subspace and is included in $\mathcal{S}'(G)$. The Sobolev spaces share many properties with their Euclidean counterparts.

**Theorem 4.4.3.** *Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu$ on a graded Lie group $G$. We consider the associated Sobolev spaces $L^p_s(G)$ for $p \in [1,\infty) \cup \{\infty_o\}$ and $s \in \mathbb{R}$.*

1. *If $s = 0$, then $L^p_0(G) = L^p(G)$ for $p \in [1,\infty)$ with $\|\cdot\|_{L^p_0(G)} = \|\cdot\|_{L^p(G)}$, and $L^{\infty_o}_0(G) = C_o(G)$ with $\|\cdot\|_{L^{\infty_o}_0(G)} = \|\cdot\|_{L^\infty(G)}$.*
2. *If $s > 0$, then for any $a \in \mathbb{C}$ with $\mathrm{Re}\, a = s$, we have*

$$L^p_s(G) = \mathrm{Dom}\left[(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}}\right] = \mathrm{Dom}(\mathcal{R}_p^{\frac{a}{\nu}}) \subsetneq L^p(G),$$

*and the following norms are equivalent to $\|\cdot\|_{L^p_s(G)}$:*

$$f \longmapsto \|f\|_{L^p(G)} + \|(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f\|_{L^p(G)}, \ f \longmapsto \|f\|_{L^p(G)} + \|\mathcal{R}_p^{\frac{s}{\nu}} f\|_{L^p(G)}.$$

3. *Let* $s \in \mathbb{R}$ *and* $f \in \mathcal{S}'(G)$.

   - *Given* $p \in (1,\infty)$, *we have* $f \in L^p_s(G)$ *if and only if the tempered distribution* $(\mathrm{I}+\mathcal{R}_p)^{s/\nu} f$ *defined in Definition 4.3.17 is in* $L^p(G)$, *in the sense that the linear mapping*

     $$\mathcal{S}(G) \ni \phi \mapsto \langle (\mathrm{I}+\mathcal{R})^{s/\nu} f, \phi\rangle = \langle f, (\mathrm{I}+\bar{\mathcal{R}}_{p'})^{s/\nu}\phi\rangle$$

     *extends to a bounded functional on* $L^{p'}(G)$ *where* $p'$ *is the conjugate exponent of* $p$.

   - $f \in L^1_s(G)$ *if and only if* $(\mathrm{I}+\mathcal{R}_1)^{s/\nu} f \in L^1(G)$ *in the sense that the linear mapping*

     $$\mathcal{S}(G) \ni \phi \mapsto \langle (\mathrm{I}+\mathcal{R})^{s/\nu} f, \phi\rangle = \langle f, (\mathrm{I}+\bar{\mathcal{R}}_{\infty_o})^{s/\nu}\phi\rangle$$

     *extends to a bounded functional on* $C_o(G)$ *and is realised as a measure given by an integrable function.*

   - $f \in L^{\infty_o}_s(G)$ *if and only if* $(\mathrm{I}+\mathcal{R}_{\infty_o})^{s/\nu} f \in C_o(G)$ *in the sense that the linear mapping*

     $$\mathcal{S}(G) \ni \phi \mapsto \langle (\mathrm{I}+\mathcal{R})^{s/\nu} f, \phi\rangle = \langle f, (\mathrm{I}+\bar{\mathcal{R}}_1)^{s/\nu}\phi\rangle$$

     *extends to a bounded functional on* $L^1(G)$ *and is realised as integration against functions in* $C_o(G)$.

4. *If* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $p \in [1,\infty)\cup\{\infty_o\}$, *then the following continuous strict inclusions hold*

   $$\mathcal{S}(G) \subsetneq L^p_b(G) \subsetneq L^p_a(G) \subsetneq \mathcal{S}'(G),$$

   *and an equivalent norm for* $L^p_b(G)$ *is*

   $$L^p_b(G) \ni f \longmapsto \|f\|_{L^p_a(G)} + \|\mathcal{R}_p^{\frac{b-a}{\nu}} f\|_{L^p_a(G)}.$$

5. *For* $p \in [1,\infty)\cup\{\infty_o\}$ *and any* $a, b, c \in \mathbb{R}$ *with* $a < c < b$, *there exists a positive constant* $C = C_{a,b,c}$ *such that for any* $f \in L^p_b$, *we have* $f \in L^p_c \cap L^p_a$ *and*

   $$\|f\|_{L^p_c} \leq C \|f\|^{1-\theta}_{L^p_a} \|f\|^{\theta}_{L^p_b},$$

   *where* $\theta := (c-a)/(b-a)$.

In Theorem 4.4.20, we will see that the definition of the Sobolev spaces and their properties given in Theorem 4.4.3 hold independently of the chosen Rockland operator $\mathcal{R}$.

From now on, we will often use the notation $L^p_0(G)$ since this allows us not to distinguish between the cases $L^p_0(G) = L^p(G)$ when $p \in [1,\infty)$ and $L^p_0(G) = C_o(G)$ when $p = \infty_o$.

In the proof of Part (2) of Theorem 4.4.3, we will need the following exercise in functional analysis:

**Lemma 4.4.4.** *Let $T_1$ and $T_2$ be two linear operators between two Banach spaces $\mathcal{X} \to \mathcal{Y}$. We assume that $T_1$ and $T_2$ are densely defined and share the same domain. We also assume that they are both closed injective operators and that $T_2$ is bijective with a bounded inverse. Then the graph norms of $T_1$ and $T_2$ are equivalent, that is,*

$$\exists C > 0 \quad \forall x \in \mathrm{Dom}(T_1) = \mathrm{Dom}(T_2)$$
$$C^{-1}(\|x\| + \|T_2 x\|) \leq \|x\| + \|T_1 x\| \leq C(\|x\| + \|T_2 x\|).$$

*Sketch of the proof of Lemma 4.4.4.* One can check easily that $T := T_1 T_2^{-1}$ defines a closed linear operator $T : \mathcal{Y} \to \mathcal{Y}$ defined on the whole space $\mathcal{Y}$. By the closed graph theorem (see, e.g., [Rud91, Theorem 2.15] or [RS80, Thm III. 12]), $T$ is bounded. Furthermore, $T$ is injective as the composition of two injective operators. It may not have a closed range in $\mathcal{Y}$ but one checks easily that the operator

$$(T_2^{-1}, T) : \left\{ \begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \times \mathcal{Y} \\ y & \longmapsto & (T_2^{-1} y, Ty) \end{array} \right. ,$$

has a closed range in $\mathcal{X} \times \mathcal{Y}$. Hence the restriction of $(T_2^{-1}, T)$ onto its image is bounded with a bounded inverse (see e.g. [RS80, Thm III. 11]). Consequently,

$$\|T_2^{-1} y\| + \|Ty\| \asymp \|y\|$$

for any element $y \in \mathcal{Y}$, in particular of the form $y = T_2 x$, $x \in \mathrm{Dom}(T_2)$. □

We can now prove Theorem 4.4.3.

*Proof of Theorem 4.4.3.* Part (1) is true since $(\mathrm{I} + \mathcal{R}_p)^{\frac{0}{\nu}} = \mathrm{I}$. Let us prove Part (2). So let $s > 0$. Clearly $L^p_s(G)$ coincides with the domain of the unbounded operator $(\mathrm{I} + \mathcal{R}_p)^{\frac{s}{\nu}}$ (see Theorem 4.3.6 (2)) hence it is a proper subspace of $L^p(G)$. As the operator $(\mathrm{I} + \mathcal{R}_p)^{-\frac{s}{\nu}}$ is bounded on $L^p(G)$, we have $\| \cdot \|_{L^p(G)} \leq C \| \cdot \|_{L^p_s(G)}$ on $L^p_s(G)$. So $\| \cdot \|_{L^p(G)} + \| \cdot \|_{L^p_s(G)}$ is a norm on $L^p_s(G)$ which is equivalent to the Sobolev norm. Theorem 4.3.6 implies that $\mathcal{R}_p^{\frac{s}{\nu}}$ and $(\mathrm{I} + \mathcal{R}_p)^{\frac{s}{\nu}}$ satisfy the hypotheses of Lemma 4.4.4. This shows part (2).

Part (3) follows from Part (2) and the duality properties of the spaces $L^p(G)$ and $C_o(G)$ in the case $s \geq 0$. We now consider the case $s < 0$. By Lemma 4.3.15 and Corollary 4.3.11 (and also Lemma 4.3.14), the mapping

$$T_{s,p',f} : \mathcal{S}(G) \ni \phi \longmapsto \langle f, (\mathrm{I} + \bar{\mathcal{R}}_{p'})^{s/\nu} \phi \rangle = \langle f, \phi * \bar{\mathcal{B}}_{-s} \rangle$$

is well defined for any $f \in \mathcal{S}'(G)$. If $T_{s,p',f}$ admits a bounded extension to a functional on $L^{p'}_0(G)$, then we denote this extension $\tilde{T}_{s,p',f}$ and we have

$$\|\tilde{T}_{s,p',f}\|_{\mathscr{L}(L^{p'}_0, \mathbb{C})} = \|f\|_{L^p_s(G)}. \tag{4.37}$$

This is certainly so if $f \in \mathcal{S}(G)$. Furthermore a sequence $\{f_\ell\}_{\ell\in\mathbb{N}}$ of Schwartz functions is convergent for the Sobolev norm $\|\cdot\|_{L^p_s(G)}$ if and only if $\{\tilde{T}_{s,p',f_\ell}\}$ is convergent in $L_0^{p'}(G)$ (see Lemma 4.3.2). In the case of convergence, by Lemma 4.4.1, $\{f_\ell\}_{\ell\in\mathbb{N}}$ converges in the sense of distributions. Denoting this limit by $f \in \mathcal{S}'(G)$, we have

$$\left[\lim_{\ell\to\infty} \tilde{T}_{s,p',f_\ell}\right]\Big|_{\mathcal{S}(G)} = T_{s,p',f}.$$

It is easy to see, by linearity of $f_1 \mapsto T_{s,p',f_1}$ and (4.37), that $T_{s,p',f}$ extends to a continuous functional on $L_0^{p'}(G)$.

Conversely, let us consider a distribution $f \in \mathcal{S}'(G)$ such that $T_{s,p',f}$ extends to a bounded functional $\tilde{T}_{s,p',f}$ on $L_0^{p'}(G)$. If $\{f_\ell\}_{\ell\in\mathbb{N}}$ is a sequence of Schwartz functions converging to $f$ in $\mathcal{S}'(G)$, then

$$\lim_{\ell\to\infty} T_{s,p',f_\ell}(\phi) = T_{s,p',f}(\phi)$$

for every $\phi \in \mathcal{S}(G)$, and using the density of $\mathcal{S}(G)$ in $L_0^{p'}(G)$ and the Banach-Steinhaus Theorem, this shows that $\{\tilde{T}_{s,p',f_\ell}\}$ converges to $\tilde{T}_{s,p',f}$ in the norm of the dual of $L_0^{p'}(G)$. This shows the case $s < 0$ and concludes the proof of Part (3).

Let us show Part (4). Let $a \le b$ and $p \in [1,\infty) \cup \{\infty_o\}$. By Theorem 4.3.6 (1), we have in the sense of operators

$$(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} \supset (\mathrm{I}+\mathcal{R}_p)^{\frac{a-b}{\nu}}(\mathrm{I}+\mathcal{R}_p)^{\frac{b}{\nu}}.$$

Since the operator $(\mathrm{I}+\mathcal{R}_p)^{\frac{a-b}{\nu}}$ is bounded, we have for any $f \in \mathcal{S}(G)$

$$\begin{aligned}\|f\|_{L^p_a(G)} &= \|(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} f\|_p = \|(\mathrm{I}+\mathcal{R}_p)^{\frac{a-b}{\nu}}(\mathrm{I}+\mathcal{R}_p)^{\frac{b}{\nu}} f\|_p \\ &\le \|(\mathrm{I}+\mathcal{R}_p)^{\frac{a-b}{\nu}}\|_{\mathscr{L}(L^p_0)} \|(\mathrm{I}+\mathcal{R}_p)^{\frac{b}{\nu}} f\|_p = \|(\mathrm{I}+\mathcal{R}_p)^{\frac{a-b}{\nu}}\|_{\mathscr{L}(L^p_0)} \|f\|_{L^p_b}.\end{aligned}$$

By density of $\mathcal{S}(G)$, this implies the continuous inclusion $L^p_b \subset L^p_a$. Note that we also have if $a < b$

$$\begin{aligned}\|f\|_{L^p_b(G)} &= \|(\mathrm{I}+\mathcal{R}_p)^{\frac{b-a}{\nu}}(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} f\|_p = \|(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} f\|_{L^p_{b-a}(G)} \\ &\asymp \|(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} f\|_{L^p(G)} + \|\mathcal{R}_p^{\frac{b-a}{\nu}}(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} f\|_{L^p(G)},\end{aligned}$$

by Part (2) above for any $f \in \mathcal{S}(G)$. By Theorem 4.3.6 (5), we can commute the operators $\mathcal{R}_p^{\frac{b-a}{\nu}}$ and $(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}}$ in this last expression. Consequently, we have obtained for any $f \in \mathcal{S}(G)$,

$$\|f\|_{L^p_b(G)} \asymp \|f\|_{L^p_a(G)} + \|\mathcal{R}_p^{\frac{b-a}{\nu}} f\|_{L^p_a(G)}.$$

By density of $\mathcal{S}(G)$, this holds for any $f \in L^p_b(G)$. Since the operator $\mathcal{R}_p^{\frac{b-a}{\nu}}$ is unbounded, this also implies the strict inclusions given in Part (4).

Part (5) follows from Theorem 4.3.6 (1f) for the case of $a = 0$. For $f \in L^p_b$, we then apply this to $b-a, c-a$ instead of $b$ and $c$ and $\phi := (\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}} f \in L^p_{b-a}$ instead of $f$.

This concludes the proof of this part and of the whole theorem. □

Theorem 4.4.3 has the two following corollaries. The first one is an easy consequence of Part (3).

**Corollary 4.4.5.** *We keep the setting and notation of Theorem 4.4.3. Let $s < 0$ and $p \in [1, \infty) \cup \{\infty_o\}$. Let $f \in \mathcal{S}'(G)$.*

*The tempered distribution $f$ is in $L^p_s(G)$ if and only if the mapping*

$$\mathcal{S}(G) \ni \phi \mapsto \langle f, \phi * \bar{\mathcal{B}}_{-s} \rangle$$

*extends to a bounded linear functional on $L^{p'}_0(G)$ with the additional property that*

- *for $p = 1$, this functional on $C_o(G)$ is realised as a measure given by an integrable function,*
- *if $p = \infty_o$, this functional on $L^1(G)$ is realised by integration against a function in $C_o(G)$.*

**Corollary 4.4.6.** *We keep the setting and notation of Theorem 4.4.3. Let $s \in \mathbb{R}$ and $p \in [1, \infty) \cup \{\infty_o\}$. Then $\mathcal{D}(G)$ is dense in $L^p_s(G)$.*

*Proof of Corollary 4.4.6.* This is certainly true for $s \geq 0$ (see the proof of Parts (1) and (2) of Theorem 4.4.3). For $s < 0$, it suffices to proceed as in the last part of the proof of Part (3) with a sequence of functions $f_\ell \in \mathcal{D}(G)$. □

Theorem 4.4.3, especially Part (3), implies the following property regarding duality of Sobolev spaces. This will be improved in Proposition 4.4.22 once we show in Theorem 4.4.20 that the Sobolev spaces are indeed independent of the considered Rockland operator.

**Lemma 4.4.7.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. We consider the associated Sobolev spaces $L^p_{s,\mathcal{R}}(G)$. If $s \in \mathbb{R}$ and $p \in (1, \infty)$, the dual space of $L^p_{s,\mathcal{R}}(G)$ is isomorphic to $L^{p'}_{-s,\bar{\mathcal{R}}}(G)$ via the distributional duality, where $p'$ is the conjugate exponent of $p$, $\frac{1}{p} + \frac{1}{p'} = 1$.*

*Proof of Lemma 4.4.7.* Clearly if $f \in L^p_{s,\mathcal{R}}(G)$ then for any $\phi \in \mathcal{S}(G)$,

$$\langle f, \phi \rangle = \langle f, (\mathrm{I} + \bar{\mathcal{R}}_{p'})^{\frac{s}{\nu}} (\mathrm{I} + \bar{\mathcal{R}}_{p'})^{-\frac{s}{\nu}} \phi \rangle = \langle (\mathrm{I} + \mathcal{R}_p)^{\frac{s}{\nu}} f, (\mathrm{I} + \bar{\mathcal{R}}_{p'})^{-\frac{s}{\nu}} \phi \rangle$$

by Theorem 4.3.6. Hence by Theorem 4.4.3 Part (3),

$$|\langle f, \phi \rangle| \leq \|(\mathrm{I} + \mathcal{R}_p)^{\frac{s}{\nu}} f\|_p \|(\mathrm{I} + \bar{\mathcal{R}}_{p'})^{-\frac{s}{\nu}} \phi\|_{p'}$$

and the linear function $\mathcal{S}(G) \ni \phi \mapsto \langle f, \phi \rangle$ extends to a bounded linear functional on $L^{p'}_{-s,\bar{\mathcal{R}}}(G)$. Conversely, let $\Psi$ be a bounded linear functional on $L^{p'}_{-s,\bar{\mathcal{R}}}(G)$. Then since

$$(\mathrm{I} + \bar{\mathcal{R}}_{p'})^{s/\nu} \mathcal{S}(G) = \mathcal{S}(G) \subset L^{p'}_{-s,\bar{\mathcal{R}}}(G),$$

see Corollary 4.3.16 and Definition 4.4.2, the linear functional $\Psi \circ (\mathrm{I} + \bar{\mathcal{R}}_{p'})^{s/\nu}$ is well defined on $\mathcal{S}(G)$ and satisfies for any $\phi \in \mathcal{S}(G)$,

$$\begin{aligned} |\Psi \circ (\mathrm{I} + \bar{\mathcal{R}}_{p'})^{s/\nu}(\phi)| &= |\Psi\left((\mathrm{I} + \bar{\mathcal{R}}_{p'})^{s/\nu}\phi\right)| \\ &\leq C\|(\mathrm{I} + \bar{\mathcal{R}}_{p'})^{s/\nu}\phi\|_{L^{p'}_{-s,\bar{\mathcal{R}}}} = C\|\phi\|_{L^{p'}_0}. \end{aligned}$$

Therefore, $\Psi \circ (\mathrm{I} + \bar{\mathcal{R}}_{p'})^{s/\nu}$ extends into a bounded linear functional on $L^p_0(G)$. □

In the next statement, we show how to produce functions and converging sequences of Sobolev spaces using the convolution:

**Proposition 4.4.8.** *We keep the setting and notation of Theorem 4.4.3. Here $a \in \mathbb{R}$ and $p \in [1, \infty) \cup \{\infty_o\}$.*

*(i) If $f \in L^p_0(G)$ and $\phi \in \mathcal{S}(G)$, then $f * \phi \in L^p_a$ for any $a$ and $p$.*

*(ii) If $f \in L^p_a(G)$ and $\psi \in \mathcal{S}(G)$, then*

$$(\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}}(\psi * f) = \psi * \left((\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}} f\right), \tag{4.38}$$

*and $\psi * f \in L^p_a(G)$ with*

$$\|\psi * f\|_{L^p_a(G)} \leq \|\psi\|_{L^1(G)} \|f\|_{L^p_a(G)}. \tag{4.39}$$

*Furthermore, if $\int \psi = 1$, writing*

$$\psi_\epsilon(x) := \epsilon^{-Q}\psi(\epsilon^{-1}x)$$

*for each $\epsilon > 0$, then $\{\psi_\epsilon * f\}$ converges to $f$ in $L^p_a(G)$ as $\epsilon \to 0$.*

*Proof of Proposition 4.4.8.* Let us prove Part (i). Here $f \in L^p_0(G)$. By density of $\mathcal{S}(G)$ in $L^p_0(G)$, we can find a sequence of Schwartz functions $\{f_\ell\}$ converging to $f$ in $L^p_0$-norm. Then $f_\ell * \phi \in \mathcal{S}(G)$ and for any $N \in \mathbb{N}$,

$$\mathcal{R}^N(f_\ell * \phi) = f_\ell * \mathcal{R}^N \phi \underset{\ell \to \infty}{\longrightarrow} f * \mathcal{R}^N \phi \quad \text{in } L^p_0(G),$$

thus $\mathcal{R}^N_p(f * \phi) = f * \mathcal{R}^N \phi \in L^p(G)$ and

$$\|f * \phi\|_{L^p_0(G)} + \|\mathcal{R}^N_p(f * \phi)\|_{L^p_0(G)} < \infty.$$

By Theorem 4.4.3 (4), this shows that $f * \phi$ is in $L^p_{\nu N}$ for any $N \in \mathbb{N}$, hence in any $p$-Sobolev spaces. This proves (i).

Let us prove Part (ii). We observe that both sides of Formula (4.38) always make sense as convolutions of a Schwartz function with a tempered distribution.

Let us first assume that $f \in \mathcal{S}(G)$. Formula (4.38) is true if $a < 0$ by Corollary 4.3.11 (ii) since then the $(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}}$ is a convolution operator with an integrable convolution kernel. Formula (4.38) is also true if $a \in \nu\mathbb{N}_0$ as in this case $(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}}$ is a left-invariant differential operator by Theorem 4.3.6 (1a). Hence Formula (4.38) holds for any $a > 0$ by writing $a = a_0 + a'$, $a_0 \in \nu\mathbb{N}_0$, $a' < 0$, and

$$(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} f = (\mathrm{I}+\mathcal{R}_p)^{\frac{a_0}{\nu}} (\mathrm{I}+\mathcal{R}_p)^{\frac{a'}{\nu}} f.$$

Together with Corollary 4.3.16, this shows that Formulae (4.38) and consequently (4.39) hold for any $a \in \mathbb{R}$ and $f \in \mathcal{S}(G)$.

By density of $\mathcal{S}(G)$ in $L^p_s(G)$ and (4.39), this shows that Formulae (4.38) and (4.39) hold for any $f \in L^p_s(G)$.

Hence $\psi * f \in L^p_a(G)$ with $L^p_a$-norm $\leq \|\psi\|_1 \|f\|_{L^p_a(G)}$.

If $\int_G \psi = 1$, by Lemma 3.1.58 (i),

$$\begin{aligned}
\|\psi_\epsilon * f - f\|_{L^p_a(G)} &= \|(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}}(\psi_\epsilon * f - f)\|_p \\
&= \|\psi_\epsilon * \left((\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} f\right) - (\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} f\|_p \longrightarrow_{\epsilon\to 0} 0,
\end{aligned}$$

that is, $\{\psi_\epsilon * f\}$ converges to $f$ in $L^p_a(G)$ as $\epsilon \to 0$. This proves (ii). □

## 4.4.2 Interpolation between inhomogeneous Sobolev spaces

In this section, we prove that interpolation between Sobolev spaces $L^p_a(G)$ works in the same way as its Euclidean counterpart.

**Theorem 4.4.9.** *Let $\mathcal{R}$ and $\mathcal{Q}$ be two positive Rockland operators on two graded Lie groups $G$ and $F$. We consider their associated Sobolev spaces $L^p_a(G)$ and $L^q_b(F)$. Let $p_0, p_1, q_0, q_1 \in (1,\infty)$ and let $a_0, a_1, b_0, b_1$ be real numbers.*

*We also consider a linear mapping $T$ from $L^{p_0}_{a_0}(G) + L^{p_1}_{a_1}(G)$ to locally integrable functions on $F$. We assume that $T$ maps $L^{p_0}_{a_0}(G)$ and $L^{p_1}_{a_1}(G)$ boundedly into $L^{q_0}_{b_0}(F)$ and $L^{q_1}_{b_1}(F)$, respectively.*

*Then $T$ extends uniquely to a bounded mapping from $L^p_{a_t}(G)$ to $L^q_{b_t}(F)$ for $t \in [0,1]$ where $a_t, b_t, p_t, q_t$ are defined by*

$$\left(a_t, b_t, \frac{1}{p_t}, \frac{1}{q_t}\right) = (1-t)\left(a_0, b_0, \frac{1}{p_0}, \frac{1}{q_0}\right) + t\left(a_1, b_1, \frac{1}{p_1}, \frac{1}{q_1}\right).$$

The idea of the proof is similar to the one of the Euclidean or stratified cases, see [Fol75, Theorem 4.7]. Some arguments will be modified since our estimates for $\|(\mathrm{I}+\mathcal{R})^{i\tau}\|_{\mathscr{L}(L^p)}$ are different from the ones obtained by Folland in [Fol75]. For this, compare Corollary 4.3.13 and Proposition 4.3.7 in this monograph with [Fol75, Proposition 4.3].

*Proof of Theorem 4.4.9.* By duality (see Lemma 4.4.7) and up to a change of notation, it suffices to prove the case

$$a_1 \geq a_0 \quad \text{and} \quad b_1 \leq b_0. \tag{4.40}$$

This fact is left to the reader to check. The idea is to interpolate between the operators formally given by

$$T_z = (\mathrm{I} + \mathcal{Q})^{\frac{b_z}{\nu_{\mathcal{Q}}}} T (\mathrm{I} + \mathcal{R})^{-\frac{a_z}{\nu_{\mathcal{R}}}}, \tag{4.41}$$

where $\nu_{\mathcal{R}}$ and $\nu_{\mathcal{Q}}$ denote the degrees of homogeneity of $\mathcal{R}$ and $\mathcal{Q}$, respectively, and the complex numbers $a_z$ and $b_z$ are defined by

$$(a_z, b_z) := z\,(a_1, b_1) + (1-z)\,(a_0, b_0),$$

for $z$ in the strip

$$S := \{z \in \mathbb{C} \ : \ \operatorname{Re} z \in [0,1]\}.$$

In (4.41), we have abused the notation regarding the fractional powers of $\mathrm{I} + \mathcal{R}_p$ and $\mathrm{I} + \mathcal{Q}_q$ and removed $p$ and $q$. This is possible by Corollary 4.3.16 and density of the Schwartz space in each Sobolev space. Hence (4.41) makes sense. We will use complex interpolation given by Theorem A.6.1, which requires to start with the space $\mathscr{B}$ of compactly supported simple functions on $G$ (see Remark A.6.2). To solve this technical problem we proceed as in the proof of [Fol75, Theorem 4.7]: we will use the convolution of a function in $\mathscr{B}$ with a bump function $\chi_\epsilon$ depending on $\epsilon$ at the end of the proof.

The hypotheses on $T$ give that the operator norms

$$\|T\|_{\mathscr{L}(L^{p_j}_{a_j}, L^{q_j}_{b_j})} = \|(\mathrm{I} + \mathcal{Q})^{\frac{b_j}{\nu_{\mathcal{Q}}}} T (\mathrm{I} + \mathcal{R})^{-\frac{a_j}{\nu_{\mathcal{R}}}}\|_{\mathscr{L}(L^{p_j}, L^{q_j})}, \qquad j = 0, 1,$$

are finite.

By Corollary 4.3.16, for any $\phi \in \mathcal{S}(G)$ and $\psi \in \mathcal{S}(F)$, we have

$$\langle T_z \phi, \psi \rangle = \langle T (\mathrm{I} + \mathcal{R})^{-N - \frac{a_z}{\nu_{\mathcal{R}}}} (\mathrm{I} + \mathcal{R})^N \phi, (\mathrm{I} + \bar{\mathcal{Q}})^{-M + \frac{b_z}{\nu_{\mathcal{Q}}}} (\mathrm{I} + \bar{\mathcal{Q}})^M \psi \rangle$$

for any $M, N \in \mathbb{Z}$. In particular, for $M$ and $N$ large enough, Theorem 4.3.6 implies that

$$S \ni z \mapsto \langle T_z \phi, \psi \rangle$$

is analytic. With $M = N \in \mathbb{N}$ large enough, for instance the smallest integer with $N > a_1, a_0, b_1, b_0$, we get

$$|\langle T_z \phi, \psi \rangle| \leq \ A(z)\ B(z)\ \|T\|_{\mathscr{L}(L^{p_1}_{a_1}, L^{q_1}_{b_1})} \|\phi\|_{L^{p_1}_N} \|\psi\|_{L^{q_1}_N},$$

where $A(z)$ and $B(z)$ denote the operator norms

$$A(z) := \|(\mathrm{I} + \mathcal{R})^{-N + \frac{-a_z + a_1}{\nu_{\mathcal{R}}}}\|_{\mathscr{L}(L^{p_1})} \quad \text{and} \quad B(z) := \|(\mathrm{I} + \bar{\mathcal{Q}})^{-M + \frac{b_z - b_1}{\nu_{\mathcal{Q}}}}\|_{\mathscr{L}(L^{q_1})}.$$

We can write

$$A(z) = \|(\mathrm{I}+\mathcal{R})^{-(\alpha+\beta z)}\|_{\mathscr{L}(L^{p_1})} \quad \text{with } \alpha = N - \frac{a_1 - a_0}{\nu_{\mathcal{R}}} > 0,\ \beta = \frac{a_1 - a_0}{\nu_{\mathcal{R}}} \geq 0.$$

Thus

$$\begin{array}{rcl} A(z) & \leq & \|(\mathrm{I}+\mathcal{R})^{-(\alpha+\beta \mathrm{Re}\, z)}\|_{\mathscr{L}(L^{p_1})}\|(\mathrm{I}+\mathcal{R})^{-\beta \mathrm{Im}\, z}\|_{\mathscr{L}(L^{p_1})} \\ & \lesssim & \|h_1\|_{L^1} e^{\theta\beta|\mathrm{Im}\, z|}, \end{array}$$

by Corollary 4.3.13 and Proposition 4.3.7 using the notation of their statements. We have a similar property for $B(z)$. This implies easily that there exists a constant $C$ depending on $\phi, \psi, a_1, a_0, b_1, b_0$ and $F, G, \mathcal{R}, \mathcal{Q}$ such that we have

$$\forall z \in S \qquad \ln|\langle T_z\phi, \psi\rangle| \leq C(1 + |\mathrm{Im}\, z|).$$

We now estimate operator norms of $T_z$ for $z$ on the boundary of the strip, that is, $z = j + iy$, $j = 0, 1$, $y \in \mathbb{R}$:

$$\begin{aligned} &\|T_z\|_{\mathscr{L}(L^{p_j},L^{q_j})} \\ &\quad = \|(\mathrm{I}+\mathcal{Q})^{\frac{b_z}{\nu_{\mathcal{Q}}}} T (\mathrm{I}+\mathcal{R})^{-\frac{a_z}{\nu_{\mathcal{R}}}}\|_{\mathscr{L}(L^{p_j},L^{q_j})} \\ &\quad = \|(\mathrm{I}+\mathcal{Q})^{\frac{b_z - b_j}{\nu_{\mathcal{Q}}}} (\mathrm{I}+\mathcal{Q})^{\frac{b_j}{\nu_{\mathcal{Q}}}} T (\mathrm{I}+\mathcal{R})^{\frac{-a_j}{\nu_{\mathcal{R}}}} (\mathrm{I}+\mathcal{R})^{\frac{a_j - a_z}{\nu_{\mathcal{R}}}}\|_{\mathscr{L}(L^{p_j},L^{q_j})} \\ &\quad \leq \|(\mathrm{I}+\mathcal{Q}_{q_j})^{\frac{b_z - b_j}{\nu_{\mathcal{Q}}}}\|_{\mathscr{L}(L^{q_j})} \|T\|_{\mathscr{L}(L^{p_j}_{a_j}, L^{q_j}_{b_j})} \|(\mathrm{I}+\mathcal{R}_{p_j})^{\frac{a_j - a_z}{\nu_{\mathcal{R}}}}\|_{\mathscr{L}(L^{p_j})} \\ &\qquad\qquad = \|(\mathrm{I}+\mathcal{Q}_{q_j})^{iy\frac{b_1 - b_0}{\nu_{\mathcal{Q}}}}\|_{\mathscr{L}(L^{q_j})} \|T\|_{\mathscr{L}(L^{p_j}_{a_j}, L^{q_j}_{b_j})} \|(\mathrm{I}+\mathcal{R}_{p_j})^{iy\frac{a_0 - a_1}{\nu_{\mathcal{R}}}}\|_{\mathscr{L}(L^{p_j})}. \end{aligned}$$

Proposition 4.3.7 then implies

$$\|T_{j+iy}\|_{\mathscr{L}(L^{p_j},L^{q_j})} \leq C\|T\|_{\mathscr{L}(L^{p_j}_{a_j}, L^{q_j}_{b_j})} e^{\theta_{\mathcal{R}}\frac{a_1 - a_0}{\nu_{\mathcal{R}}}|y|} e^{\theta_{\mathcal{Q}}\frac{b_0 - b_1}{\nu_{\mathcal{R}}}|y|},$$

where $C$, $\theta_{\mathcal{R}}$ and $\theta_{\mathcal{Q}}$ are positive constants obtained from the applications of Proposition 4.3.7 to $\mathcal{R}$ and $\mathcal{Q}$.

The end of the proof is now classical. We fix a non-negative function $\chi \in \mathcal{S}(G)$ with $\int_G \chi = 1$ and write

$$\chi_\epsilon(x) := \epsilon^{-Q}\chi(\epsilon^{-1}x)$$

for $\epsilon > 0$. If $f \in \mathscr{B}$, then $f * \chi_\epsilon \in \mathcal{S}(G)$ (see Lemma 3.1.59) and we can set for any $\epsilon > 0$, $z \in S$,

$$T_{z,\epsilon} f := T_z\left(f * \chi_\epsilon\right).$$

Clearly $T_{z,\epsilon}$ satisfy the hypotheses of Theorem A.6.1 (see also Remark A.6.2). Thus for any $t \in [0, 1]$, there exists a constant $M_t > 0$ independent of $\epsilon$ such that

$$\forall f \in \mathscr{B} \qquad \|T_{t,\epsilon} f\|_{q_t} \leq M_t \|f\|_{p_t}.$$

For $p \in (1,\infty)$, we consider the space $\mathcal{V}_p$ of functions $\phi$ of the form $\phi = f * \chi_\epsilon$, with $f \in \mathscr{B}$ and $\epsilon > 0$, satisfying $\|f\|_p \leq 2\|f * \chi_\epsilon\|_p$. By Lemma 3.1.59, the space $\mathcal{V}_p$ contains $\mathcal{S}(G)$ and is dense in $L^p(G)$ for $p \in (1,\infty)$. Going back to the proof of Theorem 4.4.9, we have obtained for any $t \in [0,1]$ and $\phi = f * \chi_\epsilon \in \mathcal{V}_{p_t}$, that

$$\|T_t\phi\|_{q_t} = \|T_{t,\epsilon} f\|_{q_t} \leq M_t \|f\|_{p_t} \leq 2M_t \|\phi\|_{p_t}.$$

This shows that $T_t$ extends to a bounded operator from $L^{p_t}(G)$ to $L^{q_t}(G)$. □

As a consequence of the interpolation properties, we have

**Corollary 4.4.10.** *Let $\kappa \in \mathcal{S}'(G)$ and let $T_\kappa$ be its associated convolution operator*

$$T_\kappa : \mathcal{S}(G) \ni \phi \mapsto \phi * \kappa.$$

*Let also $a \in \mathbb{R}$, $p \in (1,\infty)$ and let $\{\gamma_\ell,\, \ell \in \mathbb{Z}\}$ be a sequence of real numbers which tends to $\pm\infty$ as $\ell \to \pm\infty$. Assume that for any $\ell \in \mathbb{Z}$, the operator $T_\kappa$ extends continuously to a bounded operator $L^p_{\gamma_\ell}(G) \to L^p_{a+\gamma_\ell}(G)$. Then the operator $T_\kappa$ extends continuously to a bounded operator $L^p_\gamma(G) \to L^p_{a+\gamma}(G)$ for any $\gamma \in \mathbb{R}$. Furthermore, for any $c \geq 0$, we have*

$$\sup_{|\gamma|\leq c} \|T_\kappa\|_{\mathscr{L}(L^p_\gamma, L^p_{a+\gamma})} \leq C_c \max\left( \|T_\kappa\|_{\mathscr{L}(L^p_{\gamma_\ell}, L^p_{a+\gamma_\ell})}, \|T_\kappa\|_{\mathscr{L}(L^p_{\gamma_{-\ell}}, L^p_{a+\gamma_{-\ell}})} \right)$$

*where $\ell \in \mathbb{N}_0$ is the smallest integer such that $\gamma_\ell \geq c$ and $-\gamma_{-\ell} \geq c$.*

### 4.4.3 Homogeneous Sobolev spaces

Here we define the homogeneous version of our Sobolev spaces and obtain their first properties. Many proofs are obtained by adapting the corresponding inhomogeneous cases and we may therefore allow ourselves to present them more succinctly. For technical reasons explained below, the definition of homogeneous Sobolev spaces is restricted to the case $p \in (1,\infty)$.

As in the inhomogeneous case, we first need the following lemma:

**Lemma 4.4.11.** *We keep the notation of Theorem 4.3.6.*

1. *For any $s \in \mathbb{R}$ and $p \in [1,\infty) \cup \{\infty_o\}$, the map $f \mapsto \|\mathcal{R}_p^{\frac{s}{\nu}} f\|_{L^p(G)}$ defines a norm on $\mathcal{S}(G) \cap \mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$. We denote it by*

$$\|f\|_{\dot{L}^p_s(G)} := \|\mathcal{R}_p^{\frac{s}{\nu}} f\|_{L^p(G)}.$$

2. *For any $s \leq 0$ and $p \in [1,\infty) \cup \{\infty_o\}$, $\mathcal{S}(G) \cap \mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$ contains $\mathcal{R}^{\lceil |s|\nu \rceil}(\mathcal{S}(G))$ which is dense in $\mathrm{Range}(\mathcal{R}_p)$ for $\|\cdot\|_{L^p(G)}$, and any sequence in $\mathcal{S}(G) \cap \mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$ which is Cauchy for $\|\cdot\|_{\dot{L}^p_s(G)}$ is convergent in $\mathcal{S}'(G)$.*

3. *If $s>0$ and $p\in(1,\infty)$, then $\mathcal{S}(G)\subset \mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$ and any sequence in $\mathcal{S}(G)$ which is Cauchy for $\|\cdot\|_{\dot{L}^p_s(G)}$ is convergent in $\mathcal{S}'(G)$.*

*Proof of Lemma 4.4.11.* The fact that the map $f\mapsto \|\mathcal{R}_p^{\frac{s}{\nu}} f\|_{L^p(G)}$ defines a norm on $\mathcal{S}(G)$ follows easily from Theorem 4.3.6 Part (1).

In the case $s=0$, $\|\cdot\|_{\dot{L}^p_0(G)}=\|\cdot\|_{L^p(G)}$ and Part 2 is proved in this case.

Let $s<0$ and $p\in[1,\infty)\cup\{\infty_o\}$. By Theorem 4.3.6 (especially Parts (1a) and (1c)), for any $N\in\mathbb{N}$ with $N>|s|/\nu$, $\mathrm{Dom}(\mathcal{R}^{\frac{s}{\nu}})$ contains $\mathcal{R}^N(\mathcal{S}(G))$ and $\mathcal{R}^N(\mathcal{S}(G))$ is dense in $\mathrm{Range}(\mathcal{R}_p)$. Consequently $\mathcal{S}(G)\cap \mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$ contains $\mathcal{R}^N(\mathcal{S}(G))$ and is dense in $\mathrm{Range}(\mathcal{R}_p)$. Let $p'$ be the dual exponent of $p$, i.e. $\frac{1}{p}+\frac{1}{p'}=1$ with the usual extension. Theorem 4.3.6 (1), and the duality properties of $L^p$ as well as $\mathcal{R}^t=\bar{\mathcal{R}}$ imply

$$|\langle f,\phi\rangle|\leq \|\mathcal{R}_p^{\frac{s}{\nu}} f\|_{L^p(G)}\|\bar{\mathcal{R}}_{p'}^{-\frac{s}{\nu}}\phi\|_{L^{p'}(G)},$$

for any $f\in\mathcal{S}(G)\cap\mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$ and $\phi\in\mathcal{S}(G)$. Furthermore, as $\phi\in\mathcal{S}(G)\subset \mathrm{Dom}(\mathcal{R}_{p'}^{-\frac{s}{\nu}})$, Theorem 4.3.6 (1) also yields for any $\phi\in\mathcal{S}(G)$

$$\begin{aligned}\|\bar{\mathcal{R}}_{p'}^{-\frac{s}{\nu}}\phi\|_{L^{p'}(G)} &\leq \max\left(\|\bar{\mathcal{R}}_{p'}^{\lfloor\frac{|s|}{\nu}\rfloor}\phi\|_{L^{p'}(G)},\|\bar{\mathcal{R}}_{p'}^{\lceil\frac{|s|}{\nu}\rceil}\phi\|_{L^{p'}(G)}\right)\\ &\leq C\max_{[\alpha]=\lfloor\frac{|s|}{\nu}\rfloor,\lceil\frac{|s|}{\nu}\rceil}\|X^\alpha\phi\|_{L^{p'}(G)}\end{aligned}$$

for some constant $C=C_{N,\mathcal{R}}$. We have obtained that

$$|\langle f,\phi\rangle|\leq C\|\mathcal{R}_p^{\frac{s}{\nu}} f\|_{L^p(G)}\max_{[\alpha]=N,N+1}\|X^\alpha\phi\|_{L^{p'}(G)}$$

for any $f\in\mathcal{S}(G)\cap\mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$ and $\phi\in\mathcal{S}(G)$. This together with the properties of the Schwartz space (see Section 3.1.9) easily implies Part 2.

Let $s>0$. By Theorem 4.3.6 (1g), $\mathcal{S}(G)\subset\mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$.

Let $p\in(1,\infty)$. By Corollary 4.3.11 Part (i), if $s\in(0,\frac{Q}{p})$, then there exists $C>0$ such that

$$\forall f\in\mathcal{S}(G)\qquad \|f\|_{L^q(G)}\leq C\|\mathcal{R}_p^{\frac{s}{\nu}} f\|_{L^p(G)}=C\|f\|_{\dot{L}^p_s(G)},$$

where $q\in(1,\infty)$ is such that

$$\frac{1}{p}-\frac{1}{q}=\frac{s}{Q}.$$

Note that $q$ is indeed in $(1,\infty)$ as $s<\frac{Q}{p}$. Hence if $\{f_\ell\}\subset\mathcal{S}(G)$ is Cauchy for $\|\cdot\|_{\dot{L}^p_s(G)}$, then $\{f_\ell\}\subset\mathcal{S}(G)$ is Cauchy for $\|\cdot\|_{L^q(G)}$ thus in $\mathcal{S}'(G)$. This shows Part 3 for any $s>0$, $p\in(1,\infty)$ satisfying $ps<Q$.

If $s \in [N\frac{Q}{p}, (N+1)\frac{Q}{p})$ for some $N \in \mathbb{N}_0$, we write $s = s_1 + s'$ with $s' \in (0, \frac{Q}{p})$ and

$$s_1 \in [(N-1)\frac{Q}{p}, N\frac{Q}{p})$$

and by Corollary 4.3.11 Part (i) with Theorem 4.3.6 (1), we have

$$\exists C = C_{s',p} \qquad \forall f \in \mathcal{S}(G) \qquad \|\mathcal{R}_q^{\frac{s_1}{\nu}} f\|_{L^q} \leq C\|\mathcal{R}_p^{\frac{s}{\nu}} f\|_{L^p(G)},$$

where $q \in (1, \infty)$ is such that

$$\frac{1}{q} - \frac{1}{p} = \frac{s'}{Q}.$$

Hence if $\{f_\ell\} \subset \mathcal{S}(G)$ is Cauchy for $\|\cdot\|_{\dot{L}^p_s(G)}$, then $\{f_\ell\} \subset \mathcal{S}(G)$ is Cauchy for $\|\cdot\|_{\dot{L}^q_{s_1}(G)}$. Note that

$$s_1 \leq \frac{NQ}{p} < \frac{NQ}{q}.$$

Recursively, this shows Part 3. □

The use of Corollary 4.3.11 in the proof above requires $p \in (1, \infty)$. Moreover, by Corollary 4.3.4 (ii), the range of $\mathcal{R}_p$ is dense in $L^p(G)$ for $p \in (1, \infty_o]$. As we want to have a unified presentation for all the homogeneous spaces of any exponent $s \in \mathbb{R}$, we restrict the parameter $p$ to be in $(1, \infty)$ only.

**Definition 4.4.12.** Let $\mathcal{R}$ be a Rockland operator of homogeneous degree $\nu$ on a graded Lie group $G$, and let $p \in (1, \infty)$. We denote by $\dot{L}^p_{s,\mathcal{R}}(G)$ the space of tempered distribution obtained by the completion of $\mathcal{S}(G) \cap \text{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$ for the norm

$$\|f\|_{\dot{L}^p_s(G)} := \|\mathcal{R}_p^{\frac{s}{\nu}} f\|_p, \quad f \in \mathcal{S}(G) \cap \text{Dom}(\mathcal{R}_p^{s/\nu}).$$

As in the inhomogeneous case, we will write $\dot{L}^p_s(G)$ or $\dot{L}^p_{s,\mathcal{R}}$ but often omit the reference to the Rockland operator $\mathcal{R}$. We will see in Theorem 4.4.20 that the homogeneous Sobolev spaces do not depend on a specific $\mathcal{R}$. Adapting the inhomogeneous case, one obtains easily:

**Proposition 4.4.13.** *Let $G$ be a graded Lie group of homogeneous dimension $Q$. Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu$ on $G$. Let $p \in (1, \infty)$ and $s \in \mathbb{R}$.*

1. *We have*

$$\left(\mathcal{S}(G) \cap \text{Dom}(\mathcal{R}_p^{s/\nu})\right) \subsetneq \dot{L}^p_s(G) \subsetneq \mathcal{S}'(G).$$

   *Equipped with the homogeneous Sobolev norm $\|\cdot\|_{\dot{L}^p_s(G)}$, the space $\dot{L}^p_s(G)$ is a Banach space which contains $\mathcal{S}(G) \cap \text{Dom}(\mathcal{R}_p^{s/\nu})$ as dense subspace.*

2. *If $s > -Q/p$ then $\mathcal{S}(G) \subset \text{Dom}(\mathcal{R}_p^{s/\nu}) \subset \dot{L}^p_s(G)$. If $s < 0$ then $\mathcal{S}(G) \cap \text{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$ contains $\mathcal{R}^{\lceil |s|\nu \rceil}(\mathcal{S}(G))$ which is dense in $L^p(G)$.*

3. *If* $s = 0$, *then* $\dot{L}^p_0(G) = L^p(G)$ *for* $p \in (1,\infty)$ *with* $\|\cdot\|_{\dot{L}^p_0(G)} = \|\cdot\|_{L^p(G)}$.

4. *Let* $s \in \mathbb{R}$, $p \in (1,\infty)$ *and* $f \in \mathcal{S}'(G)$. *If* $f \in \dot{L}^p_s(G)$ *then* $\mathcal{R}_p^{s/\nu} f \in L^p(G)$ *in the sense that the linear mapping*

$$\left(\mathcal{S}(G) \cap \mathrm{Dom}(\bar{\mathcal{R}}_{p'}^{s/\nu})\right) \ni \phi \mapsto \langle f, \bar{\mathcal{R}}_{p'}^{s/\nu} \phi\rangle$$

*is densely defined on* $L^{p'}(G)$ *and extends to a bounded functional on* $L^{p'}(G)$ *where* $p'$ *is the conjugate exponent of* $p$. *The converse is also true.*

5. *If* $1 < p < q < \infty$ *and* $a, b \in \mathbb{R}$ *with*

$$b - a = Q(\frac{1}{p} - \frac{1}{q}),$$

*then we have the continuous inclusion*

$$\dot{L}^p_b \subset \dot{L}^q_a$$

*that is, for every* $f \in \dot{L}^p_b$, *we have* $f \in \dot{L}^q_a$ *and there exists a constant* $C = C_{a,b,p,q,G} > 0$ *independent of* $f$ *such that*

$$\|f\|_{\dot{L}^q_a} \leq C\|f\|_{\dot{L}^p_b}.$$

6. *For* $p \in (1,\infty)$ *and any* $a, b, c \in \mathbb{R}$ *with* $a < c < b$, *there exists a positive constant* $C = C_{a,b,c}$ *such that we have for any* $f \in \dot{L}^p_b$

$$\|f\|_{\dot{L}^p_c} \leq C\|f\|^{1-\theta}_{\dot{L}^p_a} \|f\|^{\theta}_{\dot{L}^p_b} \quad \text{where } \theta := (c-a)/(b-a).$$

*Proof of Proposition 4.4.13.* Parts (1), (2), and (3) follow from Lemma 4.4.11 and its proof. Part (4) follows easily by duality and Lemma 4.4.11. Parts (5) and (6) are an easy consequence of the property of the fractional powers of $\mathcal{R}$ on the $L^p$-spaces (cf. Theorem 4.3.6) and the operator $\mathcal{R}_p^{-s/\nu}$, $s \in (0, Q)$, being of type $s$ and independent of $p$ (cf. Corollary 4.3.11 (i)). □

Note that Part (2) of Proposition 4.4.13 can not be improved in general as the inclusions $\mathcal{S}(G) \subset \mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$ or $\mathcal{S}(G) \subset \dot{L}^s_p(G)$ can not hold in general for any group $G$ as they do not hold in the Euclidean case i.e. $G = (\mathbb{R}^n, +)$ with the usual dilations. Indeed in the case of $\mathbb{R}^n$, $p = 2$, one can construct Schwartz functions which can not be in $\dot{L}^2_s$ with $s < -n/2$. It suffices to consider a function $\phi \in \mathcal{S}(G)$ satisfying $\widehat{\phi}(\xi) \equiv 1$ on a neighbourhood of 0 since then $|\xi|^s \widehat{\phi}(\xi)$ is not square integrable about 0 for $s < -n/2$.

As in the homogeneous case (see Lemma 4.4.7), Part (4) of Proposition 4.4.13 above implies the following property regarding duality of Sobolev spaces. This will be improved in Proposition 4.4.22 once we know (see Theorem 4.4.20) that homogeneous Sobolev spaces are indeed independent of the considered Rockland operator.

**Lemma 4.4.14.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. We consider the associated homogeneous Sobolev spaces $\dot{L}^p_{s,\mathcal{R}}(G)$. If $s \in \mathbb{R}$ and $p \in (1,\infty)$, the dual space of $\dot{L}^p_{s,\mathcal{R}}(G)$ is isomorphic to $\dot{L}^{p'}_{-s,\bar{\mathcal{R}}}(G)$ via the distributional duality, where $p'$ is the conjugate exponent of $p$, i.e. $\frac{1}{p}+\frac{1}{p'}=1$.*

The following interpolation property can be proved after a careful modification of the inhomogeneous proof:

**Proposition 4.4.15.** *Let $\mathcal{R}$ and $\mathcal{Q}$ be two positive Rockland operators on two graded Lie groups $G$ and $F$ respectively. We consider their associated homogeneous Sobolev spaces $\dot{L}^p_a(G)$ and $\dot{L}^q_b(F)$. Let $p_0,p_1,q_0,q_1 \in (1,\infty)$ and $a_0,a_1,b_0,b_1 \in \mathbb{R}$.*

*We also consider a linear mapping $T$ from $\dot{L}^{p_0}_{a_0}(G)+\dot{L}^{p_1}_{a_1}(G)$ to locally integrable functions on $F$. We assume that $T$ maps $\dot{L}^{p_0}_{a_0}(G)$ and $\dot{L}^{p_1}_{a_1}(G)$ boundedly into $\dot{L}^{q_0}_{b_0}(F)$ and $\dot{L}^{q_1}_{b_1}(F)$, respectively.*

*Then $T$ extends uniquely to a bounded mapping from $\dot{L}^{p_t}_{a_t}(G)$ to $\dot{L}^{q_t}_{b_t}(F)$ for $t \in [0,1]$, where $a_t, b_t, p_t, q_t$ are defined by*

$$\left(a_t, b_t, \frac{1}{p_t}, \frac{1}{q_t}\right) = (1-t)\left(a_0, b_0, \frac{1}{p_0}, \frac{1}{q_0}\right) + t\left(a_1, b_1, \frac{1}{p_1}, \frac{1}{q_1}\right).$$

*Sketch of the proof of Proposition 4.4.15.* By duality (see Lemma 4.4.14) and up to a change of notation, it suffices to prove the case $a_1 \geq a_0$ and $b_1 \leq b_0$. The idea is to interpolate between the operators formally given by

$$T_z = \mathcal{Q}^{z\frac{b_1-b_0}{\nu_{\mathcal{Q}}}} \mathcal{Q}^{\frac{b_0}{\nu_{\mathcal{Q}}}} T \mathcal{R}^{-\frac{a_0}{\nu_{\mathcal{R}}}} \mathcal{R}^{z\frac{a_0-a_1}{\nu_{\mathcal{R}}}}, \quad z \in S, \tag{4.42}$$

with the same notation for $\nu_{\mathcal{R}}$, $\nu_{\mathcal{Q}}$, $a_z$, $b_z$ and $S$ as in the proof of Theorem 4.4.9. In (4.42), we have abused the notation regarding the fractional powers of $\mathcal{R}_p$ and $\mathcal{Q}_q$ and removed $p$ and $q$ thanks to by Theorem 4.3.6 (1). Moreover, Theorem 4.3.6 implies that on $\mathcal{S}(G)$, each operator $T_z$, $z \in S$, coincides with

$$T_z = \mathcal{Q}^{(1-z)\frac{b_0-b_1}{\nu_{\mathcal{Q}}}} \mathcal{Q}^{\frac{b_1}{\nu_{\mathcal{Q}}}} T \mathcal{R}^{-\frac{a_1}{\nu_{\mathcal{R}}}} \mathcal{R}^{(1-z)\frac{a_1-a_0}{\nu_{\mathcal{R}}}},$$

and that for any $\phi \in \mathcal{S}(G)$ and $\psi \in \mathcal{S}(F)$, $z \mapsto \langle T_z\phi, \psi\rangle$ is analytic on $S$. We also have

$$|\langle T_z\phi,\psi\rangle| \leq \|T\|_{\mathscr{L}(\dot{L}^{p_1}_{a_1}, \dot{L}^{q_1}_{b_1})} \|\mathcal{R}^{\frac{-a_z+a_1}{\nu_{\mathcal{R}}}}\phi\|_{L^{p_1}} \|\bar{\mathcal{Q}}^{\frac{b_z-b_1}{\nu_{\mathcal{Q}}}}\psi\|_{L^{q_1}}.$$

Note that $-\mathrm{Re}\, a_z + a_1 \geq 0$ thus we have

$$\begin{aligned} \|\mathcal{R}^{\frac{-a_z+a_1}{\nu_{\mathcal{R}}}}\phi\|_{L^{p_1}} &\leq \|\mathcal{R}^{\frac{-\mathrm{Re}\, a_z+a_1}{\nu_{\mathcal{R}}}}\phi\|_{L^{p_1}} \|\mathcal{R}^{\frac{-\mathrm{Im}\, a_z}{\nu_{\mathcal{R}}}}\phi\|_{L^{p_1}} \\ &\lesssim \|\phi\|_{L^{p_1}}^{1-\alpha} \|\mathcal{R}^N\phi\|_{L^{p_1}}^{\alpha} e^{\theta\frac{|\mathrm{Im}\, a_z|}{\nu_{\mathcal{R}}}}, \end{aligned}$$

by Theorem 4.3.6 (1f) with $N$ the smallest integer strictly greater than $-\mathrm{Re}\, a_z + a_1$ and $\alpha = (-\mathrm{Re}\, a_z + a_1)/N$, and by Proposition 4.3.9 using the notation of its statement. We have similar bounds for $\|\bar{\mathcal{Q}}^{\frac{b_z-b_1}{\nu_{\mathcal{Q}}}}\psi\|_{q_1}$ and all these estimates imply easily that there exists a constant depending on $\phi,\psi,a_1,a_0,b_1,b_0$ such that

$$\forall z\in S \qquad \ln|\langle T_z\phi,\psi\rangle|\leq C(1+|\mathrm{Im}\,z|).$$

For the estimate on the boundary of the strip, that is, $z=j+iy$, $j=0,1$, $y\in\mathbb{R}$, we see as in the proof of Theorem 4.4.9:

$$\|T_z\|_{\mathscr{L}(L^{p_j},L^{q_j})}\leq \|\mathcal{Q}_{q_j}^{iy\frac{b_1-b_0}{\nu_{\mathcal{Q}}}}\|_{\mathscr{L}(L^{q_j})}\|T\|_{\mathscr{L}(\dot{L}^{p_j}_{a_j},\dot{L}^{q_j}_{b_j})}\|\mathcal{R}_{p_j}^{iy\frac{a_0-a_1}{\nu_{\mathcal{R}}}}\|_{\mathscr{L}(L^{p_j})}.$$

Proposition 4.3.9 then implies

$$\|T_{j+iy}\|_{\mathscr{L}(L^{p_j},L^{q_j})}\leq C\|T\|_{\mathscr{L}(\dot{L}^{p_j}_{a_j},\dot{L}^{q_j}_{b_j})}e^{\theta_{\mathcal{R}}\frac{a_1-a_0}{\nu_{\mathcal{R}}}|y|}e^{\theta_{\mathcal{Q}}\frac{b_0-b_1}{\nu_{\mathcal{R}}}|y|},$$

where $C$, $\theta_{\mathcal{R}}$ and $\theta_{\mathcal{Q}}$ are positive constants obtained from the applications of Proposition 4.3.9 to $\mathcal{R}$ and $\mathcal{Q}$. We conclude the proof in the same way as for Theorem 4.4.9. □

### 4.4.4 Operators acting on Sobolev spaces

In this section we show that left-invariant differential operators act continuously on homogeneous and inhomogeneous Sobolev spaces. We will also show a similar property for operators of type $\nu$, $\mathrm{Re}\,\nu=0$.

In the statements and in the proofs of this section, we keep the same notation for an operator defined on a dense subset of some $L^p$-space and its possible bounded extensions to some Sobolev spaces in order to ease the notation.

**Theorem 4.4.16.** *Let $G$ be a graded Lie group.*

1. *Let $T$ be a left-invariant differential operator of homogeneous degree $\nu_T$. Then for every $p\in(1,\infty)$ and $s\in\mathbb{R}$, $T$ maps continuously $L^p_{s+\nu_T}(G)$ to $L^p_s(G)$. Fixing a positive Rockland operator $\mathcal{R}$ in order to define the Sobolev norms, it means that*

$$\exists C=C_{s,p,T}>0 \qquad \forall\phi\in\mathcal{S}(G) \qquad \|T\phi\|_{L^p_s(G)}\leq C\|\phi\|_{L^p_{s+\nu_T}(G)}.$$

2. *Let $T$ be a $\nu_T$-homogeneous left-invariant differential operator. Then for every $p\in(1,\infty)$ and $s\in\mathbb{R}$, $T$ maps continuously $\dot{L}^p_{s+\nu_T}(G)$ to $\dot{L}^p_s(G)$. Fixing a positive Rockland operator $\mathcal{R}$ in order to define the Sobolev norms, it means that*

$$\exists C=C_{s,p,T}>0 \qquad \forall\phi\in\dot{L}^p_{s+\nu_T}(G) \qquad \|T\phi\|_{\dot{L}^p_s(G)}\leq C\|\phi\|_{\dot{L}^p_{s+\nu_T}(G)}.$$

We start the proof of Theorem 4.4.16 with studying the case of $T=X_j$. This uses the definition and properties of kernel of type 0, see Section 3.2.5.

**Lemma 4.4.17.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$ and $\mathcal{I}_a$ the kernel of its Riesz operator as in Corollary 4.3.11.*

1. *For any $j = 1, \ldots, n$, $X_j\mathcal{I}_{\upsilon_j}$ is a kernel of type 0.*
2. *If $\kappa$ is a kernel of type 0, then, for any $j = 1, \ldots, n$, $X_j\big(\kappa * \mathcal{I}_{\upsilon_j}\big)$ is a kernel of type 0 and, more generally, for any multi-index $\alpha \in \mathbb{N}_0^n$, the kernel*
$$X^\alpha\Big(\kappa * \mathcal{I}_{[\upsilon_1]}^{(*)^{\alpha_1}} * \ldots * \mathcal{I}_{[\upsilon_n]}^{(*)^{\alpha_n}}\Big)$$
*is of type 0.*
3. *If $T$ is an operator of type 0, then, for any $N \in \mathbb{N}$, $\mathcal{R}^N T \mathcal{R}_2^{-N}$ is an operator of type 0 hence it is bounded on $L^p(G)$, $p \in (1, \infty)$.*
4. *For any $j = 1, \ldots, n$ and for any $N \in \mathbb{N}_0$, $\mathcal{R}^N X_j \mathcal{R}_2^{-\frac{\upsilon_j}{\nu}-N}$ is an operator of type 0.*

In Part 2, we have used the notation
$$f^{(*)^m} = \underbrace{f * \ldots * f}_{m \text{ times}}$$

*Proof of Lemma 4.4.17.* We adopt the notation of the statement. By Corollary 4.3.11 (i), $\mathcal{I}_{\upsilon_j}$ is a kernel of type $\upsilon_j \in (0, Q)$ hence, by Lemma 3.2.33, $X_j\mathcal{I}_{\upsilon_j}$ is a kernel of type 0. This shows Part 1.

More generally, if $\kappa$ is a kernel of type 0, then $\kappa * \mathcal{I}_{\upsilon_j}$ is a kernel of type $\upsilon_j$ by Proposition 3.2.35 (ii) hence by Lemma 3.2.33, $X_j(\kappa * \mathcal{I}_{\upsilon_j})$ is a kernel of type 0. Iterating this procedure shows Part 2.

Let $T$ be an operator of type 0. We denote by $\kappa$ its kernel. Let $N \in \mathbb{N}$. The operator $\mathcal{R}^N$ can be written as a linear combination of $X^\alpha$, $\alpha \in \mathbb{N}_0^n$ with $[\alpha] = \nu N$. Using the spectral calculus of $\mathcal{R}$ to define and decompose $\mathcal{R}_2^{-N}$, this shows that the operator $\mathcal{R}^N T \mathcal{R}_2^{-N}$ can be written as a linear combination over $[\alpha] = \nu N$ of the operators $X^\alpha T \mathcal{R}_2^{-\frac{\upsilon_1}{\nu}\alpha_1} \ldots \mathcal{R}_2^{-\frac{\upsilon_n}{\nu}\alpha_n}$ whose kernel can be written as $X^\alpha\big(\kappa * \mathcal{I}_{[\upsilon_1]}^{(*)^{\alpha_1}} * \ldots * \mathcal{I}_{[\upsilon_n]}^{(*)^{\alpha_n}}\big)$. Part 2 implies that the operator $\mathcal{R}^N T \mathcal{R}_2^{-N}$ is of type 0. By Theorem 3.2.30, it is a bounded operator on $L^p(G)$, $p \in (1, \infty)$. This shows Part 3.

Part 4 follows from combining Parts 1 and 3. □

We can now finish the proof of Theorem 4.4.16.

*Proof of Theorem 4.4.16.* By Lemma 4.4.17, Part 4, $\mathcal{R}^N X_j \mathcal{R}_2^{-\frac{\upsilon_j}{\nu}-N}$ is an operator of type 0, hence bounded on $L^p(G)$, $p \in (1, \infty)$. The transpose of this operator is
$$(\mathcal{R}^N X_j \mathcal{R}_2^{-\frac{\upsilon_j}{\nu}-N})^t = -\bar{\mathcal{R}}_2^{-\frac{\upsilon_j}{\nu}-N} X_j \bar{\mathcal{R}}^N,$$

since $X_j^t = -X_j$ and $\mathcal{R}^t = \bar{\mathcal{R}}$. By duality, this operator is $L^{p'}$-bounded where $\frac{1}{p'} + \frac{1}{p} = 1$. As $\bar{\mathcal{R}}$ is also a positive Rockland operator, see Lemma 4.1.11, we can exchange the rôle of $\mathcal{R}$ and $\bar{\mathcal{R}}$. Hence we have obtained that the operators $\mathcal{R}^N X_j \mathcal{R}_2^{-\frac{\upsilon_j}{\nu}-N}$ and $\mathcal{R}_2^{-\frac{\upsilon_j}{\nu}-N} X_j \mathcal{R}^N$ are bounded on $L^p(G)$ for any $p \in (1,\infty)$ and $N \in \mathbb{N}$. This shows that $X_j$ maps $\dot{L}^p_{\upsilon_j+N\nu}$ to $\dot{L}^p_{N\nu}$ and $\dot{L}^p_{-N\nu}$ to $\dot{L}^p_{-\upsilon_j-N\nu}$ continuously. The properties of interpolation, cf. Proposition 4.4.15, imply that $X_j$ maps $\dot{L}^p_{\upsilon_j+s}$ to $\dot{L}^p_s$ continuously for any $s \in \mathbb{R}$, $p \in (1,\infty)$ and $j = 1, \ldots, n$.

Interpreting any $X^\alpha$ as a composition of operators $X_j$ shows Part (2) for any $T = X^\alpha$, $\alpha \in \mathbb{N}_0^n$, with $\nu_T = [\alpha]$. As any $\nu_T$-homogeneous left-invariant differential operator is a linear combination of $X^\alpha$, $\alpha \in \mathbb{N}_0^n$, with $\nu_T = [\alpha]$, this shows Part (2).

Let us show Part (1). Let $\alpha \in \mathbb{N}_0^n$. If $s > 0$, then by Theorem 4.4.3 (4) and Part (2), we have for any $\phi \in \mathcal{S}(G)$

$$\begin{aligned}
\|X^\alpha \phi\|_{L^p_s} &\lesssim \|X^\alpha \phi\|_{L^p} + \|X^\alpha \phi\|_{\dot{L}^p_s} \\
&\lesssim \|\phi\|_{\dot{L}^p_{[\alpha]}} + \|\phi\|_{\dot{L}^p_{s+[\alpha]}} \\
&\lesssim \|\phi\|_{L^p_{[\alpha]}} + \|\phi\|_{L^p_{s+[\alpha]}} \\
&\lesssim \|\phi\|_{L^p_{s+[\alpha]}}.
\end{aligned}$$

This shows that $X^\alpha$ maps $L^p_{s+[\alpha]}$ to $L^p_s$ continuously for any $s > 0$, $p \in (1,\infty)$ and any $\alpha \in \mathbb{N}_0^n$. The transpose $(X^\alpha)^t$ of $X^\alpha$ is a linear combination of $X^\beta$, $[\beta] = [\alpha]$, and will also have the same properties. By duality, this shows that $X^\alpha$ maps $L^p_{-s}$ to $L^p_{-(s+[\alpha])}$ continuously for any $s > 0$, $p \in (1,\infty)$ and any $\alpha \in \mathbb{N}_0^n$. Together with the properties of interpolation (cf. Theorem 4.4.9), this shows that $X^\alpha$ maps $L^p_{s+[\alpha]}$ to $L^p_s$ continuously for any $s \in \mathbb{R}$, $p \in (1,\infty)$ and any $\alpha \in \mathbb{N}_0^n$.

As any left invariant differential operator can be written as a linear combination of monomials $X^\alpha$, this implies Part (1) and concludes the proof of Theorem 4.4.16. □

The ideas of the proofs above can be adapted to the proof of the following properties for the operators of type 0:

**Theorem 4.4.18.** *Let $T$ be an operator of type $\nu \in \mathbb{C}$ on a graded Lie group $G$ with $\operatorname{Re}\nu = 0$. Then for every $p \in (1,\infty)$ and $s \in \mathbb{R}$, $T$ maps continuously $L^p_s(G)$ to $L^p_s(G)$ and $\dot{L}^p_s(G)$ to $\dot{L}^p_s(G)$. Fixing a positive Rockland operator $\mathcal{R}$ in order to define the Sobolev norms, it means that there exists $C = C_{s,p,T} > 0$ satisfying*

$$\forall \phi \in \mathcal{S}(G) \qquad \|T\phi\|_{L^p_s(G)} \leq C\|\phi\|_{L^p_s(G)}$$

*and*

$$\forall \phi \in \dot{L}^p_s \qquad \|T\phi\|_{\dot{L}^p_s(G)} \leq C\|\phi\|_{\dot{L}^p_s(G)}.$$

*Proof.* Let $T$ be a operator of type $\nu_T \in \mathbb{C}$ with $\operatorname{Re}\nu_T = 0$. Proceeding as in the proof of Lemma 4.4.17 Part 3 yields that for any $N \in \mathbb{N}$, the operator $\mathcal{R}^N T \mathcal{R}_2^{-N}$

is of type $\nu_T$. We can apply this to the transpose $T^t$ of $T$ as well as the operator $T^t$ is also of type $\nu$. By Theorem 3.2.30, the operators $\mathcal{R}^N T \mathcal{R}_2^{-N}$ and $\mathcal{R}^N T^t \mathcal{R}_2^{-N}$ are bounded on $L^p(G)$. This shows that $T$ maps $\dot{L}^p_s$ to $\dot{L}^p_s$ continuously for $s = N$ and $s = -N$, $N \in \mathbb{N}_0$. By interpolation, this holds for any $s \in \mathbb{R}$ and this shows the statement for the homogeneous Sobolev spaces. If $s > 0$, then by Theorem 4.4.3 (4), using the continuity on homogeneous Sobolev spaces which has just been proven, we have for any $\phi \in \mathcal{S}(G)$

$$\|T\phi\|_{L^p_s} \lesssim \|T\phi\|_{L^p} + \|T\phi\|_{\dot{L}^p_s} \lesssim \|\phi\|_{L^p} + \|\phi\|_{\dot{L}^p_s} \lesssim \|\phi\|_{L^p_s}.$$

This shows that $T$ maps $L^p_s$ to $L^p_s$ continuously for any $s > 0$, $p \in (1, \infty)$. Applying this to $T^t$, by duality, we also obtain this property for $s < 0$. The case $s = 0$ follows from Theorem 3.2.30. This concludes the proof of Theorem 4.4.18. □

Theorem 4.4.18 extends the result of Theorem 3.2.30, that is, the boundedness on $L^p(G)$ of an operator of type $\nu_T$, $\operatorname{Re}\nu_T = 0$, from $L^p$-spaces to Sobolev spaces. Let us comment on similar results in related contexts:

- In the case of $\mathbb{R}^n$ (and similarly for compact Lie groups), the continuity on Sobolev spaces would be easy since $T_\kappa$ would commute with the Laplace operator but the homogeneous setting requires a more substantial argument.
- Theorem 4.4.18 was shown by Folland in [Fol75, Theorem 4.9] on any stratified Lie group and for $\nu = 0$. However, the proof in that context uses the existence of a positive Rockland operator with a unique homogeneous fundamental solution, namely 'the' (any) sublaplacian. If we wanted to follow closely the same line of arguments, we would have to assume that the group is equipped with a Rockland operator with homogeneous degree $\nu$ with $\nu < Q$, see Remark 4.3.12. This is not always the case for a graded Lie group as the example of the three dimensional Heisenberg group with gradation (3.1) shows.
- The proof above is valid under no restriction in the graded case. Somehow the use of the homogeneous fundamental solution in the stratified case is replaced by the kernel of the Riesz potentials together with the properties of the Sobolev spaces proved so far.

### 4.4.5 Independence in Rockland operators and integer orders

In this Section, we show that the homogeneous and inhomogeneous Sobolev spaces do not depend on a particular choice of a Rockland operator. Consequently Theorems 4.4.3, 4.4.9, 4.4.16, and 4.4.18, Corollaries 4.4.6 and 4.4.10, Propositions 4.4.8 and 4.4.13 and 4.4.15, hold independently of any chosen Rockland operator $\mathcal{R}$.

We will need the following property:

**Lemma 4.4.19.** *Let $\mathcal{R}$ be a Rockland operator on $G$ of homogeneous degree $\nu$ and let $\ell \in \mathbb{N}_0$, $p \in (1, \infty)$. Then the space $L^p_{\nu\ell}(G)$ is the collection of functions $f \in L^p(G)$ such that $X^\alpha f \in L^p(G)$ for any $\alpha \in \mathbb{N}_0^n$ with $[\alpha] = \nu\ell$. Moreover, the map*

$$\phi \mapsto \sum_{[\alpha]=\nu\ell} \|X^\alpha \phi\|_p$$

*is a norm on $\dot{L}^p_{\nu\ell}(G)$ which is equivalent to the homogeneous Sobolev norm and the map*

$$\phi \mapsto \|\phi\|_p + \sum_{[\alpha]=\nu\ell} \|X^\alpha \phi\|_p$$

*is a norm on $L^p_{\nu\ell}(G)$ which is equivalent to the Sobolev norm.*

*Proof of Lemma 4.4.19.* Writing

$$\mathcal{R}^\ell = \sum_{[\alpha]=\ell\nu} c_{\alpha,\ell} X^\alpha$$

we have on one hand,

$$\forall \phi \in \mathcal{S}(G) \qquad \|\mathcal{R}^\ell \phi\|_p \leq \max |c_{\alpha,\ell}| \sum_{[\alpha]=\ell\nu} \|X^\alpha \phi\|_p. \tag{4.43}$$

On the other hand, by Theorem 4.4.16 (2), for any $\alpha \in \mathbb{N}_0^n$, the operator $X^\alpha$ maps continuously $\dot{L}^p_{[\alpha]}(G)$ to $\dot{L}^p(G)$, hence

$$\exists C > 0 \quad \forall \phi \in \mathcal{S}(G) \qquad \sum_{[\alpha]=\ell\nu} \|X^\alpha \phi\|_p \leq C \|\phi\|_{\dot{L}^p_{[\alpha]}}.$$

This shows the property of Lemma 4.4.19 for homogeneous Sobolev spaces.

Adding $\|\phi\|_{L^p}$ on both sides of (4.43) implies by Theorem 4.4.3, Part (2):

$$\exists C > 0 \quad \forall \phi \in \mathcal{S}(G) \qquad \|\phi\|_{L^p_{\ell\nu}} \leq C \left( \|\phi\|_{L^p} + \sum_{[\alpha]=\ell\nu} \|X^\alpha \phi\|_p \right).$$

On the other hand, by Theorem 4.4.16 (1), for any $\alpha \in \mathbb{N}_0^n$, the operator $X^\alpha$ maps continuously $L^p_{[\alpha]}(G)$ to $L^p(G)$, hence

$$\exists C > 0 \quad \forall \phi \in \mathcal{S}(G) \qquad \sum_{[\alpha]=\ell\nu} \|X^\alpha \phi\|_p \leq C \|\phi\|_{L^p_{[\alpha]}}.$$

This shows the property of Lemma 4.4.19 for inhomogeneous Sobolev spaces and concludes the proof of Lemma 4.4.19. $\square$

One may wonder whether Lemma 4.4.19 would be true not only for integer exponents of the form $s = \nu\ell$ but for any integer $s$. In fact other inhomogeneous Sobolev spaces on a graded Lie group were defined by Goodman in [Goo76, Section III. 5.4] following this idea. More precisely the $L^p$ Goodman-Sobolev space of order $s \in \mathbb{N}_0$ is given via the norm

$$\phi \longmapsto \sum_{[\alpha]\leq s} \|X^\alpha \phi\|_p \tag{4.44}$$

Goodman's definition does not use Rockland operators but makes sense only for integer exponents.

The $L^p$ Goodman-Sobolev space of integer order $s$ certainly contains $L^p_s(G)$. Indeed, proceeding almost as in the proof of Lemma 4.4.19, using Theorem 4.4.16 and Theorem 4.4.3, we have

$$\forall s \in \mathbb{N}_0 \quad \exists C = C_s > 0 \quad \forall \phi \in \mathcal{S}(G) \quad \sum_{[\alpha]\leq s} \|X^\alpha \phi\|_p \leq C\|\phi\|_{L^p_s}.$$

In fact, adapting the rest of the proof of Lemma 4.4.19, one could show easily that the $L^p$ Goodman-Sobolev space of order $s \in \mathbb{N}_0$ with $s$ proportional to the homogeneous degree $\nu$ of a positive Rockland operator coincides with our Sobolev spaces $L^p_s(G)$. Moreover, on any stratified Lie group, for any non-negative integer $s$ without further restriction, they would coincide as well, see [Fol75, Theorem 4.10].

However, this equality between Goodman-Sobolev spaces and our Sobolev spaces is not true on any general graded Lie group. For instance this does not hold on a graded Lie groups whose weights are all strictly greater than 1. Indeed the $L^p$ Goodman-Sobolev space of order $s = 1$ is $L^p(G)$ which contains $L^p_1(G)$ strictly (see Theorem 4.4.3 (4)). An example of such a graded Lie group was given by the gradation of the three dimensional Heisenberg group via (3.1).

We can now show the main result of this section, that is, that the Sobolev spaces on graded Lie groups are independent of the chosen positive Rockland operators.

**Theorem 4.4.20.** *Let $G$ be a graded Lie group and $p \in (1, \infty)$. The homogeneous $L^p$-Sobolev spaces on $G$ associated with any positive Rockland operators coincide. The inhomogeneous $L^p$-Sobolev spaces on $G$ associated with any positive Rockland operators coincide. Moreover, in the homogeneous and inhomogeneous cases, the Sobolev norms associated to two positive Rockland operators are equivalent.*

*Proof of Theorem 4.4.20.* Positive Rockland operators always exist, see Remark 4.2.4 Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two positive Rockland operators on $G$ of homogeneous degrees $\nu_1$ and $\nu_2$, respectively. By Lemma 4.2.5, $\mathcal{R}_1^{\nu_2}$ and $\mathcal{R}_2^{\nu_1}$ are two positive Rockland operators with the same homogeneous degree $\nu = \nu_1\nu_2$. Their associated homogeneous (respectively inhomogeneous) Sobolev spaces of exponent $\nu\ell = \nu_1\nu_2\ell$

for any $\ell \in \mathbb{N}_0$ coincide and have equivalent norms by Lemma 4.4.19. By interpolation (see Proposition 4.4.15, respectively Theorem 4.4.9), this is true for any Sobolev spaces of exponent $s \geq 0$, and by duality (see Lemma 4.4.14, respectively Lemma 4.4.7) for any exponent $s \in \mathbb{R}$. □

**Corollary 4.4.21.** *Let $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ be two positive Rockland operators on a graded Lie group $G$ with degrees of homogeneity $\nu_1$ and $\nu_2$, respectively. Then for any $s \in \mathbb{C}$ and $p \in (1, \infty)$, the operators $(\mathrm{I} + \mathcal{R}^{(1)})^{\frac{s}{\nu_1}} (\mathrm{I} + \mathcal{R}^{(2)})^{-\frac{s}{\nu_2}}$ and $(\mathcal{R}^{(1)})^{\frac{s}{\nu_1}} (\mathcal{R}^{(2)})^{-\frac{s}{\nu_2}}$ extend boundedly on $L^p(G)$.*

*Proof of Corollary 4.4.21.* Let us prove the inhomogeneous case first. For any $a \in \mathbb{R}$, we view the operator $(\mathrm{I}+\mathcal{R}_p^{(2)})^{-\frac{a}{\nu_2}}$ as a bounded operator from $L^p(G)$ to $L^p_a(G)$ and use the norm $f \mapsto \|(\mathrm{I} + \mathcal{R}_p^{(1)})^{\frac{a}{\nu_1}} f\|_p$ on $L^p_a(G)$. This shows that the operator $(\mathrm{I} + \mathcal{R}^{(1)})^{\frac{s}{\nu_1}} (\mathrm{I} + \mathcal{R}^{(2)})^{-\frac{s}{\nu_2}}$ is bounded on $L^p(G)$, $p \in (1, \infty)$ for $s = a \in \mathbb{R}$. The case of $s \in \mathbb{C}$ follows from Proposition 4.3.7.

Let us prove the homogeneous case. For any $a \in \mathbb{R}$, we view the operator $(\mathcal{R}_p^{(2)})^{-\frac{a}{\nu_2}}$ as a bounded operator from $L^p(G)$ to $\dot{L}^p_a(G)$ and use the norm $f \mapsto \|(\mathcal{R}_p^{(1)})^{\frac{a}{\nu_1}} f\|_p$ on $\dot{L}^p_a(G)$. This shows that the operator $(\mathcal{R}^{(1)})^{\frac{s}{\nu_1}} (\mathcal{R}^{(2)})^{-\frac{s}{\nu_2}}$ is bounded on $L^p(G)$, $p \in (1, \infty)$ for $s = a \in \mathbb{R}$. The case of $s \in \mathbb{C}$ follows from Proposition 4.3.9. □

Thanks to Theorem 4.4.20, we can now improve our duality result given in Lemmata 4.4.7 and 4.4.14:

**Proposition 4.4.22.** *Let $L^p_s(G)$ and $\dot{L}^p_s(G)$, $p \in (1, \infty)$ and $s \in \mathbb{R}$, be the inhomogeneous and homogeneous Sobolev spaces on a graded Lie group $G$, respectively.*

*For any $s \in \mathbb{R}$ and $p \in (1, \infty)$, the dual space of $L^p_s(G)$ is isomorphic to $L^{p'}_{-s}(G)$ via the distributional duality, and the dual space of $\dot{L}^p_s(G)$ is isomorphic to $\dot{L}^{p'}_{-s}(G)$ via the distributional duality. Here $p'$ is the conjugate exponent of $p$ if $p \in (1, \infty)$, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Consequently the Banach spaces $L^p_s(G)$ and $\dot{L}^p_s(G)$ are reflexive.*

### 4.4.6 Sobolev embeddings

In this section, we show local embeddings between the (inhomogeneous) Sobolev spaces and their Euclidean counterparts, and global embeddings in the form of an analogue of the classical fractional integration theorems of Hardy-Littlewood and Sobolev.

#### Local results

Recalling that $G$ has a local topological structure of $\mathbb{R}^n$, one can wonder what is the relation between our Sobolev spaces $L^p_s(G)$ and their Euclidean counterparts $L^p_s(\mathbb{R}^n)$. The latter can also be seen as Sobolev spaces associated by the described construction to the abelian group $(\mathbb{R}^n, +)$, with Rockland operator being the Laplacian on $\mathbb{R}^n$.

By Proposition 3.1.28 the coefficients of vector fields $X_j$ with respect to the abelian derivatives $\partial_{x_k}$ are polynomials in the coordinate functions $x_\ell$, and conversely the coefficients of $\partial_{x_j}$'s with respect to derivatives $X_k$ are polynomials in the coordinate functions $x_\ell$'s. Hence, we can not expect any global embeddings between $L^p_s(G)$ and $L^p_s(\mathbb{R}^n)$.

It is convenient to define the local Sobolev spaces for $s \in \mathbb{R}$ and $p \in (1, \infty)$ as

$$L^p_{s,loc}(G) := \{f \in \mathcal{D}'(G) : \phi f \in L^p_s(G) \text{ for all } \phi \in \mathcal{D}(G)\}. \tag{4.45}$$

The following proposition shows that $L^p_{s,loc}(G)$ contains $L^p_s(G)$.

**Proposition 4.4.23.** *For any $\phi \in \mathcal{D}(G)$, $p \in (1, \infty)$ and $s \in \mathbb{R}$, the operator $f \mapsto f\phi$ defined for $f \in \mathcal{S}(G)$ extends continuously into a bounded map from $L^p_s(G)$ to itself. Consequently, we have*

$$L^p_s(G) \subset L^p_{s,loc}(G).$$

*Proof.* The Leibniz' rule for the $X_j$'s and the continuous inclusions in Theorem 4.4.3 (4) imply easily that for any fixed $\alpha \in \mathbb{N}_0^n$ there exist a constant $C = C_{\alpha,\phi} > 0$ and a constant $C' = C'_{\alpha,\phi} > 0$ such that

$$\forall f \in \mathcal{D}(G) \quad \|X^\alpha (f\phi)\|_p \le C \sum_{[\beta] \le [\alpha]} \|X^\beta f\|_p \le C' \|f\|_{L^p_{[\alpha]}(G)}.$$

Lemma 4.4.19 yields the existence of a constant $C'' = C''_{\alpha,\phi} > 0$ such that

$$\forall f \in \mathcal{D}(G) \quad \|f\phi\|_{L^p_{\ell\nu}(G)} \le C'' \|f\|_{L^p_{\ell\nu}(G)}$$

for any integer $\ell \in \mathbb{N}_0$ and any degree of homogeneity $\nu$ of a Rockland operator.

This shows the statement for the case $s = \nu\ell$. The case $s > 0$ follows by interpolation (see Theorem 4.4.9), and the case $s < 0$ by duality (see Proposition 4.4.22). □

We can now compare locally the Sobolev spaces on graded Lie groups and on their abelian counterpart:

**Theorem 4.4.24** (Local Sobolev embeddings)**.** *For any $p \in (1, \infty)$ and $s \in \mathbb{R}$,*

$$L^p_{s/\upsilon_1,loc}(\mathbb{R}^n) \subset L^p_{s,loc}(G) \subset L^p_{s/\upsilon_n,loc}(\mathbb{R}^n).$$

Above, $L^p_{s,loc}(\mathbb{R}^n)$ denotes the usual local Sobolev spaces, or equivalently the spaces defined by (4.45) in the case of the abelian (graded) Lie group $(\mathbb{R}^n, +)$. Recall that $\upsilon_1$ and $\upsilon_n$ are respectively the smallest and the largest weights of the dilations. In particular, in the stratified case, $\upsilon_1 = 1$ and $\upsilon_n$ coincides with the number of steps in the stratification, and with the step of the nilpotent Lie group $G$. Hence in the stratified case we recover Theorem 4.16 in [Fol75].

*Proof of Theorem 4.4.24.* It suffices to show that the mapping $f \mapsto f\phi$ defined on $\mathcal{D}(G)$ extends boundedly from $L^p_{s/\upsilon_1}(\mathbb{R}^n)$ to $L^p_s(G)$ and from $L^p_s(G)$ to $L^p_{s/\upsilon_n,loc}(\mathbb{R}^n)$. By duality and interpolation (see Theorem 4.4.9 and Proposition 4.4.22), it suffices to show this for a sequence of increasing positive integers $s$.

For the $L^p_{s/\upsilon_1}(\mathbb{R}^n) \to L^p_s(G)$ case, we assume that $s$ is divisible by the homogeneous degree of a positive Rockland operator. Then we use Lemma 4.4.19, the fact that the $X^\alpha$ may be written as a combination of the $\partial_x^\beta$ with polynomial coefficients in the $x_\ell$'s and that $\max_{[\beta]\leq s}|\beta| = s/\upsilon_1$.

For the case of $L^p_s(G) \to L^p_{s/\upsilon_n,loc}(\mathbb{R}^n)$, we use the fact that the abelian derivative $\partial_x^\alpha$, $|\alpha| \leq s$, may be written as a combination over the $X^\beta$, $|\beta| \leq s$, with polynomial coefficients in the $x_\ell$'s, that $X^\beta$ maps $L^p \to L^p_{[\beta]}$ boundedly together with $\max_{|\beta|\leq s}[\beta] = s\upsilon_n$. □

Proceeding as in [Fol75, p.192], one can convince oneself that Theorem 4.4.24 can not be improved.

### Global results

In this section, we show the analogue of the classical fractional integration theorems of Hardy-Littlewood and Sobolev. The stratified case was proved by Folland in [Fol75] (mainly Theorem 4.17 therein).

**Theorem 4.4.25** (Sobolev embeddings)**.** *Let $G$ be a graded Lie group with homogeneous dimension $Q$.*

(i) *If $1 < p < q < \infty$ and $a, b \in \mathbb{R}$ with*

$$b - a = Q(\frac{1}{p} - \frac{1}{q})$$

*then we have the continuous inclusion*

$$L^p_b \subset L^q_a,$$

*that is, for every $f \in L^p_b$, we have $f \in L^q_a$ and there exists a constant $C = C_{a,b,p,q,G} > 0$ independent of $f$ such that*

$$\|f\|_{L^q_a} \leq C\|f\|_{L^p_b}.$$

(ii) *If $p \in (1,\infty)$ and*

$$s > Q/p$$

*then we have the inclusion*

$$L^p_s \subset (C(G) \cap L^\infty(G)),$$

*in the sense that any function $f \in L^p_s(G)$ admits a bounded continuous representative on $G$ (still denoted by $f$). Furthermore, there exists a constant $C = C_{s,p,G} > 0$ independent of $f$ such that*

$$\|f\|_\infty \leq C \|f\|_{L^p_s(G)}.$$

*Proof.* Let us first prove Part (i). We fix a positive Rockland operator $\mathcal{R}$ of homogeneous degree $\nu$ and we assume that $b > a$ and $p, q \in (1,\infty)$ satisfy $b-a = Q(\frac{1}{p} - \frac{1}{q})$. By Proposition 4.4.13 (5),

$$\|\mathcal{R}_q^{\frac{a}{\nu}} \phi\|_{L^q} \leq C \|\mathcal{R}_p^{\frac{b}{\nu}} \phi\|_{L^p}.$$

We can apply this to $(a, b)$ and to $(0, b-a)$. Adding the two corresponding estimates, we obtain

$$\|\phi\|_{L^q} + \|\mathcal{R}_q^{\frac{a}{\nu}} \phi\|_{L^q} \leq C \left( \|\mathcal{R}_p^{\frac{b-a}{\nu}} \phi\|_{L^p} + \|\mathcal{R}_p^{\frac{b}{\nu}} \phi\|_{L^p} \right).$$

Since $b$, $a$, and $b-a$ are positive, by Theorem 4.4.3 (4), the left-hand side is equivalent to $\|\phi\|_{L^q_a}$ and both terms in the right-hand side are $\leq C\|\phi\|_{L^p_b}$. Therefore, we have obtained that

$$\exists C = C_{a,b,p,q,\mathcal{R}} \quad \forall \phi \in \mathcal{S}(G) \qquad \|\phi\|_{L^q_a} \leq C \|\phi\|_{L^p_b}.$$

By density of $\mathcal{S}(G)$ in the Sobolev spaces, this shows Part (i).

Let us prove Part (ii). Let $p \in (1,\infty)$ and $s > Q/p$. By Corollary 4.3.13, we know that

$$\mathcal{B}_s \in L^1(G) \cap L^{p'}(G),$$

where $p'$ is the conjugate exponent of $p$. For any $f \in L^p_s(G)$, we have

$$f_s := (\mathrm{I} + \mathcal{R}_p)^{\frac{s}{\nu}} f \in L^p$$

and

$$f = (\mathrm{I} + \mathcal{R}_p)^{-\frac{s}{\nu}} f_s = f_s * \mathcal{B}_s.$$

Therefore, by Hölder's inequality,

$$\|f\|_\infty \leq \|f_s\|_p \|\mathcal{B}_s\|_{p'} = \|\mathcal{B}_s\|_{p'} \|f\|_{L^p_s}.$$

Moreover, for almost every $x$, we have

$$f(x) = \int_G f_s(y) \mathcal{B}_s(y^{-1}x) dy = \int_G f_s(xz^{-1}) \mathcal{B}_s(z) dz.$$

Thus for almost every $x, x'$, we have

$$\begin{aligned} |f(x) - f(x')| &= \left| \int_G \left( f_s(xz^{-1}) - f_s(x'z^{-1}) \right) \mathcal{B}_s(z) dz \right| \\ &\leq \|\mathcal{B}_s\|_{p'} \|f_s(x\,\cdot) - f_s(x'\,\cdot)\|_p. \end{aligned}$$

As the left regular representation is continuous (see Example 1.1.2) we have

$$\|f_s(x\,\cdot) - f_s(x'\,\cdot)\|_{L^p(G)} \longrightarrow_{x'\to x} 0,$$

thus almost surely

$$|f(x) - f(x')| \longrightarrow_{x'\to x} 0.$$

Hence we can modify $f$ so that it becomes a continuous function. This concludes the proof. □

From the Sobolev embeddings (Theorem 4.4.25 (ii)) and the description of Sobolev spaces with integer exponent (Lemma 4.4.19) the following property follows easily:

**Corollary 4.4.26.** *Let $G$ be a graded Lie group, $p \in (1,\infty)$ and $s \in \mathbb{N}$. We assume that $s$ is proportional to the homogeneous degree $\nu$ of a positive Rockland operator, that is, $\frac{s}{\nu} \in \mathbb{N}$, and that $s > Q/p$.*

*Then if $f$ is a distribution on $G$ such that $f \in L^p(G)$ and $X^\alpha f \in L^p(G)$ when $\alpha \in \mathbb{N}_0^n$ satisfies $[\alpha] = s$, then $f$ admits a bounded continuous representative (still denoted by $f$). Furthermore, there exists a constant $C = C_{s,p,G} > 0$ independent of $f$ such that*

$$\|f\|_\infty \le C \left( \|f\|_p + \sum_{[\alpha]=s} \|X^\alpha f\|_p \right).$$

The Sobolev embeddings, especially Corollary 4.4.26, enables us to define Schwartz seminorms not only in terms of the supremum norm, but also in terms of any $L^p$-norms:

**Proposition 4.4.27.** *Let $|\cdot|$ be a homogeneous norm on a graded Lie group $G$. For any $p \in [1,\infty]$, $a > 0$ and $k \in \mathbb{N}_0$, the mapping*

$$\mathcal{S}(G) \ni \phi \mapsto \|\phi\|_{\mathcal{S},a,k,p} := \sum_{[\alpha]\le k} \|(1+|\cdot|)^a X^\alpha \phi\|_p$$

*is a continuous seminorm on the Fréchet space $\mathcal{S}(G)$.*

*Moreover, let us fix $p \in [1,\infty]$ and two sequences $\{k_j\}_{j\in\mathbb{N}}$, $\{a_j\}_{j\in\mathbb{N}}$, of non-negative integers and positive numbers, respectively, which go to infinity. Then the family of seminorms $\|\cdot\|_{\mathcal{S},a_j,k_j,p}$, $j \in \mathbb{N}$, yields the usual topology on $\mathcal{S}(G)$.*

*Proof of Proposition 4.4.27.* One can check easily that the property

$$\begin{aligned} \forall 1 \le p,q \le \infty,\ a>0,\ k\in\mathbb{N}_0, \quad \exists a'>0,\ k'\in\mathbb{N}_0,\ C>0, \\ \|\cdot\|_{\mathcal{S},a,k,p} \le \|\cdot\|_{\mathcal{S},a',k',q}, \end{aligned} \tag{4.46}$$

is a consequence of the following observations (applied to $X^\alpha\phi$ instead of $\phi$):

1. If $p$ and $q$ are finite, by Hölder's inequality, we have

$$\|(1+|\cdot|)^a\phi\|_p \le C\|(1+|\cdot|)^{a'}\phi\|_q$$

where $C$ is a finite constant of the group $G$, $p$ and $q$. In fact $C$ is explicitly given by

$$C = \|(1+|\cdot|)^{-\frac{Q+1}{r}}\|_r = \left(|B(0,1)|\int_0^\infty (1+\rho)^{-(Q+1)}\rho^{Q-1}d\rho\right)^{\frac{1}{r}},$$

with $r \in (1,\infty)$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

2. If $p$ is finite and $q = \infty$, we also have

$$\|(1+|\cdot|)^a\phi\|_p \le C\|(1+|\cdot|)^{a+Q+1}\phi\|_\infty$$

where $C = \|(1+|\cdot|)^{-Q-1}\|_p$ is a finite constant.

3. In the case $q$ is finite and $p = \infty$, let us prove that

$$\|(1+|\cdot|)^a\phi\|_\infty \le C_{s,p}\sum_{[\alpha]\le s}\|(1+|\cdot|)^a X^\alpha\phi\|_p. \tag{4.47}$$

Indeed first we notice that, by equivalence of the homogeneous quasi-norms (see Proposition 3.1.35), we may assume that the quasi-norm is smooth away from 0. We fix a function $\psi \in \mathcal{D}(G)$ such that

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

We have easily

$$\|(1+|\cdot|)^a\phi\|_\infty \le C_\psi\left(\|\phi\psi\|_\infty + \|\phi(1-\psi)|\cdot|^a\|_\infty\right). \tag{4.48}$$

By Corollary 4.4.26, there exist an integer $s \in \mathbb{N}$ such that

$$\|\phi\psi\|_\infty \le C_{s,p}\sum_{[\alpha]\le s}\|X^\alpha(\phi\psi)\|_p.$$

By the Leibniz rule (which is valid for any vector field) and Hölder's inequality, we have

$$\begin{aligned}\|X^\alpha(\phi\psi)\|_p &\le C_\alpha \sum_{[\alpha_1]+[\alpha_2]\le[\alpha]}\|X^{\alpha_1}\phi\, X^{\alpha_2}\psi\|_p \\ &\le C_{\alpha,p}\sum_{[\alpha_1]+[\alpha_2]\le[\alpha]}\|X^{\alpha_1}\phi\|_p\|X^{\alpha_2}\psi\|_\infty.\end{aligned}$$

Hence
$$\|\phi\psi\|_\infty \leq C_{s,p,\psi} \sum_{[\alpha]\leq s} \|X^\alpha \phi\|_p. \tag{4.49}$$

Following the same line of arguments, we have

$$\begin{aligned}\|\phi(1-\psi)|\cdot|^a\|_\infty &\leq C_{s,p} \sum_{[\alpha]\leq s} \|X^\alpha(\phi(1-\psi)|\cdot|^a)\|_p \\ &\leq C_{s,p} \sum_{[\alpha_1]+[\alpha_2]\leq s} \|X^{\alpha_1}\phi\, X^{\alpha_2}\{(1-\psi)|\cdot|^a\}\|_p \\ &\leq C_{s,p} \sum_{[\alpha_1]+[\alpha_2]\leq s} \|(1+|\cdot|)^a X^{\alpha_1}\phi\|_p \|(1+|\cdot|)^{-a} X^{\alpha_2}\{(1-\psi)|\cdot|^a\}\|_\infty.\end{aligned}$$

All the $\|\cdot\|_\infty$-norms above are finite since $X^{\alpha_2}\{(1-\psi)|\cdot|^a\}(x) = 0$ if $|x|\leq 1$ and for $|x|\geq 1$,

$$\begin{aligned}|X^{\alpha_2}\{(1-\psi)|\cdot|^a\}(x)| &\leq C_{\alpha_2} \sum_{[\alpha_3]+[\alpha_4]=[\alpha_2]} |X^{\alpha_3}(1-\psi)(x)|\, |X^{\alpha_4}|\cdot|^a|(x) \\ &\leq C_{\alpha_2} \sum_{[\alpha_3]+[\alpha_4]=[\alpha_2]} \|X^{\alpha_3}(1-\psi)\|_\infty |x|^{a-[\alpha_4]},\end{aligned}$$

since $X^{\alpha_4}|\cdot|^a$ is a homogeneous function of degree $a-[\alpha_4]$. Hence we have obtained

$$\|\phi(1-\psi)|\cdot|^a\|_\infty \leq C_{s,p,\psi} \sum_{[\alpha]\leq s} \|(1+|\cdot|)^a X^\alpha \phi\|_p.$$

Together with (4.48) and (4.49), this shows (4.47).

4. If $p=q$ is finite or infinite, (4.46) is trivial.

Hence Property (4.46) holds. We also have directly for $p=q\in[1,\infty]$ and any $0<a\leq a'$, $k\leq k'$,
$$\|\cdot\|_{\mathcal{S},a,k,p} \leq \|\cdot\|_{\mathcal{S},a',k',p}.$$
Consequently we can assume $a'$ to be an integer in (4.46). This clearly implies that any family of seminorms $\|\cdot\|_{\mathcal{S},a_j,k_j,p}$, $j\in\mathbb{N}$, yields the same topology as the family of seminorms $\|\cdot\|_{\mathcal{S},N,N,\infty}$, $N\in\mathbb{N}$. The latter is easily equivalent to the topology given by the family of seminorms $\|\cdot\|_{\mathcal{S}(G),N}$ defined in Section 3.1.9. This is the usual topology on $\mathcal{S}(G)$. □

### 4.4.7 List of properties for the Sobolev spaces

In this section, we list the important properties of Sobolev spaces we have already obtained and also give some easy consequences regarding the special case of $p=2$.

**Theorem 4.4.28.** *Let $G$ be a graded Lie group with homogeneous dimension $Q$.*

1. *Let $p \in (1, \infty)$ and $s \in \mathbb{R}$. The inhomogeneous Sobolev space $L^p_s(G)$ is a Banach space satisfying*

$$\mathcal{S}(G) \subsetneq L^p_s(G) \subset \mathcal{S}'(G).$$

   *The homogeneous Sobolev space $\dot{L}^p_s(G)$ is a Banach space satisfying*

$$(\mathcal{S}(G) \cap \mathrm{Dom}(\mathcal{R}_p^{s/\nu})) \subsetneq \dot{L}^p_s(G) \subsetneq \mathcal{S}'(G).$$

   *Norms on the Banach spaces $L^p_s(G)$ and $\dot{L}^p_s(G)$ are given respectively by*

$$\phi \mapsto \|(\mathrm{I} + \mathcal{R}_p)^{\frac{s}{\nu}} \phi\|_{L^p(G)} \qquad \text{and} \qquad \phi \mapsto \|\mathcal{R}_p^{\frac{s}{\nu}} \phi\|_{L^p(G)},$$

   *for any positive Rockland operator $\mathcal{R}$ (whose homogeneous degree is denoted by $\nu$). All these homogeneous norms are equivalent, all these inhomogeneous norms are equivalent.*

   *The continuous inclusions $L^p_a(G) \subset L^p_b(G)$ holds for any $a \geq b$ and $p \in (1, \infty)$.*

2. *If $s = 0$ and $p \in (1, \infty)$, then $\dot{L}^p_0(G) = L^p_0(G) = L^p(G)$ with $\| \cdot \|_{\dot{L}^p_0(G)} = \| \cdot \|_{L^p_0(G)} = \| \cdot \|_{L^p(G)}$.*

3. *If $s > 0$ and $p \in (1, \infty)$, then we have*

$$L^p_s(G) = \dot{L}^p_s(G) \cap L^p(G),$$

   *and the inhomogeneous Sobolev norm (associated with a positive Rockland operator) is equivalent to*

$$\| \cdot \|_{L^p_s(G)} \asymp \| \cdot \|_{L^p(G)} + \| \cdot \|_{\dot{L}^p_s(G)}.$$

4. *If $T$ is a left-invariant differential operator of homogeneous degree $\nu_T$, then $T$ maps continuously $L^p_{s+\nu_T}(G)$ to $L^p_s(G)$ for every $s \in \mathbb{R}$, $p \in (1, \infty)$.*

   *If $T$ is a $\nu_T$-homogeneous left-invariant differential operator, then $T$ maps continuously $\dot{L}^p_{s+\nu_T}(G)$ to $\dot{L}^p_s(G)$ for every $s \in \mathbb{R}$, $p \in (1, \infty)$.*

5. *If $1 < p < q < \infty$ and $a, b \in \mathbb{R}$ with $b - a = Q(\frac{1}{p} - \frac{1}{q})$, then we have the continuous inclusions*

$$\dot{L}^p_b \subset \dot{L}^q_a \qquad \text{and} \qquad L^p_b \subset L^q_a.$$

   *If $p \in (1, \infty)$ and $s > Q/p$ then we have the following inclusion:*

$$L^p_s \subset (C(G) \cap L^\infty(G)),$$

*in the sense that any function $f \in L^p_s(G)$ admits a bounded continuous representative on $G$ (still denoted by $f$). Furthermore, there exists a constant $C = C_{s,p,G} > 0$ independent of $f$ such that*

$$\|f\|_\infty \le C\|f\|_{L^p_s(G)}.$$

6. *For $p \in (1,\infty)$ and any $a, b, c \in \mathbb{R}$ with $a < c < b$, there exists a positive constant $C = C_{a,b,c}$ such that we have for any $f \in \dot{L}^p_b$*

$$\|f\|_{\dot{L}^p_c} \le C\|f\|^{1-\theta}_{\dot{L}^p_a}\|f\|^{\theta}_{\dot{L}^p_b}$$

   *and for any $f \in L^p_b$*

$$\|f\|_{L^p_c} \le C\|f\|^{1-\theta}_{L^p_a}\|f\|^{\theta}_{L^p_b}$$

   *where $\theta := (c-a)/(b-a)$.*

7. *(Gagliardo-Nirenberg inequality) If $q, r \in (1,\infty)$ and $0 < \sigma < s$ then there exists $C > 0$ such that we have*

$$\forall f \in L^q(G) \cap \dot{L}^r_s(G) \qquad \|f\|_{\dot{L}^p_\sigma} \le C\|f\|^{\theta}_{L^q}\|f\|^{1-\theta}_{\dot{L}^r_s},$$

   *where $\theta := 1 - \frac{\sigma}{s}$ and $p \in (1,\infty)$ is given via $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}$.*

8. *Let $s$ be an integer which is proportional to the homogeneous degree of a positive Rockland operator. Let $p \in (1,\infty)$. Let $f \in \mathcal{S}'(G)$.*

   *The membership of $f$ in $L^p_s(G)$ is equivalent to $f \in L^p(G)$ and $X^\alpha f \in L^p(G)$, $\alpha \in \mathbb{N}_0^n$, $[\alpha] = s$. Furthermore*

$$\phi \mapsto \|\phi\|_p + \sum_{[\alpha]=\nu\ell} \|X^\alpha \phi\|_p$$

   *is a norm on the Banach space $L^p_s(G)$.*

   *The membership of $f$ in $\dot{L}^p_s(G)$ is equivalent to $X^\alpha f \in L^p(G)$, $\alpha \in \mathbb{N}_0^n$, $[\alpha] = s$. Furthermore*

$$\phi \mapsto \sum_{[\alpha]=\nu\ell} \|X^\alpha \phi\|_p$$

   *is a norm on the Banach space $\dot{L}^p_s(G)$.*

9. *(Interpolation) The inhomogeneous and homogeneous Sobolev spaces satisfy the properties of interpolation in the sense of Theorem 4.4.9 and Proposition 4.4.15 respectively.*

10. *(Duality) Let $s \in \mathbb{R}$. Let $p \in (1,\infty)$ and $p'$ its conjugate exponent. The dual space of $\dot{L}^p_s(G)$ is isomorphic to $\dot{L}^{p'}_{-s}(G)$ via the distributional duality, and the dual space of $L^p_s(G)$ is isomorphic to $\dot{L}^{p'}_{-s}(G)$ via the distributional duality, Consequently, the Banach spaces $L^p_s(G)$ and $\dot{L}^p_s(G)$ are reflexive.*

*Proof.* Parts (1), (2), (3), and (6) follow from Theorem 4.4.3, Proposition 4.4.13 and Theorem 4.4.20.

Part (4) follows from Theorem 4.4.16 and Proposition 4.4.13.

Part (5) follows from Theorem 4.4.25 and Proposition 4.4.13 (5).

Part (7) follows from Parts (5) and (6).

Part (8) follows from Theorem 4.4.20.

For Part (9), see Theorem 4.4.9 and Proposition 4.4.15.

Part (10) follows from Lemmata 4.4.7 and 4.4.14 together with Theorem 4.4.20. □

**Properties of $L^2_s(G)$**

Here we discuss some special feature of the case $L^p(G)$, $p = 2$. Indeed $L^2(G)$ is a Hilbert space where one can use the spectral analysis of a positive Rockland operator.

Many of the proofs in Chapter 4 could be simplified if we had restricted the study to the case $L^p$ with $p = 2$. For instance, let us consider a positive Rockland operator $\mathcal{R}$ and its self-adjoint extension $\mathcal{R}_2$ on $L^2(G)$. One can define the fractional powers of $\mathcal{R}_2$ and $\mathrm{I} + \mathcal{R}_2$ by functional analysis. Then one can obtain the properties of the kernels of the Riesz and Bessel potentials with similar methods as in Corollary 4.3.11.

In this case, one would not need to use the general theory of fractional powers of an operator recalled in Section A.3. Even if it is not useful, let us mention that the proof that $\mathcal{R}_2$ satisfies the hypotheses of Theorem A.3.4 is easy in this case: it follows directly from the Lumer-Phillips Theorem (see Theorem A.2.5) together with the heat semi-group $\{e^{-t\mathcal{R}_2}\}_{t>0}$ being an $L^2(G)$-contraction semi-group by functional analysis.

The proof of the properties of the associated Sobolev spaces $L^2_s(G)$ would be the same in this particular case, maybe slightly helped occasionally by the Hölder inequality being replaced by the Cauchy-Schwartz inequality. A noticeable exception is that Lemma 4.4.19 can be obtained directly in the case $L^p$, $p = 2$, from the estimates due to Helffer and Nourrigat (see Corollary 4.1.14).

The main difference between $L^2$ and $L^p$ Sobolev spaces is the structure of Hilbert spaces of $L^2_s(G)$ whereas the other Sobolev spaces $L^p_s(G)$ are 'only' Banach spaces:

**Proposition 4.4.29** (Hilbert space $L^2_s$)**.** *Let $G$ be a graded Lie group.*

*For any $s \in \mathbb{R}$, $L^2_s(G)$ is a Hilbert space with the inner product given by*

$$(f, g)_{L^2_s(G)} := \int_G (\mathrm{I} + \mathcal{R}_2)^{\frac{s}{\nu}} f(x)\ \overline{(\mathrm{I} + \mathcal{R}_2)^{\frac{s}{\nu}} g(x)} dx,$$

*and $\dot{L}^2_s(G)$ is a Hilbert space with the inner product given by*

$$(f, g)_{\dot{L}^2_s(G)} := \int_G \mathcal{R}_2^{\frac{s}{\nu}} f(x)\ \overline{\mathcal{R}_2^{\frac{s}{\nu}} g(x)} dx,$$

*where $\mathcal{R}$ is a positive Rockland operator of homogeneous degree $\nu$.*

*If $s > 0$, an equivalent inner product on $L^2_s(G)$ is*

$$(f,g)_{L^2_s(G)} := \int_G f(x)\ \overline{g(x)}dx\ +\ \int_G \mathcal{R}_2^{\frac{s}{\nu}} f(x)\ \overline{\mathcal{R}_2^{\frac{s}{\nu}} g(x)}dx.$$

*If $s = \nu\ell$ with $\ell \in \mathbb{N}_0$, an equivalent inner product on $L^2_s(G)$ is*

$$(f,g) = (f,g)_{L^2(G)} + \sum_{[\alpha]=\nu\ell} (X^\alpha f, X^\alpha g)_{L^2(G)},$$

*and an equivalent inner product on $\dot{L}^2_s(G)$ is*

$$(f,g) = \sum_{[\alpha]=\nu\ell} (X^\alpha f, X^\alpha g)_{L^2(G)}.$$

Proposition 4.4.29 is easily checked, using the structure of Hilbert space of $L^2(G)$ and, for the last property, simplifying the proof of Lemma 4.4.19.

### 4.4.8 Right invariant Rockland operators and Sobolev spaces

We could have started with right-invariant (homogeneous) Rockland operators $\tilde{\mathcal{R}}$ instead of $\mathcal{R}$. We discuss here some links between the two operators and their Sobolev spaces.

Since both left and right invariant Rockland operators are differential operators, we can relate them by Formulae (1.11) for the derivatives $X^\alpha$ and $\tilde{X}^\alpha$. Then, given our analysis of $\mathcal{R}$, we can give some immediate properties of the right-invariant operator $\tilde{\mathcal{R}}$:

**Proposition 4.4.30.** *Let $\mathcal{R}$ be a positive Rockland operator. For any $\phi \in \mathcal{S}(G)$,*

$$\tilde{\mathcal{R}}\phi(x) = (\mathcal{R}^t\{\phi(\cdot^{-1})\})(x^{-1}) = (\bar{\mathcal{R}}\{\phi(\cdot^{-1})\})(x^{-1}),$$

*because $\mathcal{R}^t = \bar{\mathcal{R}}$. Therefore, the spectral measure $\tilde{E}$ of $\tilde{\mathcal{R}}$ is given by*

$$\tilde{E}(\phi)(x) = (\bar{E}\{\phi(\cdot^{-1})\})(x^{-1}), \quad \phi \in L^2(G),\ x \in G.$$

*Consequently, the multipliers of $\tilde{\mathcal{R}}$ and $\mathcal{R}$ are linked by*

$$m(\tilde{\mathcal{R}})(\phi)(x) = (m(\bar{\mathcal{R}})\{\phi(\cdot^{-1})\})(x^{-1}). \tag{4.50}$$

*The operators $\mathcal{R}$ and $\tilde{\mathcal{R}}$ commute strongly, that is, their spectral measures $E$ and $\tilde{E}$ commute. Moreover, for functions $f, g \in \mathcal{S}'(G)$ and $a \in \mathbb{C}$, we have*

$$\begin{aligned} \mathcal{R}^a(f*g) &= f*\mathcal{R}^a g, \\ \tilde{\mathcal{R}}^a(f*g) &= (\tilde{\mathcal{R}}^a f)*g, \\ (\mathcal{R}^a f)*g &= f*\tilde{\mathcal{R}}^a g. \end{aligned}$$

We can give a right-invariant version of Definition 4.3.17:

**Definition 4.4.31.** Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu$ and let $s \in \mathbb{R}$. For any tempered distribution $f \in \mathcal{S}'(G)$, we denote by $(\mathrm{I}+\tilde{\mathcal{R}})^{s/\nu} f$ the tempered distribution defined by

$$\langle (\mathrm{I}+\tilde{\mathcal{R}})^{s/\nu} f, \phi\rangle := \langle f, (\mathrm{I}+\tilde{\tilde{\mathcal{R}}})^{s/\nu}\phi\rangle, \quad \phi \in \mathcal{S}(G).$$

The Sobolev spaces that we have introduced are based on the Sobolev spaces corresponding to left-invariant vector fields and left-invariant positive Rockland operators. We could have considered the right Sobolev spaces $\tilde{L}^p_s(G)$ defined via the Sobolev norms

$$f \mapsto \|(\mathrm{I}+\tilde{\mathcal{R}})^{s/\nu} f\|_{L^p}.$$

The relations between left and right vector fields in (1.11) easily implies that if $f \in L^p(G)$ is such that $X^\alpha f \in L^p(G)$ then $\tilde{f} : x \mapsto f(x^{-1})$ is in $L^p(G)$ and satisfies $\tilde{X}^\alpha \tilde{f} \in L^p(G)$. By Lemma 4.4.19, we see that the map $f \mapsto \tilde{f}$ must map continuously $L^p_s \to \tilde{L}^p_s$ for any $p \in (1,\infty)$ and $s$ a multiple of the homogeneous degrees of positive Rockland operators.

More generally, the spectral calculus, see (4.50), implies

$$(\mathrm{I}+\tilde{\mathcal{R}}_2)^{s/\nu} f(x) = (\mathrm{I}+\mathcal{R}_2)^{s/\nu}\tilde{f}(x^{-1}), \qquad f \in \mathcal{S}(G),$$

where, again, $\tilde{f}(x) = f(x^{-1})$, and thus for any $p \in (1,\infty_o)$,

$$\|(\mathrm{I}+\tilde{\mathcal{R}}_p)^{s/\nu} f\|_{L^p(G)} = \|(\mathrm{I}+\mathcal{R}_p)^{s/\nu}\tilde{f}\|_{L^p(G)}, \qquad f \in \mathcal{S}(G).$$

This easily implies that $f \mapsto \tilde{f}$ maps continuously $L^p_s \to \tilde{L}^p_s$ for any $p \in (1,\infty)$ and any real exponent $s \in \mathbb{R}$. This is also an involution: $\tilde{\tilde{f}} = f$. Hence the map

$$\begin{cases} L^p_s(G) & \longrightarrow & \tilde{L}^p_s(G) \\ f & \longmapsto & \tilde{f} \end{cases}$$

is an isomorphism of vector spaces.

Even if the left and right Sobolev spaces are isomorphic, they are not equal in general. Note that in the commutative case of $G = \mathbb{R}^n$, both left and right Sobolev spaces coincide. It is also the case on compact Lie groups, where the Sobolev spaces are associated with the Laplace-Beltrami operator (which is central) and coincide with localisation of the Euclidean Sobolev spaces [RT10a]. This is no longer the case in the nilpotent setting. Indeed, below we give an example of functions $f$ (necessarily not symmetric, that is, $\tilde{f} \neq f$), in some $L^p_s(G)$ but not in $\tilde{L}^p_s(G)$.

*Example* 4.4.32. Let us consider the three dimensional Heisenberg group $\mathbb{H}_1$ and the canonical basis $X, Y, T$ of its Lie algebra (see Example 1.6.4). Then $X = \partial_x - \frac{y}{2}\partial_t$ whereas $\tilde{X} = \partial_x + \frac{y}{2}\partial_t$ thus $\tilde{X} - X = y\partial_t$.

The Sobolev spaces are then associated with the natural sub-Laplacian $X^2 + Y^2$, see Example 6.1.1. Hence it is covered by the work of Folland [Fol75] on Sobolev spaces associated with sub-Laplacian on stratified Lie groups and consequently, $L_1^2(G)$ is the space of functions $f \in L^2(\mathbb{H}_1)$ such that $Xf$ and $Yf$ are both in $L^2(\mathbb{H}_1)$ [Fol75, Corollary 4.13].

One can find a smooth function $\phi \in C^\infty(\mathbb{R})$ such that $\phi, \phi' \in L^2(\mathbb{R})$ but $\int_{\mathbb{R}} |z|^2 |\phi'(z)|^2 dz = \infty$. For instance, we consider $\phi = \phi_1 * \psi$ where $\psi$ is a suitable smoothing function (i.e. $\psi \in \mathcal{D}(G)$ is valued in $[0,1]$ with a 'small' support around 0), and the graph of the function $\phi_1$ is given by isosceles triangles parametrised by $\ell \in \mathbb{N}$, with vertex at points $(\ell, \ell^\beta)$, and base on the horizontal axis and with length $2/\ell^\alpha$. We then choose $\alpha, \beta \in \mathbb{R}$ with $2\beta \in (-3,-1)$ and $2\alpha > 2\beta + 1$. We also fix a smooth function $\chi : \mathbb{R} \to [0,1]$ supported on $[1/2, 2]$ with $\chi(1) = 1$. We define $f \in C^\infty(\mathbb{R}^3)$ via

$$f(x,y,t) = \phi\left(\frac{yx}{2} + t\right)\chi(x)\chi(t).$$

One checks easily that $f$, $Xf$ and $Yf$ are square integrable hence $f \in L_1^2(\mathbb{H}_1)$. However $y\partial_t f$ is not square integrable. As $\tilde{X} - X = y\partial_t f$, this shows that $(-X + \tilde{X})f \notin L^2(\mathbb{H}_1)$ and $\tilde{X}f$ can not be in $L^2$ thus $f$ is not in $\tilde{L}_1^2(\mathbb{H}_1)$.

## 4.5 Hulanicki's theorem

We now turn our attention to Hulanicki's theorem which will be useful in the next chapter when we deal with pseudo-differential operators on graded Lie groups. An important consequence of Hulanicki's theorem is the fact that a Schwartz multiplier in (the $L^2$-self-adjoint extension of) a positive Rockland operator has a Schwartz kernel. This section is devoted to the statement and the proof of Hulanicki's theorem and its consequence regarding Schwartz multiplier.

From now on, we will allow ourselves to keep the same notation $\mathcal{R}$ for a positive Rockland operator and its self-adjoint extension $\mathcal{R}_2$ on $L^2(G)$ when no confusion is possible. In particular, when we define functions of $\mathcal{R}_2$ (see Corollary 4.1.16), that is, a multiplier $m(\mathcal{R}_2)$ defined using the spectral measure of $\mathcal{R}_2$ where $m \in L^\infty(\mathbb{R}_+)$ is a function, we may often write

$$m(\mathcal{R}_2) = m(\mathcal{R}),$$

in order to ease the notation. Furthermore, we denote the corresponding right-convolution kernel of this operator by

$$m(\mathcal{R})\delta_o.$$

### 4.5.1 Statement

Hulanicki proved in [Hul84] that if multipliers $m$ satisfy Marcinkiewicz properties, then the kernels of $m(\mathcal{R})$ satisfy certain estimates:

**Theorem 4.5.1** (Hulanicki)**.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. Let $|\cdot|$ be a fixed homogeneous quasi-norm on $G$. For any $M_1 \in \mathbb{N}, M_2 \geq 0$ there exist $C = C_{M_1,M_2} > 0$ and $k = k_{M_1,M_2} \in \mathbb{N}_0$, $k' = k'_{M_1,M_2} \in \mathbb{N}_0$ such that for any $m \in C^k[0,\infty)$, we have*

$$\sum_{[\alpha]\leq M_1} \int_G |X^\alpha m(\mathcal{R})\delta_o(x)| \; (1+|x|)^{M_2} dx \leq C \sup_{\substack{\lambda>0 \\ \ell=0,\ldots,k \\ \ell'=0,\ldots,k'}} (1+\lambda)^{\ell'} |\partial_\lambda^\ell m(\lambda)|,$$

*in the sense that if the right-hand side is finite then the left-hand side is also finite and the inequality holds.*

The main consequence of Theorem 4.5.1 is the following:

**Corollary 4.5.2.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. If $\phi \in \mathcal{S}(\mathbb{R})$ then the kernel $\phi(\mathcal{R})\delta_o$ of $\phi(\mathcal{R})$ is Schwartz. Furthermore, the map associating a multiplier function with its kernel*

$$\mathcal{S}(\mathbb{R}) \ni \phi \longmapsto \phi(\mathcal{R})\delta_o \in \mathcal{S}(G), \tag{4.51}$$

*is continuous between the Schwartz spaces.*

The continuity of (4.51) means that for any continuous seminorm $\|\cdot\|$ on $\mathcal{S}(G)$ there exist $C > 0$ and $N \in \mathbb{N}$ such that for any $m \in \mathcal{S}(\mathbb{R})$ we have

$$\|m(\mathcal{R})\delta_o\| \leq C \sup_{x\in\mathbb{R},\ell\leq N} |(1+|x|)^N \partial^\ell m(x)|.$$

Examples of such Schwartz seminorms are $\|\cdot\|_{\mathcal{S}(G),N}$, $N \in \mathbb{N}$, defined in Section 3.1.9, and $\|\cdot\|_{\mathcal{S},a,k,p}$, $a > 0$, $k \in \mathbb{N}_0$, $p \in [1,\infty]$, defined in Proposition 4.4.27.

For completeness' sake, we include the proofs of Theorem 4.5.1 and Corollary 4.5.2 below. Before this, let us notice that Corollary 4.5.2 implies that the heat kernel of any Rockland operator is Schwartz. However, we will see that the proofs of Theorem 4.5.1 and Corollary 4.5.2 rely on the properties of the Bessel potentials which have been shown, in turn, using the properties of the heat kernel. Beside the properties of the Bessel potentials, the proof uses the functional calculus of $\mathcal{R}$ and the structure of $G$.

## 4.5.2 Proof of Hulanicki's theorem

This section is devoted to the proof of Theorem 4.5.1 and can be skipped at first reading.

We follow the essence of [Hul84], but we modify the original proof to take into account our presentation of the properties of Rockland operators as well as to bring some (small) simplifications. We also do not present some results obtained in [Hul84] on groups of polynomial growth. One of these simplifications is the fact that we fix a quasi-norm $|\cdot|$ which we assume to be a norm. Indeed, it is clear from the equivalence of quasi-norms (see Proposition 3.1.35) that it suffices to prove Hulanicki's theorem for one quasi-norm for it to hold for any quasi-norm. As a homogeneous norm exists by Theorem 3.1.39, we may assume that $|\cdot|$ is a norm without loss of generality. We could do without this but it simplifies the constants in the next pages.

### First step

The first step in the proof can be summarised with the following lemma:

**Lemma 4.5.3.** *Let $m : [0, +\infty) \to \mathbb{C}$ be a function and let $\ell_o \in \mathbb{N}$. We define the function $F : (-\infty, 1) \to \mathbb{C}$ by*

$$F(\xi) := \begin{cases} m\left(\xi^{-\frac{1}{\ell_o}} - 1\right) & \text{if } 0 < \xi < 1, \\ 0 & \text{if } \xi \leq 0, \end{cases}$$

*and we have*

$$\forall \lambda \in [0, \infty) \qquad m(\lambda) = F\left((1+\lambda)^{-\ell_o}\right).$$

*Furthermore, the following holds.*

1. *The function $F$ extends to a continuous function on $\mathbb{R}$ if and only if $m$ is continuous on $[0, \infty)$ and $\lim_{\lambda \to +\infty} m(\lambda) = 0$.*
2. *The function $F$ extends to a $C^1$ function on $\mathbb{R}$ if and only if $m$ is $C^1$ on $[0, \infty)$ with $\lim_{\lambda \to +\infty} m(\lambda) = 0$ and $\lim_{\lambda \to +\infty} (1+\lambda)^{1+\ell_o} m'(\lambda) = 0$.*

   *Let $k \in \mathbb{N}$. If $m \in C^k[0, +\infty)$ and*

   $$\lim_{\lambda \to +\infty} (1+\lambda)^{1+j+k\ell_o} |m^{(j)}(\lambda)| = 0 \text{ for } j = 1, \ldots, k',$$

   *then the function $F$ extends to a function in $C^k(\mathbb{R})$*
3. *Let $k \in \mathbb{N}$ and $m \in C^k[0, \infty)$. We assume that the suprema*

   $$\sup_{\lambda \geq 0} (1+\lambda)^{2+j+k\ell_o} |m^{(j)}(\lambda)|, \qquad j = 0, \ldots, k.$$

   *are finite. Then we can construct an extension to $\mathbb{R}$, still denoted by $F$, such that the function $F \in C^k(\mathbb{R})$ is supported in $[0, 2]$ and satisfies $\widehat{F}(0) = 0$ and for every $\ell \in \mathbb{Z}$,*

   $$\left|\widehat{F}(\ell)\right| \leq C(1+|\ell|)^{-k} \sup_{\substack{\lambda \geq 0 \\ j=0,\ldots,k}} (1+\lambda)^{1+j+k\ell_o} |m^{(j)}(\lambda)|,$$

   *where $C = C_{k,\ell_o}$ is a positive constant independent of $m$. Here $\widehat{F}(\ell)$, $\ell \in \mathbb{Z}$, denotes the Fourier coefficients of $F$ in the sense of*

   $$\widehat{F}(\ell) := \int_{-\pi}^{\pi} F(\xi) e^{-i\xi\ell} \frac{d\xi}{2\pi}.$$

*Proof.* Part (1) is easy to prove. Part (2) in the case of $k = 1$ follows easily from the following observations.

- If $\xi = (1+\lambda)^{-\ell_0}$, $\lambda > 0$ then
$$\frac{F(\xi) - F(0)}{\xi} = (1+\lambda)^{\ell_o} m(\lambda).$$

- We can compute formally for $\xi \in (0,1)$:
$$F'(\xi) = \frac{-\frac{1}{\ell_o}}{\xi^{\frac{1}{\ell_o}+1}} m'\left(\xi^{-\frac{1}{\ell_o}} - 1\right),$$
and in particular if $\xi = (1+\lambda)^{-\ell_0}$, $\lambda > 0$, then
$$F'(\xi) = -\frac{1}{\ell_o}(1+\lambda)^{1+\ell_o} m'(\lambda).$$

The general case of Part (2) follows from the following observation: $F^{(k')}(\xi)$ is a linear combination over $j = 1, \ldots, k'$ of
$$\xi^{-\frac{1}{\ell_o}-(k'-j)-j(\frac{1}{\ell_o}+1)} m^{(j)}\left(\xi^{-\frac{1}{\ell_o}} - 1\right) = \xi^{-\frac{1+j}{\ell_o}-k'} m^{(j)}\left(\xi^{-\frac{1}{\ell_o}} - 1\right).$$
The details are left to the reader.

Let us prove Part (3). Let $m \in C^k[0,\infty)$. Let $P_k$ be the Taylor expansion of $m$ at 0, that is, $P_k$ is the polynomial of degree $k$ such that we have for $\lambda > 0$ small,
$$m(\lambda) = P_k(\lambda) + o(|\lambda|^k).$$
We fix an arbitrary smooth function $\chi$ supported in $[0,2]$ and satisfying $\chi \equiv 1$ on $[0,1]$. We construct an extension of $F$, still denoted $F$, by setting
$$F(\xi) := \begin{cases} 0 & \text{if } \xi \leq 0, \\ m\left(\xi^{-\frac{1}{\ell_o}} - 1\right) & \text{if } 0 < \xi < 1, \\ P_k\left(\xi^{-\frac{1}{\ell_o}} - 1\right)\chi(\xi) & \text{if } \xi \geq 1. \end{cases}$$

We assume that the suprema given in the statement of Part 3 are finite. Clearly $F \in C^k(\mathbb{R})$ is supported in $[0,2]$. The proof of Part 2 implies easily
$$\|F^{(k')}\|_\infty \leq C \sum_{j=1}^{k'} \sup_{\lambda \geq 0} (1+\lambda)^{1+j+\ell_o k'} |m^{(j)}(\lambda)|, \tag{4.52}$$
where the constant $C = C_{k',\ell_o,\chi} > 0$ is independent on $m$.

The Fourier coefficient of $F$ at 0 is

$$\begin{aligned}\widehat{F}(0) &= \int_{-\pi}^{\pi} F(\xi)\frac{d\xi}{2\pi} \\ &= \int_0^1 m(\xi^{-\frac{1}{\ell_o}}-1)\frac{d\xi}{2\pi} + \int_1^2 P_k\left(\xi^{-\frac{1}{\ell_o}}-1\right)\chi(\xi)\frac{d\xi}{2\pi} \\ &= \int_0^{\infty} m(\lambda)\frac{-\ell_o}{2\pi}\frac{d\lambda}{(1+\lambda)^{\ell_o+1}} + \int_1^2 P_k\left(\xi^{-\frac{1}{\ell_o}}-1\right)\chi(\xi)\frac{d\xi}{2\pi}.\end{aligned}$$

We can always assume that the function $\chi$ was chosen so that

$$\int_1^2 P_k\left(\xi^{-\frac{1}{\ell_o}}-1\right)\chi(\xi)\frac{d\xi}{2\pi} = \int_0^{\infty} m(\lambda)\frac{\ell_o}{2\pi}\frac{d\lambda}{(1+\lambda)^{\ell_o+1}}.$$

Indeed, it suffices to replace $\chi$ by $\chi + c\chi_1$ where $\chi_1 \in \mathcal{D}(\mathbb{R})$ is supported in $(1,2)$ and $c$ a well chosen constant.

It is a simple exercise using integration by parts to show that the Fourier coefficients may be estimated by

$$\forall k' = 0,\ldots,k \quad \exists C = C_{k'} > 0 \quad \forall \ell \in \mathbb{Z} \qquad |\widehat{F}(\ell)| \leq C(1+|\ell|)^{-k'}\|F^{(k')}\|_\infty.$$

This together with (4.52) concludes the proof of Part (3). $\square$

**Second step**

The second step consists in noticing that, with the notation of Lemma 4.5.3, studying the multiplier $m(\mathcal{R})$ and using the Fourier series of $F$ leads to consider the operator $e^{i\ell(\mathrm{I}+\mathcal{R})^{-\ell_o}}$ and, more precisely, the properties of its convolution kernel.

**Lemma 4.5.4.** *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $G$. Let $\ell_o \in \mathbb{N}$ and $F_o(\xi) := e^{i\xi} - 1$, $\xi \in \mathbb{R}$. Then, for any $\ell \in \mathbb{Z}$, the convolution kernel of $F_o(\ell(\mathrm{I}+\mathcal{R})^{-\ell_o})$ is an integrable function:*

$$F_o(\ell(\mathrm{I}+\mathcal{R})^{-\ell_o})\delta_o \in L^1(G).$$

*Proof of Lemma 4.5.4.* Since $F_o(\ell\xi) = \sum_{j=1}^{\infty} \frac{(i\ell\xi)^j}{j!}$, we have at least formally

$$\begin{aligned}\kappa_\ell &:= \left\{F_o(\ell(\mathrm{I}+\mathcal{R})^{-\ell_o})\right\}\delta_o \\ &= \sum_{j=1}^{\infty}\frac{(i\ell)^j}{j!}(\mathrm{I}+\mathcal{R})^{-j\ell_o}\delta_0 = \sum_{j=1}^{\infty}\frac{(i\ell)^j}{j!}\mathcal{B}_{\nu j\ell_o},\end{aligned}$$

where $\mathcal{B}_a$ is the convolution kernel of the Bessel potentials, see Section 4.3.4, and $\nu$ is the degree of homogeneity of $\mathcal{R}$. In fact, by Corollary 4.3.11, we know that

$$\forall a \in \mathbb{C}_+ \quad \mathcal{B}_a \in L^1(G) \quad \text{and} \quad \mathcal{B}_{\nu j\ell_o} = \mathcal{B}_{\nu\ell_o} * \ldots * \mathcal{B}_{\nu\ell_o} := \mathcal{B}_{\nu\ell_o}^{*j}.$$