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Quantization on Nilpotent Lie Groups



Ferran Sunyer i Balaguer
Award winning monograph

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Ferran Sunyer i Balaguer (1912–1967) was a self-taught Catalan mathematician who, in spite of a serious physical disability, was very active in research in classical mathematical analysis, an area in which he acquired international recognition. His heirs created the Fundació Ferran Sunyer i Balaguer inside the Institut d'Estudis Catalans to honor the memory of Ferran Sunyer i Balaguer and to promote mathematical research.

Each year, the Fundació Ferran Sunyer i Balaguer and the Institut d'Estudis Catalans award an international research prize for a mathematical monograph of expository nature. The prize-winning monographs are published in this series. Details about the prize and the Fundació Ferran Sunyer i Balaguer can be found at

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This book has been awarded the Ferran Sunyer i Balaguer 2014 prize.



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Preface

The purpose of this monograph is to give an exposition of the global quantization of operators on nilpotent homogeneous Lie groups. We also present the background analysis on homogeneous and graded nilpotent Lie groups. The analysis on homogeneous nilpotent Lie groups drew a considerable attention from the 70's onwards. Research went in several directions, most notably in harmonic analysis and in the study of hypoellipticity and solvability of partial differential equations. Over the decades the subject has been developing on different levels with advances in the analysis on the Heisenberg group, stratified Lie groups, graded Lie groups, and general homogeneous Lie groups.

In the last years analysis on homogeneous Lie groups and also on other types of Lie groups has received another boost with newly found applications and further advances in many topics. Examples of this boost are subelliptic estimates, multiplier theorems, index formulae, nonlinear problems, potential theory, and symbolic calculi tracing full symbols of operators. In particular, the latter has produced further applications in the study of linear and nonlinear partial differential equations, requiring the knowledge of lower order terms of the operators.

Because of the current advances, it seems to us that a systematic exposition of the recently developed quantizations on Lie groups is now desirable. This requires bringing together various parts of the theory in the right generality, and extending notions and techniques known in particular cases, for instance on compact Lie groups or on the Heisenberg group.

In order to do so, we start with a review of the recent developments in the global quantization on compact Lie groups. In this, we follow mostly the development of this subject in the monograph [RT10a] by Turunen and the second author, as well as its further progress in subsequent papers. After a necessary exposition of the background analysis on graded and homogeneous Lie groups, we present the quantization on general graded Lie groups. As the final part of the monograph, we work out details of the general theory developed in this book in the particular case of the Heisenberg group.

In the introduction, we will provide a link between, on one hand, the symbolic calculus of matrix valued symbols on compact Lie groups with, on the other hand,

different approaches to the symbolic calculus on the Heisenberg group for instance. We will also motivate further our choices of presentation from the point of view of the development of the theory and of its applications.

We would like to thank Fulvio Ricci for discussions and for useful comments on the historical overview of parts of the subject that we tried to present in the introduction. We would also like to thank Gerald Folland for comments leading to improvements of some parts of the monograph.

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