Part I
Valuation Adjustments

Nonlinearity Valuation Adjustment

Nonlinear Valuation Under Collateralization, Credit Risk, and Funding Costs

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Abstract We develop a consistent, arbitrage-free framework for valuing derivative trades with collateral, counterparty credit risk, and funding costs. Credit, debit, liquidity, and funding valuation adjustments (CVA, DVA, LVA, and FVA) are simply introduced as modifications to the payout cash flows of the trade position. The framework is flexible enough to accommodate actual trading complexities such as asymmetric collateral and funding rates, replacement close-out, and re-hypothecation of posted collateral—all aspects which are often neglected. The generalized valuation equation takes the form of a forward–backward SDE or semi-linear PDE. Nevertheless, it may be recast as a set of iterative equations which can be efficiently solved by our proposed least-squares Monte Carlo algorithm. We implement numerically the case of an equity option and show how its valuation changes when including the above effects. In the paper we also discuss the financial impact of the proposed valuation framework and of nonlinearity more generally. This is fourfold: First, the valuation equation is only based on observable market rates, leaving the value of a derivatives transaction invariant to any theoretical risk-free rate. Secondly, the presence of funding costs makes the valuation problem a highly recursive and nonlinear one. Thus, credit and funding risks are non-separable in general, and despite common practice in banks, CVA, DVA, and FVA cannot be treated as purely additive adjustments without running the risk of double counting. To quantify the valuation error that can be attributed to double counting, we introduce a "nonlinearity valuation adjustment" (NVA) and show that its magnitude can be significant under asymmetric funding rates and replacement close-out at default. Thirdly, as trading

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parties cannot observe each others' liquidity policies nor their respective funding costs, the bilateral nature of a derivative price breaks down. The value of a trade to a counterparty will not be just the opposite of the value seen by the bank. Finally, valuation becomes aggregation-dependent and portfolio values cannot simply be added up. This has operational consequences for banks, calling for a holistic, consistent approach across trading desks and asset classes.

Keywords Nonlinear valuation · Nonlinear valuation adjustment NVA · Credit risk · Credit valuation adjustment CVA · Funding costs · Funding valuation adjustment FVA · Consistent valuation · Collateral

1 Introduction

Recent years have seen an unprecedented interest among banks in understanding the risks and associated costs of running a derivatives business. The financial crisis in 2007–2008 made banks painfully aware that derivative transactions involve a number of risks, e.g., credit or liquidity risks that they had previously overlooked or simply ignored. The industry practice for dealing with these issues comes in the form of a series of price adjustments to the classic, risk-neutral price definition of a contingent claim, often coined under mysteriously sounding acronyms such as CVA, DVA, or FVA.[1] The credit valuation adjustment (CVA) corrects the price for the expected costs to the dealer due to the possibility that the counterparty may default, while the so-called debit valuation adjustment (DVA) is a correction for the expected benefits to the dealer due to his own default risk. Dealers also make adjustments due to the costs of funding the trade. This practice is known as a liquidity and funding valuation adjustment (LVA, FVA). Recent headlines such as J.P. Morgan taking a hit of $1.5 billion in its 2013 fourth-quarter earnings due to funding valuation adjustments underscores the sheer importance of accounting for FVA.

In this paper we develop an arbitrage-free valuation approach of collateralized as well as uncollateralized trades that consistently accounts for credit risk, collateral, and funding costs. We derive a general valuation equation where CVA, DVA, collateral, and funding costs are introduced simply as modifications of payout cash flows. This approach can also be tailored to address trading through a central clearing house (CCP) with initial and variation margins as investigated in Brigo and Pallavicini [6]. In addition, our valuation approach does not put any restrictions on the banks' liquidity policies and hedging strategies, while accommodating asymmetric collateral and funding rates, collateral rehypothecation, and risk-free/replacement close-out conventions. We present an invariance theorem showing that our valuation equa-

[1] Recently, a new adjustment, the so-called KVA or capital valuation adjustment, has been proposed to account for the capital cost of a derivatives transaction (see e.g. Green et al. [26]). Following the financial crisis, banks are faced by more severe capital requirements and leverage constraints put forth by the Basel Committee and local authorities. Despite being a key issue for the industry, we will not consider costs of capital in this paper.

tions do not depend on some unobservable risk-free rates; valuation is purely based on observable market rates. The invariance theorem has appeared first implicitly in Pallavicini et al. [33], and is studied in detail in Brigo et al. [15], a version of which is in this same volume.

Several studies have analyzed the various valuation adjustments separately, but few have tried to build a valuation approach that consistently takes collateralization, counterparty credit risk, and funding costs into account. Under unilateral default risk, i.e., when only one party is defaultable, Brigo and Masetti [4] consider valuation of derivatives with CVA, while particular applications of their approach are given in Brigo and Pallavicini [5], Brigo and Chourdakis [3], and Brigo et al. [8]; see Brigo et al. [11] for a summary. Bilateral default risk appears in Bielecki and Rutkowski [1], Brigo and Capponi [2], Brigo et al. [9] and Gregory [27] who price both the CVA and DVA of a derivatives deal. The impact of collateralization on default risk has been investigated in Cherubini [20] and more recently in Brigo et al. [7, 12]. Assuming no default risk, Piterbarg [36] provides an initial analysis of collateralization and funding risk in a stylized Black–Scholes economy. Morini and Prampolini [31], Fries [25] and Castagna [19] consider basic implications of funding in presence of default risk. However, the most comprehensive attempts to develop a consistent valuation framework are those of Burgard and Kjaer [16, 17], Crépey [21–23], Crépey et al. [24], Pallavicini et al. [33, 34], and Brigo et al. [13, 14].

We follow the works of Pallavicini et al. [34], Brigo et al. [13, 14], and Sloth [37] and consider a general valuation framework that fully and consistently accounts for collateralization, counterparty credit risk, and funding risk when pricing a derivatives trade. We find that the precise patterns of funding-adjusted values depend on a number of factors, including the asymmetry between borrowing and lending rates. Moreover, the introduction of funding risk creates a highly recursive and nonlinear valuation problem. The inherent nonlinearity manifests itself in the valuation equations by taking the form of semi-linear PDEs or BSDEs.

Thus, valuation under funding risk poses a computationally challenging problem; funding and credit costs do not split up in a purely additive way. A consequence of this is that valuation becomes aggregation-dependent. Portfolio values do not simply add up, making it difficult for banks to create CVA and FVA desks with separate and clear-cut responsibilities. Nevertheless, banks often make such simplifying assumptions when accounting for the various price adjustments. This can be done, however, only at the expense of tolerating some degree of double counting in the different valuation adjustments.

We introduce the concept of nonlinearity valuation adjustment (NVA) to quantify the valuation error that one makes when treating CVA, DVA, and FVA as separate, additive terms. In particular, we examine the financial error of neglecting nonlinearities such as asymmetric borrowing and lending funding rates and by substituting replacement close-out at default by the more stylized risk-free close-out assumption. We analyze the large scale implications of nonlinearity of the valuation equations: non-separability of risks, aggregation dependence in valuation, and local valuation measures as opposed to universal ones. Finally, our numerical results confirm that

NVA and asymmetric funding rates can have a non-trivial impact on the valuation of financial derivatives.

To summarize, the financial implications of our valuation framework are fourfold:

- Valuation is invariant to any theoretical risk-free rate and only based on observable market rates.
- Valuation is a nonlinear problem under asymmetric funding and replacement close-out at default, making funding and credit risks non-separable.
- Valuation is no longer bilateral because counterparties cannot observe each others' liquidity policies nor their respective funding costs.
- Valuation is aggregation-dependent and portfolio values can no longer simply be added up.

The above points stress the fact that we are dealing with values rather than prices. By this, we mean to distinguish between the unique *price* of an asset in a complete market with a traded risk-free bank account and the *value* a bank or market participant attributes to the particular asset. Nevertheless, in the following, we will use the terms price and value interchangeably to mean the latter. The paper is organized as follows. Section 2 describes the general valuation framework with collateralized credit, debit, liquidity, and funding valuation adjustments. Section 3 derives an iterative solution of the pricing equation as well as a continuous-time approximation. Section 4 introduces the nonlinearity valuation adjustment and provides numerical results for specific valuation examples. Finally, Sect. 5 concludes the paper.

2 Trading Under Collateralization, Close-Out Netting, and Funding Risk

In this section we develop a general risk-neutral valuation framework for OTC derivative deals. The section clarifies how the traditional pre-crisis derivative price is consistently adjusted to reflect the new market realities of collateralization, counterparty credit risk, and funding risk. We refer to the two parties of a credit-risky deal as the investor or dealer ("I") on one side and the counterparty or client ("C") on the other.

We now introduce the mathematical framework we will use. We point out that the focus here is not on mathematics but on building the valuation framework. Full mathematical subtleties are left for other papers and may motivate slightly different versions of the cash flows, see for example Brigo et al. [15]. More details on the origins of the cash flows used here are in Pallavicini et al. [33, 34].

Fixing the time horizon $T \in \mathbb{R}_+$ of the deal, we define our risk-neutral valuation model on the probability space $(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\in[0,T]}, \mathbb{Q})$. $\mathbb{Q}$ is the risk-neutral probability measure ideally associated with the locally risk-free bank account numeraire growing at the risk-free rate r. The filtration $(\mathscr{G}_t)_{t\in[0,T]}$ models the flow of information of the whole market, including credit, such that the default times of the investor τ_I and the counterparty τ_C are $\mathscr{G}$-stopping times. We adopt the notational convention that $\mathbb{E}_t$ is the risk-neutral expectation conditional on the information $\mathscr{G}_t$. Moreover, we exclude the possibility of simultaneous defaults for simplicity and define the time of the first default event among the two parties as the stopping time

$$\tau \triangleq (\tau_I \wedge \tau_C).$$

In the sequel we adopt the view of the investor and consider the cash flows and consequences of the deal from her perspective. In other words, when we price the deal we obtain the value of the position to the investor. As we will see, with funding risk this price will not be the value of the deal to the counterparty with opposite sign, in general.

The gist of the valuation framework is conceptually simple and rests neatly on the classical finance disciplines of risk-neutral valuation and discounting cash flows. When a dealer enters into a derivatives deal with a client, a number of cash flows are exchanged, and just like valuation of any other financial claim, discounting these cash in- or outflows gives us a price of the deal. Post-crisis market practice includes four (or more) different types of cash flow streams occurring once a trading position has been entered: (i) Cash flows coming directly from the derivatives contract, such as payoffs, coupons, dividends, etc. We denote by $\pi(t, T)$ the sum of the discounted cash flows happening over the time period $(t, T]$ without including any credit, collateral, and funding effects. This is where classical derivatives valuation would usually stop and the price of a derivative contract with maturity T would be given by

$$V_t = \mathbb{E}_t\left[\pi(t, T)\right].$$

This price assumes no credit risk of the parties involved and no funding risk of the trade. However, present-day market practice requires the price to be adjusted by taking further cash-flow transactions into account: (ii) Cash flows required by collateral margining. If the deal is collateralized, cash flows happen in order to maintain a collateral account that in the case of default will be used to cover any losses. $\gamma(t, T; C)$ is the sum of the discounted margining costs over the period $(t, T]$ with C denoting the collateral account. (iii) Cash flows exchanged once a default event has occurred. We let $\theta_\tau(C, \varepsilon)$ denote the on-default cash-flow with ε being the residual value of the claim traded at default. Lastly, (iv) cash flows required for funding the deal. We denote the sum of the discounted funding costs over the period $(t, T]$ by $\varphi(t, T; F)$ with F being the cash account needed for funding the deal. Collecting the terms we obtain a consistent price $\bar{V}$ of a derivative deal taking into account counterparty credit risk, margining costs, and funding costs

$$\begin{aligned}\bar{V}_t(C, F) = \mathbb{E}_t\big[&\pi(t, T \wedge \tau) + \gamma(t, T \wedge \tau; C) + \varphi(t, T \wedge \tau; F) \\ &+ \mathbf{1}_{\{t<\tau<T\}} D(t, \tau)\theta_\tau(C, \varepsilon)\big],\end{aligned} \tag{1}$$

where $D(t, \tau) = \exp(-\int_t^\tau r_s ds)$ is the risk-free discount factor.

By using a risk-neutral valuation approach, we see that only the payout needs to be adjusted under counterparty credit and funding risk. In the following paragraphs we expand the terms of (1) and carefully discuss how to compute them.

2.1 Collateralization

The ISDA master agreement is the most commonly used framework for full and flexible documentation of OTC derivative transactions and is published by the International Swaps and Derivatives Association (ISDA [29]). Once agreed between two parties, the master agreement sets out standard terms that apply to all deals entered into between those parties. The ISDA master agreement lists two tools to mitigate counterparty credit risk: *collateralization* and *close-out netting*. Collateralization of a deal means that the party which is out-of-the-money is required to post collateral—usually cash, government securities, or highly rated bonds—corresponding to the amount payable by that party in the case of a default event. The credit support annex (CSA) to the ISDA master agreement defines the rules under which the collateral is posted or transferred between counterparties. Close-out netting means that in the case of default, all transactions with the counterparty under the ISDA master agreement are consolidated into a single net obligation which then forms the basis for any recovery settlements.

Collateralization of a deal usually happens according to a margining procedure. Such a procedure involves that both parties post collateral amounts to or withdraw amounts from the collateral account C according to their current exposure on prefixed dates $\{t_1, \ldots, t_n = T\}$ during the life of the deal, typically daily. Let α_i be the year fraction between t_i and t_{i+1}. The terms of the margining procedure may, furthermore, include independent amounts, minimum transfer amounts, thresholds, etc., as described in Brigo et al. [7]. However, here we adopt a general description of the margining procedure that does not rely on the particular terms chosen by the parties.

We consider a collateral account C held by the investor. Moreover, we assume that the investor is the collateral taker when $C_t > 0$ and the collateral provider when $C_t < 0$. The CSA ensures that the collateral taker remunerates the account C at an accrual rate. If the investor is the collateral taker, he remunerates the collateral account by the accrual rate $c_t^+(T)$, while if he is the collateral provider, the counterparty remunerates the account at the rate $c_t^-(T)$.[2] The effective accrual collateral rate $\tilde{c}_t(T)$ is defined as

$$\tilde{c}_t(T) \triangleq c_t^-(T)\mathbf{1}_{\{C_t<0\}} + c_t^+(T)\mathbf{1}_{\{C_t>0\}}. \tag{2}$$

[2] We stress the slight abuse of notation here: A plus and minus sign does not indicate that the rates are positive or negative parts of some other rate, but instead it tells which rate is used to accrue interest on the collateral according to the sign of the collateral account.

More generally, to understand the cash flows originating from collateralization of the deal, let us consider the consequences of the margining procedure to the investor. At the first margin date, say t_1, the investor opens the account and posts collateral if he is out-of-the-money, i.e. if $C_{t_1} < 0$, which means that the counterparty is the collateral taker. On each of the following margin dates t_k, the investor posts collateral according to his exposure as long as $C_{t_k} < 0$. As collateral taker, the counterparty pays interest on the collateral at the accrual rate $c^-_{t_k}(t_{k+1})$ between the following margin dates t_k and t_{k+1}. We assume that interest accrued on the collateral is saved into the account and thereby directly included in the margining procedure and the close-out. Finally, if $C_{t_n} < 0$ on the last margin date t_n, the investor closes the collateral account, given no default event has occurred in between. Similarly, for positive values of the collateral account, the investor is instead the collateral taker and the counterparty faces corresponding cash flows at each margin date. If we sum up all the discounted margining cash flows of the investor and the counterparty, we obtain

$$\gamma(t, T \wedge \tau; C) \triangleq \sum_{k=1}^{n-1} \mathbf{1}_{\{t \leqslant t_k < (T \wedge \tau)\}} D(t, t_k) C_{t_k} \left(1 - \frac{P_{t_k}(t_{k+1})}{P^{\tilde{c}}_{t_k}(t_{k+1})} \right), \tag{3}$$

with the zero-coupon bond $P^{\tilde{c}}_t(T) \triangleq [1 + (T - t)\tilde{c}_t(T)]^{-1}$, and the risk-free zero coupon bond, related to the risk-free rate r, given by $P_t(T)$. If we adopt a first order expansion (for small c and r), we can approximate

$$\gamma(t, T \wedge \tau; C) \approx \sum_{k=1}^{n-1} \mathbf{1}_{\{t \leqslant t_k < (T \wedge \tau)\}} D(t, t_k) C_{t_k} \alpha_k \left(r_{t_k}(t_{k+1}) - \tilde{c}_{t_k}(t_{k+1}) \right), \tag{4}$$

where with a slight abuse of notation we call $\tilde{c}_t(T)$ and $r_t(T)$ the continuously (as opposed to simple) compounded interest rates associated with the bonds $P^{\tilde{c}}$ and P. This last expression clearly shows a cost of carry structure for collateral costs. If C is positive to "I", then "I" is holding collateral and will have to pay (hence the minus sign) an interest c^+, while receiving the natural growth r for cash, since we are in a risk-neutral world. In the opposite case, if "I" posts collateral, C is negative to "I" and "I" receives interest c^- while paying the risk-free rate, as should happen when one shorts cash in a risk-neutral world.

A crucial role in collateral procedures is played by rehypothecation. We discuss rehypothecation and its inherent liquidity risk in the following.

Rehypothecation

Often the CSA grants the collateral taker relatively unrestricted use of the collateral for his liquidity and trading needs until it is returned to the collateral provider. Effectively, the practice of rehypothecation lowers the costs of remuneration of the provided collateral. However, while without rehypothecation the collateral provider can expect to get any excess collateral returned after honoring the amount payable on the deal, if rehypothecation is allowed the collateral provider runs the risk of losing a fraction or all of the excess collateral in case of default on the collateral taker's part.

We denote the recovery fraction on the rehypothecated collateral by R'_I when the investor is the collateral taker and by R'_C when the counterparty is the collateral taker. The general recovery fraction on the market value of the deal that the investor receives in the case of default of the counterparty is denoted by R_C, while R_I is the recovery fraction received by the counterparty if the investor defaults. The collateral provider typically has precedence over other creditors of the defaulting party in getting back any excess capital, which means $R_I \leqslant R'_I \leqslant 1$ and $R_C \leqslant R'_C \leqslant 1$. If no rehypothecation is allowed and the collateral is kept safe in a segregated account, we have that $R'_I = R'_C = 1$.

2.2 Close-Out Netting

In case of default, all terminated transactions under the ISDA master agreement with a given counterparty are netted and consolidated into a single claim. This also includes any posted collateral to back the transactions. In this context the close-out amount plays a central role in calculating the on-default cash flows. The close-out amount is the costs or losses that the surviving party incurs when replacing the terminated deal with an economic equivalent. Clearly, the size of these costs will depend on which party survives so we define the close-out amount as

$$\varepsilon_\tau \triangleq \mathbf{1}_{\{\tau=\tau_C<\tau_I\}}\varepsilon_{I,\tau} + \mathbf{1}_{\{\tau=\tau_I<\tau_C\}}\varepsilon_{C,\tau}, \tag{5}$$

where $\varepsilon_{I,\tau}$ is the close-out amount on the counterparty's default priced at time τ by the investor and $\varepsilon_{C,\tau}$ is the close-out amount if the investor defaults. Recall that we always consider the deal from the investor's viewpoint in terms of the sign of the cash flows involved. This means that if the close-out amount $\varepsilon_{I,\tau}$ as measured by the investor is positive, the investor is a creditor of the counterpaty, while if it is negative, the investor is a debtor of the counterparty. Analogously, if the close-out amount $\varepsilon_{C,\tau}$ to the counterparty but viewed from the investor is positive, the investor is a creditor of the counterparty, and if it is negative, the investor is a debtor to the counterparty.

We note that the ISDA documentation is, in fact, not very specific in terms of how to actually calculate the close-out amount. Since 2009, ISDA has allowed for the possibility to switch from a risk-free close-out rule to a replacement rule that includes the DVA of the surviving party in the recoverable amount. Parker and McGarry[35] and Weeber and Robson [40] show how a wide range of values of the close-out amount can be produced within the terms of ISDA. We refer to Brigo et al. [7] and the references therein for further discussions on these issues. Here, we adopt the approach of Brigo et al. [7] listing the cash flows of all the various scenarios that can occur if default happens. We will net the exposure against the pre-default value of the collateral $C_{\tau-}$ and treat any remaining collateral as an unsecured claim.

If we aggregate all these cash flows and the pre-default value of collateral account, we reach the following expression for the on-default cash-flow

$$\theta_\tau(C,\varepsilon) \triangleq \mathbf{1}_{\{\tau=\tau_C<\tau_I\}}\left(\varepsilon_{I,\tau} - \mathrm{LGD}_C(\varepsilon^+_{I,\tau} - C^+_{\tau^-})^+ - \mathrm{LGD}'_C(\varepsilon^-_{I,\tau} - C^-_{\tau^-})^+\right) \quad (6)$$
$$+ \mathbf{1}_{\{\tau=\tau_I<\tau_C\}}\left(\varepsilon_{C,\tau} - \mathrm{LGD}_I(\varepsilon^-_{C,\tau} - C^-_{\tau^-})^- - \mathrm{LGD}'_I(\varepsilon^+_{C,\tau} - C^+_{\tau^-})^-\right).$$

We use the short-hand notation $\mathscr{X}^+ := \max(\mathscr{X}, 0)$ and $\mathscr{X}^- := \min(\mathscr{X}, 0)$, and define the loss-given-default as $\mathrm{LGD}_C \triangleq 1 - R_C$, and the collateral loss-given-default as $\mathrm{LGD}'_C \triangleq 1 - R'_C$. If both parties agree on the exposure, namely $\varepsilon_{I,\tau} = \varepsilon_{C,\tau} = \varepsilon_\tau$, when we take the risk-neutral expectation in (1), we see that the price of the discounted on-default cash-flow,

$$\mathbb{E}_t[\mathbf{1}_{\{t<\tau<T\}}D(t,\tau)\theta_\tau(C,\varepsilon)] = \mathbb{E}_t[\mathbf{1}_{\{t<\tau<T\}}D(t,\tau)\,\varepsilon_\tau] - \mathrm{CVA}(t,T;C) + \mathrm{DVA}(t,T;C), \quad (7)$$

is the present value of the close-out amount reduced by the positive collateralized CVA and DVA terms

$$\Pi_{\mathrm{CVAcoll}}(s) = \left(\mathrm{LGD}_C(\varepsilon^+_{I,s} - C^+_{s^-})^+ + \mathrm{LGD}'_C(\varepsilon^-_{I,s} - C^-_{s-})^+\right) \geq 0,$$
$$\Pi_{\mathrm{DVAcoll}}(s) = -\left(\mathrm{LGD}_I(\varepsilon^-_{C,s} - C^-_{s-})^- + \mathrm{LGD}'_I(\varepsilon^+_{C,s} - C^+_{s-})^-\right) \geq 0,$$

and

$$\mathrm{CVA}(t,T;C) \triangleq \mathbb{E}_t\left[\mathbf{1}_{\{\tau=\tau_C<T\}}D(t,\tau)\Pi_{\mathrm{CVAcoll}}(\tau)\right],$$
$$\mathrm{DVA}(t,T;C) \triangleq \mathbb{E}_t\left[\mathbf{1}_{\{\tau=\tau_I<T\}}D(t,\tau)\Pi_{\mathrm{DVAcoll}}(\tau)\right]. \quad (8)$$

Also, observe that if rehypothecation of the collateral is not allowed, the terms multiplied by LGD'_C and LGD'_I drop out of the CVA and DVA calculations.

2.3 Funding Risk

The hedging strategy that perfectly replicates the no-arbitrage price of a derivative is formed by a position in cash and a position in a portfolio of hedging instruments. When we talk about a derivative deal's funding, we essentially mean the cash position that is required as part of the hedging strategy, and with funding costs we refer to the costs of maintaining this cash position. If we denote the cash account by F and the risky asset account by H, we get

$$\bar{V}_t = F_t + H_t.$$

In the classical Black–Scholes–Merton theory, the risky part H of the hedge would be a delta position in the underlying stock, whereas the locally risk-free (cash) part F would be a position in the risk-free bank account. If the deal is collateralized, the margining procedure is included in the deal definition insuring that funding of

the collateral is automatically taken into account. Moreover, if rehypothecation is allowed for the collateralized deal, the collateral taker can use the posted collateral as a funding source and thereby reduce or maybe even eliminate the costs of funding the deal. Thus, we have the following two definitions of the funding account:
If rehypothecation of the posted collateral is allowed,

$$F_t \triangleq \bar{V}_t - C_t - H_t, \tag{9}$$

and if such rehypothecation is forbidden, we have

$$F_t \triangleq \bar{V}_t - H_t. \tag{10}$$

By implication of (9) and (10) it is obvious that if the funding account $F_t > 0$, the dealer needs to borrow cash to establish the hedging strategy at time t. Correspondingly, if the funding account $F_t < 0$, the hedging strategy requires the dealer to invest surplus cash. Specifically, we assume the dealer enters a funding position on a discrete time-grid $\{t_1, \ldots, t_m\}$ during the life of the deal. Given two adjacent funding times t_j and t_{j+1}, for $1 \leq j \leq m-1$, the dealer enters a position in cash equal to F_{t_j} at time t_j. At time t_{j+1} the dealer redeems the position again and either returns the cash to the funder if it was a long cash position and pays funding costs on the borrowed cash, or he gets the cash back if it was a short cash position and receives funding benefits as interest on the invested cash. We assume that these funding costs and benefits are determined at the start date of each funding period and charged at the end of the period.

Let $P_t^{\tilde{f}}(T)$ represent the price of a borrowing (or lending) contract measurable at t where the dealer pays (or receives) one unit of cash at maturity $T > t$. We introduce the effective funding rate $\tilde{f}_t$ as a function: $\tilde{f}_t = f(t, F, H, C)$, assuming that it depends on the cash account F_t, hedging account H_t, and collateral account C_t. Moreover, the zero-coupon bond corresponding to the effective funding rate is defined as

$$P_t^{\tilde{f}}(T) \triangleq [1 + (T-t)\tilde{f}_t(T)]^{-1},$$

If we assume that the dealer hedges the derivatives position by trading in the spot market of the underlying asset(s), and the hedging strategy is implemented on the same time-grid as the funding procedure of the deal, the sum of discounted cash flows from funding the hedging strategy during the life of the deal is equal to

$$\begin{aligned}
&\varphi(t, T \wedge \tau; F, H) \\
&\quad = \sum_{j=1}^{m-1} \mathbf{1}_{\{t \leqslant t_j < (T \wedge \tau)\}} D(t, t_j) \left(F_{t_j} - (F_{t_j} + H_{t_j}) \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{\tilde{f}}(t_{j+1})} + H_{t_j} \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{\tilde{f}}(t_{j+1})} \right) \\
&\quad = \sum_{j=1}^{m-1} \mathbf{1}_{\{t \leqslant t_j < (T \wedge \tau)\}} D(t, t_j) F_{t_j} \left(1 - \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{\tilde{f}}(t_{j+1})} \right).
\end{aligned} \tag{11}$$

This is, strictly speaking, a discounted payout and the funding cost or benefit at time t is obtained by taking the risk-neutral expectation of the above cash flows. For a trading example giving more details on how the above formula for φ originates, see Brigo et al. [15].

As we can see from Eq. (11), the dependence of hedging account dropped off from the funding procedure. For modeling convenience, we can define the effective funding rate $\tilde{f}_t$ faced by the dealer as

$$\tilde{f}_t(T) \triangleq f_t^-(T)\mathbf{1}_{\{F_t<0\}} + f_t^+(T)\mathbf{1}_{\{F_t>0\}}. \tag{12}$$

A related framework would be to consider the hedging account H as being perfectly collateralized and use the collateral to fund hedging, so that there is no funding cost associated with the hedging account.

As with collateral costs mentioned earlier, we may rewrite the cash flows for funding as a first order approximation in continuously compounded rates $\tilde{f}$ and r associated to the relevant bonds. We obtain

$$\varphi(t, T\wedge\tau; F) \approx \sum_{j=1}^{m-1} \mathbf{1}_{\{t\leqslant t_j<(T\wedge\tau)\}} D(t,t_j) F_{t_j} \alpha_j \left(r_{t_j}(t_{j+1}) - \tilde{f}_{t_j}(t_{j+1}) \right), \tag{13}$$

We should also mention that, occasionally, we may include the effects of repo markets or stock lending in our framework. In general, we may borrow/lend the cash needed to establish H from/to our treasury, and we may then use the risky asset in H for repo or stock lending/borrowing in the market. This means that we could include the funding costs and benefits coming from this use of the risky asset. Here, we assume that the bank's treasury automatically recognizes this benefit or cost at the same rate $\tilde{f}$ as used for cash, but for a more general analysis involving repo rate $\tilde{h}$ please refer to, for example, Pallavicini et al. [34], Brigo et al. [15].

The particular positions entered by the dealer to either borrow or invest cash according to the sign and size of the funding account depend on the bank's liquidity policy. In the following we discuss two possible cases: One where the dealer can fund at rates set by the bank's treasury department, and another where the dealer goes to the market directly and funds his trades at the prevailing market rates. As a result, the funding rates and therefore the funding effect on the price of a derivative deal depends intimately on the chosen liquidity policy.

Treasury Funding

If the dealer funds the hedge through the bank's treasury department, the treasury determines the funding rates $f^{\pm}$ faced by the dealer, often assuming average funding costs and benefits across all deals. This leads to two curves as functions of maturity; one for borrowing funds f^+ and one for lending funds f^-. After entering a funding position F_{t_j} at time t_j, the dealer faces the following discounted cash-flow

$$\Phi_j(t_j, t_{j+1}; F) \triangleq -N_{t_j} D(t_j, t_{j+1}),$$

with

$$N_{t_j} \triangleq \frac{F_{t_j}^-}{P_{t_j}^{f^-}(t_{j+1})} + \frac{F_{t_j}^+}{P_{t_j}^{f^+}(t_{j+1})}.$$

Under this liquidity policy, the treasury—and not the dealer himself—is in charge of debt valuation adjustments due to funding-related positions. Also, being entities of the same institution, both the dealer and the treasury disappear in case of default of the institution without any further cash flows being exchanged and we can neglect the effects of funding in this case. So, when default risk is considered, this leads to following definition of the funding cash flows

$$\bar{\Phi}_j(t_j, t_{j+1}; F) \triangleq \mathbf{1}_{\{\tau > t_j\}} \Phi_j(t_j, t_{j+1}; F).$$

Thus, the risk-neutral price of the cash flows due to the funding positions entered at time t_j is

$$\mathbb{E}_{t_j}\left[\, \bar{\Phi}_j(t_j, t_{j+1}; F)\,\right] = -\mathbf{1}_{\{\tau > t_j\}} \left(F_{t_j}^- \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{f^-}(t_{j+1})} + F_{t_j}^+ \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{f^+}(t_{j+1})} \right).$$

If we consider a sequence of such funding operations at each time t_j during the life of the deal, we can define the sum of cash flows coming from all the borrowing and lending positions opened by the dealer to hedge the trade up to the first-default event

$$\begin{aligned} \varphi(t, T \wedge \tau; F) &\triangleq \sum_{j=1}^{m-1} \mathbf{1}_{\{t \leqslant t_j < (T\wedge\tau)\}} D(t, t_j) \left(F_{t_j} + \mathbb{E}_{t_j}\left[\, \bar{\Phi}_j(t_j, t_{j+1}; F)\,\right] \right) \qquad (14) \\ &= \sum_{j=1}^{m-1} \mathbf{1}_{\{t \leqslant t_j < (T\wedge\tau)\}} D(t, t_j) \left(F_{t_j} - F_{t_j}^- \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{f^-}(t_{j+1})} - F_{t_j}^+ \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{f^+}(t_{j+1})} \right). \end{aligned}$$

In terms of the effective funding rate, this expression collapses to (11).

Market Funding

If the dealer funds the hedging strategy in the market—and not through the bank's treasury—the funding rates are determined by prevailing market conditions and are often deal-specific. This means that the rate f^+ the dealer can borrow funds at may be different from the rate f^- at which funds can be invested. Moreover, these rates may differ across deals depending on the deals' notional, maturity structures, dealer-client relationship, and so forth. Similar to the liquidity policy of treasury funding, we assume a deal's funding operations are closed down in the case of default. Furthermore, as the dealer now operates directly on the market, he needs to include a DVA due to his funding positions when he marks-to-market his trading books. For simplicity, we assume that the funder in the market is default-free so no funding CVA needs to be accounted for. The discounted cash-flow from the borrowing or lending position between two adjacent funding times t_j and t_{j+1} is given by

$$\begin{aligned}\bar{\Phi}_j(t_j, t_{j+1}; F) \triangleq \mathbf{1}_{\{\tau > t_j\}} \mathbf{1}_{\{\tau_I > t_{j+1}\}} \Phi_j(t_j, t_{j+1}; F) \\ - \mathbf{1}_{\{\tau > t_j\}} \mathbf{1}_{\{\tau_I < t_{j+1}\}} (\mathrm{LGD}_I \varepsilon^-_{F,\tau_I} - \varepsilon_{F,\tau_I}) D(t_j, \tau_I),\end{aligned}$$

where $\varepsilon_{F,t}$ is the close-out amount calculated by the funder on the dealer's default

$$\varepsilon_{F,\tau_I} \triangleq -N_{t_j} P_{\tau_I}(t_{j+1}).$$

To price this funding cash-flow, we take the risk-neutral expectation

$$\mathbb{E}_{t_j}\left[\, \bar{\Phi}_j(t_j, t_{j+1}; F)\,\right] = -\mathbf{1}_{\{\tau > t_j\}} \left(F^-_{t_j} \frac{P_{t_j}(t_{j+1})}{P^{f^-}_{t_j}(t_{j+1})} + F^+_{t_j} \frac{P_{t_j}(t_{j+1})}{\bar{P}^{f^+}_{t_j}(t_{j+1})} \right).$$

Here, the zero-coupon funding bond $\bar{P}^{f^+}_t(T)$ for borrowing cash is adjusted for the dealer's credit risk

$$\bar{P}^{f^+}_t(T) \triangleq \frac{P^{f^+}_t(T)}{\mathbb{E}^T_t\left[\, \mathrm{LGD}_I \mathbf{1}_{\{\tau_I > T\}} + R_I \,\right]},$$

where the expectation on the right-hand side is taken under the T-forward measure. Naturally, since the seniority could be different, one might assume a different recovery rate on the funding position than on the derivatives deal itself (see Crépey [21]). Extensions to this case are straightforward.

Next, summing the discounted cash flows from the sequence of funding operations through the life of the deal, we get a new expression for φ that is identical to (14) where the $P^{f^+}_t(T)$ in the denominator is replaced by $\bar{P}^{f^+}_t(T)$. To avoid cumbersome notation, we will not explicitly write $\bar{P}^{f^+}$ in the sequel, but just keep in mind that when the dealer funds directly in the market then P^{f^+} needs to be adjusted for *funding DVA*. Thus, in terms of the effective funding rate, we obtain (11).

3 Generalized Derivatives Valuation

In the previous section we analyzed the discounted cash flows of a derivatives trade and we developed a framework for consistent valuation of such deals under collateralized counterparty credit and funding risk. The arbitrage-free valuation framework is captured in the following theorem.

Theorem 1 (Generalized Valuation Equation)
The consistent arbitrage-free price $\bar{V}_t(C, F)$ of a contingent claim under counterparty credit risk and funding costs takes the form

$$\bar{V}_t(C, F) = \mathbb{E}_t\Big[\pi(t, T \wedge \tau) + \gamma(t, T \wedge \tau; C) + \varphi(t, T \wedge \tau; F) \\ + \mathbf{1}_{\{t<\tau<T\}} D(t, \tau)\theta_\tau(C, \varepsilon)\Big], \tag{15}$$

where

1. $\pi(t, T \wedge \tau)$ *is the discounted cash flows from the contract's payoff structure up to the first-default event.*
2. $\gamma(t, T \wedge \tau; C)$ *is the discounted cash flows from the collateral margining procedure up to the first-default event and is defined in (3).*
3. $\varphi(t, T \wedge \tau; F)$ *is the discounted cash flows from funding the hedging strategy up to the first-default event and is defined in (11).*
4. $\theta_\tau(C, \varepsilon)$ *is the on-default cash-flow with close-out amount* ε *and is defined in (6).*

Note that in general a nonlinear funding rate may lead to arbitrages since the choice of the martingale measure depends on the funding/hedging strategy (see Remark 4.2). One has to be careful in order to guarantee that the relevant valuation equation admits solutions. Existence and uniqueness of solutions in the framework of this paper are discussed from a fully mathematical point of view in Brigo et al. [15], a version of which, from the same authors, appears in this volume.

In general, while the valuation equation is conceptually clear—we simply take the expectation of the sum of all discounted cash flows of the trade under the risk-neutral measure—solving the equation poses a recursive, nonlinear problem. The future paths of the effective funding rate $\tilde{f}$ depend on the future signs of the funding account F, i.e. whether we need to borrow or lend cash on each future funding date. At the same time, through the relations (9) and (10), the future sign and size of the funding account F depend on the adjusted price $\bar{V}$ of the deal which is the quantity we are trying to compute in the first place. One crucial implication of this nonlinear structure of the valuation problem is the fact that FVA is generally not just an additive adjustment term, as often assumed. More importantly, we see that the celebrated conjecture identifying the DVA of a deal with its funding benefit is not fully general. Only in the unrealistic setting where the dealer can fund an uncollateralized trade at equal borrowing and lending rates, i.e. $f^+ = f^-$, do we achieve the additive structure often assumed by practitioners. If the trade is collateralized, we need to impose even further restrictions as to how the collateral is linked to the price of the trade $\bar{V}$. It should be noted here that funding DVA (as referred to in the previous section) is similar to the DVA2 in Hull and White [28] and the concept of "windfall funding benefit at own default" in Crépey [22, 23]. In practice, however, funds transfer pricing and similar operations conducted by banks' treasuries clearly weaken the link between FVA and this source of DVA. The DVA of the funding instruments does not regard the bank's funding positions, but the derivatives position, and in general it does not match the FVA mainly due to the presence of funding netting sets.

Remark 1 (*The Law of One Price*.)
On the theoretical side, the generalized valuation equation shakes the foundation of the celebrated Law of One Price prevailing in classical derivatives pricing. Clearly, if we assume no funding costs, the dealer and counterparty agree on the price of

the deal as both parties can—at least theoretically—observe the credit risk of each other through CDS contracts traded in the market and the relevant market risks, thus agreeing on CVA and DVA. In contrast, introducing funding costs, they will not agree on the FVA for the deal due to asymmetric information. The parties cannot observe each others' liquidity policies nor their respective funding costs associated with a particular deal. As a result, the value of a deal position will not generally be the same to the counterparty as to the dealer just with opposite sign.

Finally, as we adopt a risk-neutral valuation framework, we implicitly assume the existence of a risk-free interest rate. Indeed, since the valuation adjustments are included as additional cash flows and not as ad-hoc spreads, all the cash flows in (15) are discounted by the risk-free discount factor $D(t, T)$. Nevertheless, the risk-free rate is merely an instrumental variable of the general valuation equation. We clearly distinguish market rates from the theoretical risk-free rate avoiding the dubious claim that the over-night rates are risk free. In fact, as we will show in continuous time, if the dealer funds the hedging strategy of the trade through cash accounts available to him—whether as rehypothecated collateral or funds from the treasury, repo market, etc.—the risk-free rate vanishes from the valuation equation.

3.1 Discrete-Time Solution

Our purpose here is to turn the generalized valuation equation (15) into a set of iterative equations that can be solved by least-squares Monte Carlo methods. These methods are already standard in CVA and DVA calculations (Brigo and Pallavicini [5]). To this end, we introduce the auxiliary function

$$\begin{aligned}\bar{\pi}(t_j, t_{j+1}; C) \triangleq \pi(t_j, t_{j+1} \wedge \tau) + \gamma(t_j, t_{j+1} \wedge \tau; C) \\ + \mathbf{1}_{\{t_j < \tau < t_{j+1}\}} D(t_j, \tau)\theta_\tau(C, \varepsilon)\end{aligned} \tag{16}$$

which defines the cash flows of the deal occurring between time t_j and t_{j+1} adjusted for collateral margining costs and default risks. We stress the fact that the close-out amount used for calculating the on-default cash flow still refers to a deal with maturity T. If we then solve valuation equation (15) at each funding date t_j in the time-grid $\{t_1, \ldots, t_n = T\}$, we obtain the deal price $\bar{V}$ at time t_j as a function of the deal price on the next consecutive funding date t_{j+1}

$$\begin{aligned}\bar{V}_{t_j} &= \mathbb{E}_{t_j}\left[\, \bar{V}_{t_{j+1}} D(t_j, t_{j+1}) + \bar{\pi}(t_j, t_{j+1}; C) \,\right] \\ &\quad + \mathbf{1}_{\{\tau > t_j\}} \left(F_{t_j} - F^-_{t_j} \frac{P_{t_j}(t_{j+1})}{P^{f^-}_{t_j}(t_{j+1})} - F^+_{t_j} \frac{P_{t_j}(t_{j+1})}{P^{f^+}_{t_j}(t_{j+1})} \right),\end{aligned}$$

where, by definition, $\bar{V}_{t_n} \triangleq 0$ on the final date t_n. Recall the definitions of the funding account in (9) if no rehypothecation of collateral is allowed and in (10) if rehypothecation is permitted, we can then solve the above for the positive and negative parts of the funding account. The outcome of this exercise is a discrete-time iterative solution of the recursive valuation equation, provided in the following theorem.

Theorem 2 (Discrete-time Solution of the Generalized Valuation Equation)
We may solve the full recursive valuation equation in Theorem 1 as a set of backward-iterative equations on the time-grid $\{t_1, \dots, t_n = T\}$ *with* $\bar{V}_{t_n} \triangleq 0$*. For* $\tau < t_j$*, we have*

$$\bar{V}_{t_j} = 0,$$

while for $\tau > t_j$*, we have*

(i) if rehypothecation is forbidden:

$$\left(\bar{V}_{t_j} - H_{t_j}\right)^{\pm} = P_{t_j}^{\tilde{f}}(t_{j+1}) \left(\mathbb{E}_{t_j}^{t_{j+1}}\left[\bar{V}_{t_{j+1}} + \frac{\bar{\pi}(t_j, t_{j+1}; C) - H_{t_j}}{D(t_j, t_{j+1})}\right]\right)^{\pm},$$

(ii) if rehypothecation is allowed:

$$\begin{aligned}(\bar{V}_{t_j} - C_{t_j} - H_{t_j})^{\pm} \\ = P_{t_j}^{\tilde{f}}(t_{j+1}) \left(\mathbb{E}_{t_j}^{t_{j+1}}\left[\bar{V}_{t_{j+1}} + \frac{\bar{\pi}(t_j, t_{j+1}; C) - C_{t_j} - H_{t_j}}{D(t_j, t_{j+1})}\right]\right)^{\pm},\end{aligned}$$

where the expectations are taken under the $\mathbb{Q}^{t_{j+1}}$*-forward measure.*

The $\pm$ sign in the theorem is supposed to stress the fact that the sign of the funding account, which determines the effective funding rate, depends on the sign of the conditional expectation. Further intuition may be gained by going to continuous time, which is the case we will now turn to.

3.2 Continuous-Time Solution

Let us consider a continuous-time approximation of the general valuation equation. This implies that collateral margining, funding, and hedging strategies are executed in continuous time. Moreover, we assume that rehypothecation is allowed, but similar results hold if this is not the case. By taking the time limit, we have the following expressions for the discounted cash flow streams of the deal

$$\pi(t, T \wedge \tau) = \int_t^{T \wedge \tau} \pi(s, s + ds) D(t, s),$$

$$\gamma(t, T \wedge \tau; C) = \int_t^{T \wedge \tau} (r_s - \tilde{c}_s) C_s D(t, s) ds,$$

$$\varphi(t, T \wedge \tau; F) = \int_t^{T \wedge \tau} (r_s - \tilde{f}_s) F_s D(t, s) ds,$$

where as mentioned earlier $\pi(t, t + dt)$ is the pay-off coupon process of the derivative contract and r_t is the risk-free rate. These equations can also be immediately derived by looking at the approximations given in Eqs. (4) and (13).

Then, putting all the above terms together with the on-default cash flow as in Theorem 1, the recursive valuation equation yields

$$\begin{aligned} \bar{V}_t = & \int_t^T \mathbb{E}_t\big[\left(\mathbf{1}_{\{s<\tau\}}\pi(s, s+ds) + \mathbf{1}_{\{\tau \in ds\}}\theta_s(C, \varepsilon)\right) D(t,s) \big] \\ & + \int_t^T \mathbb{E}_t\big[\mathbf{1}_{\{s<\tau\}}(r_s - \tilde{c}_s) C_s D(t,s) \big] ds \\ & + \int_t^T \mathbb{E}_t\Big[\mathbf{1}_{\{s<\tau\}}(r_s - \tilde{f}_s) F_s \Big] D(t,s) ds. \end{aligned} \tag{17}$$

By recalling Eq. (7), we can write the following

Proposition 1 *The value $\bar{V}_t$ of the claim under credit gap risk, collateral, and funding costs can be written as*

$$\bar{V}_t = V_t - CVA_t + DVA_t + LVA_t + FVA_t \tag{18}$$

where V_t is the price of the deal when there is no credit risk, no collateral, and no funding costs; LVA is a liquidity valuation adjustment accounting for the costs/benefits of collateral margining; FVA is the funding cost/benefit of the deal hedging strategy, and CVA and DVA are the familiar credit and debit valuation adjustments after collateralization. These different adjustments can be obtained by rewriting (17). One gets

$$V_t = \int_t^T \mathbb{E}_t\left\{ D(t,s) 1_{\{\tau > s\}} \Big[\pi(s, s+ds) + 1_{\{\tau \in ds\}} \varepsilon_s \Big] \right\} \tag{19}$$

and the valuation adjustments

$$\begin{aligned} CVA_t &= -\int_t^T \mathbb{E}\Big\{ D(t,s) 1_{\{\tau>s\}} \big[-\mathbf{1}_{\{s=\tau_C<\tau_I\}} \Pi_{CVAcoll}(s) \big] \Big\} du \\ DVA_t &= \int_t^T \mathbb{E}\Big\{ D(t,s) 1_{\{\tau>s\}} \big[\mathbf{1}_{\{s=\tau_I<\tau_C\}} \Pi_{DVAcoll}(s) \big] \Big\} du \\ LVA_t &= \int_t^T \mathbb{E}_t\Big\{ D(t,s) 1_{\{\tau>s\}} (r_s - \tilde{c}_s) C_s \Big\} ds \\ FVA_t &= \int_t^T \mathbb{E}\Big\{ D(t,s) 1_{\{\tau>s\}} \big[(r_s - \tilde{f}_s) F_s \big] \Big\} ds \end{aligned}$$

As usual, CVA and DVA are both positive, while LVA and FVA can be either positive or negative. Notice that if $\tilde{c}$ *equals the risk-free rate, LVA vanishes. Similarly, FVA vanishes if the funding rate* $\tilde{f}$ *is equal to the risk-free rate.*

We note that there is no general consensus on our definition of LVA and other authors may define it differently. For instance, Crépey [21–23] refers to LVA as the liquidity component (i.e., net of credit) of the funding valuation adjustment.

We now take a number of heuristic steps. A more formal analysis in terms of FBSDEs or PDEs is, for example, provided in Brigo et al. [15]. For simplicity, we first switch to the default-free market filtration $(\mathscr{F}_t)_{t\geq 0}$. This step implicitly assumes a separable structure of our complete filtration $(\mathscr{G}_t)_{t\geq 0}$. We are also assuming that the basic portfolio cash flows $\pi(0,t)$ are $\mathscr{F}_t$-measurable and that default times of all parties are conditionally independent, given filtration $\mathscr{F}$.

Assuming the relevant technical conditions are satisfied, the Feynman–Kac theorem now allows us to write down the corresponding pre-default partial differential equation (PDE) of the valuation problem (further details may be found in Brigo et al. [13, 14], and Sloth [37]). This PDE could be solved directly as in Crépey [22]. However, if we apply the Feynman–Kac theorem again—this time going from the pre-default PDE to the valuation expectation—and integrate by parts, we arrive at the following result

Theorem 3 (Continuous-time Solution of the Generalized Valuation Equation) *If we assume collateral rehypothecation and delta-hedging, we can solve the iterative equations of Theorem 2 in continuous time. We obtain*

$$\bar{V}_t = \int_t^T \mathbb{E}^{\tilde{f}}\{D(t,u;\tilde{f}+\lambda)[\pi_u + \lambda_u\theta_u + (\tilde{f}_u - \tilde{c}_u)C_u]|\mathscr{F}_t\}du \tag{20}$$

where λ_t *is the first-to-default intensity,* $\pi_t\,dt$ *is shorthand for* $\pi(t,t+dt)$*, and the discount factor is defined as* $D(t,s;\xi) \triangleq e^{-\int_t^s \xi_u du}$*. The expectations are taken under the pricing measure* $\mathbb{Q}^{\tilde{f}}$ *for which the underlying risk factors grow at the rate* $\tilde{f}$ *when the underlying pays no dividend.*

Theorem 3 decomposes the deal price $\bar{V}$ into three intuitive terms. The first term is the value of the deal cash flows, discounted at the funding rate plus credit. The second term is the price of the on-default cash-flow in excess of the collateral, which includes the CVA and DVA of the deal after collateralization. The last term collects the cost of collateralization. At this point it is very important to appreciate once again that $\tilde{f}$ depends on F, and hence on V.

Remark 2 (*Deal-dependent Valuation Measure, Local Risk-neutral Measures*). Since the pricing measure depends on $\tilde{f}$ which in turn depends on the very value $\bar{V}$ we are trying to compute, we have that the valuation measure becomes deal/portfolio-dependent. Claims sharing a common set of hedging instruments can be priced under a common measure.

Finally, we stress once again a very important invariance result that first appeared in Pallavicini et al. [34] and studied in detail in a more mathematical setting in Brigo et al. [15]. The proof is immediate by inspection.

Theorem 4 (Invariance of the Valuation Equation wrt. the Short Rate r_t).
Equation (20) *for valuation under credit, collateral, and funding costs is completely governed by market rates; there is no dependence on a risk-free rate r_t. Whichever initial process is postulated for r, the final price is invariant to it.*

4 Nonlinear Valuation: A Numerical Analysis

This section provides a numerical case study of the valuation framework outlined in the previous sections. We investigate the impact of funding risk on the price of a derivatives trade under default risk and collateralization. Also, we analyze the valuation error of ignoring nonlinearties of the general valuation problem. Specifically, to quantify this error, we introduce the concept of a nonlinearity valuation adjustment (NVA). A generalized least-squares Monte Carlo algorithm is proposed inspired by the simulation methods of Carriere [18], Longstaff and Schwartz [30], Tilley [38], and Tsitsiklis and Van Roy [39] for pricing American-style options. As the purpose is to understand the fundamental implications of funding risk and other nonlinearities, we focus on trading positions in relatively simple derivatives. However, the Monte Carlo method we propose below can be applied to more complex derivative contracts, including derivatives with bilateral payments.

4.1 Monte Carlo Pricing

Recall the recursive structure of the general valuation: The deal price depends on the funding decisions, while the funding strategy depends on the future price itself. The intimate relationship among the key quantities makes the valuation problem computationally challenging.

We consider K default scenarios during the life of the deal—either obtained by simulation, bootstrapped from empirical data, or assumed in advance. For each first-to-default time τ corresponding to a default scenario, we compute the price of the deal $\bar{V}$ under collateralization, close-out netting, and funding costs. The first step of our simulation method entails simulating a large number of sample paths N of the underlying risk factors X. We simulate these paths on the time-grid $\{t_1, \ldots, t_m = T^*\}$ with step size $\Delta t = t_{j+1} - t_j$ from the assumed dynamics of the risk factors. T^* is equal to the final maturity T of the deal or the consecutive time-grid point following the first-default time τ, whichever occurs first. For simplicity, we assume the time periods for funding decisions and collateral margin payments coincide with the simulation time grid.

Given the set of simulated paths, we solve the funding strategy recursively in a dynamic programming fashion. Starting one period before T^*, we compute for each simulated path the funding decision F and the deal price $\bar{V}$ according to the set of backward-inductive equations of Theorem 2. Note that while the reduced formulation of Theorem 3 may look simpler at first sight, avoiding the implicit recursive structure of Theorem 2, it would instead give us a forward–backward SDE problem to solve since the underlying asset now accrues at the funding rate which itself depends on $\bar{V}$. The algorithm then proceeds recursively until time zero. Ultimately, the total price of the deal is computed as the probability-weighted average of the individual prices obtained in each of the K default scenarios.

The conditional expectations in the backward-inductive funding equations are approximated by across-path regressions based on least squares estimation similar to Longstaff and Schwartz [30]. We regress the present value of the deal price at time t_{j+1}, the adjusted payout cash flow between t_j and t_{j+1}, the collateral account and funding account at time t_j on basis functions ψ of realizations of the underlying risk factors at time t_j across the simulated paths. To keep notation simple, let us assume that we are exposed to only one underlying risk factor, e.g. a stock price. Specifically, the conditional expectations in the iterative equations of Theorem 2, taken under the risk-neutral measure, are equal to

$$\mathbb{E}_{t_j}\left[\varXi_{t_j}(\bar{V}_{t_{j+1}})\right] = \theta'_{t_j}\,\psi(X_{t_j}), \tag{21}$$

where we have defined $\varXi_{t_j}(\bar{V}_{t_{j+1}}) \triangleq D(t_j, t_{j+1})\bar{V}_{t_{j+1}} + \bar{\pi}(t_j, t_{j+1}; C) - C_{t_j} - H_{t_j}$. Note the C_{t_j} term drops out if rehypothecation is not allowed. The usual least-squares estimator of θ is then given by

$$\hat{\theta}_{t_j} \triangleq \left[\psi(X_{t_j})\psi(X_{t_j})'\right]^{-1}\psi(X_{t_j})\,\varXi_{t_j}(\bar{V}_{t_{j+1}}). \tag{22}$$

Orthogonal polynomials such as Chebyshev, Hermite, Laguerre, and Legendre may all be used as basis functions for evaluating the conditional expectations. We find, however, that simple power series are quite effective and that the order of the polynomials can be kept relatively small. In fact, linear or quadratic polynomials, i.e. $\psi(X_{t_j}) = (\mathbf{1}, X_{t_j}, X_{t_j}^2)'$, are often enough.

Further complexities are added, as the dealer may—realistically—decide to hedge the full deal price $\bar{V}$. Now, the hedge H itself depends on the funding strategy through $\bar{V}$, while the funding decision depends on the hedging strategy. This added recursion requires that we solve the funding and hedging strategies simultaneously. For example, if the dealer applies a delta-hedging strategy we can write, heuristically,

$$H_{t_j} = \frac{\partial \bar{V}}{\partial X}\bigg|_{t_j} X_{t_j} \approx \frac{\bar{V}_{t_{j+1}} - (1 + \varDelta t_j \tilde{f}_{t_j})\bar{V}_{t_j}}{X_{t_{j+1}} - (1 + \varDelta t_j \tilde{f}_{t_j})X_{t_j}}\,X_{t_j}, \tag{23}$$

and we obtain, in the case of rehypothecation, the following system of nonlinear equations

$$\begin{cases} F_{t_j} - \frac{P_{t_j}^{\tilde{f}}(t_{j+1})}{P_{t_j}(t_{j+1})} \mathbb{E}_{t_j}\left[\Xi_{t_j}(\bar{V}_{t_{j+1}})\right] = 0, \\ H_{t_j} - \frac{\bar{V}_{t_{j+1}} - (1+\Delta t_j \tilde{f}_{t_j})\bar{V}_{t_j}}{X_{t_{j+1}} - (1+\Delta t_j \tilde{\tilde{f}}_{t_j})X_{t_j}} X_{t_j} = 0, \\ \bar{V}_{t_j} = F_{t_j} + C_{t_j} + H_{t_j}, \end{cases} \tag{24}$$

where all matrix operations are on an element-by-element basis. An analogous result holds when rehypothecation of the posted collateral is forbidden.

Each period and for each simulated path, we find the funding and hedging decisions by solving this system of equations, given the funding and hedging strategies for all future periods until the end of the deal. We apply a simple Newton–Raphson method to solve the system of nonlinear equations numerically, but instead of using the exact Jacobian, we approximate it by finite differences. As initial guess, we use the Black–Scholes delta position

$$H_{t_j}^0 = \Delta_{t_j}^{BS} X_{t_j}.$$

The convergence is quite fast and only a small number of iterations are needed in practice. Finally, if the dealer decides to hedge only the risk-free price of the deal, i.e. the classic derivative price V, the valuation problem collapses to a much simpler one. The hedge H no longer depends on the funding decision and can be computed separately, and the numerical solution of the nonlinear equation system can be avoided altogether.

In the following we apply our valuation framework to the case of a stock or equity index option. Nevertheless, the methodology extends fully to any other derivatives transaction. For instance, applications to interest rate swaps can be found in Pallavicini and Brigo [32] and Brigo and Pallavicini [6].

4.2 Case Outline

Let S_t denote the price of some stock or equity index and assume it evolves according to a geometric Brownian motion $dS_t = rS_t dt + \sigma S_t dW_t$ where W is a standard Brownian motion under the risk-neutral measure. The risk-free interest rate r is 100 bps, the volatility σ is 25 %, and the current price of the underlying is $S_0 = 100$. The European call option is in-the-money and has strike $K = 80$. The maturity T of the deal is 3 years and, in the full case, we assume that the investor delta-hedges the deal according to (23). The usual default-free funding-free and collateral-free Black–Scholes price V_0 of the call option deal is given by

$$V_t = S_t \Phi(d_1(t)) - Ke^{-r(T-t)}\Phi(d_2(t)), \quad d_{1,2} = \frac{\ln(S_t/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

and for $t = 0$ we get

$$V_0 = 28.9$$

with our choice of inputs. As usual, Φ is the cumulative distribution function of the standard normal random variable. In the usual setting, the hedge would not be (23) but a classical delta-hedging strategy based on $\Phi(d_1(t))$.

We consider two simple discrete probability distributions of default. Both parties of the deal are considered default risky but can only default at year 1 or at year 2. The localized joint default probabilities are provided in the matrices below. The rows denote the default time of the investor, while the columns denote the default times of the counterparty. For example, in matrix D_{low} the event $(\tau_I = 2yr, \tau_C = 1yr)$ has a 3 % probability and the first-to-default time is 1 year. Simultaneous defaults are introduced as an extension of our previous assumptions, and we determine the close-out amount by a random draw from a uniform distribution. If the random number is above 0.5, we compute the close-out as if the counterparty defaulted first, and vice versa.

For the first default distribution, we have a low dependence between the default risk of the counterparty and the default risk of the investor

$$D_{\text{low}} = \begin{array}{c c} & \begin{array}{ccc} 1yr & 2yr & n.d. \end{array} \\ \begin{array}{c} 1yr \\ 2yr \\ n.d. \end{array} & \begin{pmatrix} 0.01 & 0.01 & 0.03 \\ 0.03 & 0.01 & 0.05 \\ 0.07 & 0.09 & 0.70 \end{pmatrix} \end{array}, \qquad \tau_K(D_{\text{low}}) = 0.21 \tag{25}$$

where $n.d.$ means no default and τ_K denotes the rank correlation as measured by Kendall's tau. In the second case, we have a high dependence between the two parties' default risk

$$D_{\text{high}} = \begin{array}{c c} & \begin{array}{ccc} 1yr & 2yr & n.d. \end{array} \\ \begin{array}{c} 1yr \\ 2yr \\ n.d. \end{array} & \begin{pmatrix} 0.09 & 0.01 & 0.01 \\ 0.03 & 0.11 & 0.01 \\ 0.01 & 0.03 & 0.70 \end{pmatrix} \end{array}, \qquad \tau_K(D_{\text{high}}) = 0.83 \tag{26}$$

Note also that the distributions are skewed in the sense that the counterparty has a higher default probability than the investor. The loss, given default, is 50 % for both the investor and the counterparty and the loss on any posted collateral is considered the same. The collateral rates are chosen to be equal to the risk-free rate. We assume that the collateral account is equal to the risk-free price of the deal at each margin date, i.e. $C_t = V_t$. This is reasonable as the dealer and client will be able to agree on this price, in contrast to $\bar{V}_t$ due to asymmetric information. Also, choosing the collateral this way has the added advantage that the collateral account C works as a control variate, reducing the variance of the least-squares Monte Carlo estimator of the deal price.

4.3 *Preliminary Valuation Under Symmetric Funding and Without Credit Risk*

To provide some ball-park figures on the effect of funding risk, we first look at the case without default risk and without collateralization of the deal. We compare our Monte Carlo approach to the following two alternative (simplified) approaches:

(a) The Black–Scholes price where both discounting and the growth of the underlying happens at the symmetric funding rate

$$V_t^{(a)} = \left(S_t \Phi(g_1(t)) - K e^{-\hat{f}(T-t)} \Phi(g_2(t))\right),$$
$$g_{1,2} = \frac{\ln(S_t/K) + (\hat{f} \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

(b) We use the above FVA formula in Proposition 1 with some approximations. Since in a standard Black–Scholes setting $F_t = -K e^{-r(T-t)} \Phi(d_2(t))$, we compute

$$\begin{aligned} \mathrm{FVA}^{(b)} &= (r - \hat{f}) \int_0^T \mathbb{E}_0 \left\{ e^{-rs} [F_s] \right\} ds \\ &= (\hat{f} - r) K e^{-rT} \int_0^T \mathbb{E}_0 \left\{ \Phi(d_2(s)) \right\} ds \end{aligned}$$

We illustrate the two approaches for a long position in an equity call option. Moreover, let the funding valuation adjustment in each case be defined by $\mathrm{FVA}^{(a,b)} = V^{(a,b)} - V$. Figure 1 plots the resulting funding valuation adjustment with credit and collateral switched off under both simplified approaches and under the full valuation approach. Recall that if the funding rate is equal to the risk-free rate, the value of the call option collapses to the Black–Scholes price and the funding valuation adjustment is zero.

Remark 3 (*Current Market Practice for FVA*).
Looking at Fig. 1, it is important to realize that at the time of writing this paper, most market players would adopt a methodology like (a) or (b) for a simple call option. Even if borrowing or lending rates were different, most market players would average them and apply a common rate to borrowing and lending, in order to avoid nonlinearities. We notice that method (b) produces the same results as the quicker method (a) which simply replaces the risk-free rate by the funding rate. In the simple case without credit and collateral, and with symmetric borrowing and lending rates, we can show that this method is sound since it stems directly from (20). We also see that both methods (a) and (b) are quite close to the full numerical method we adopt. Overall both simplified methods (a) and (b) work well here, and there would be no need to implement the full machinery under these simplifying assumptions. However, once collateral, credit, and funding risks are in the picture, we have to abandon approximations like (a) or (b) and implement the full methodology instead.

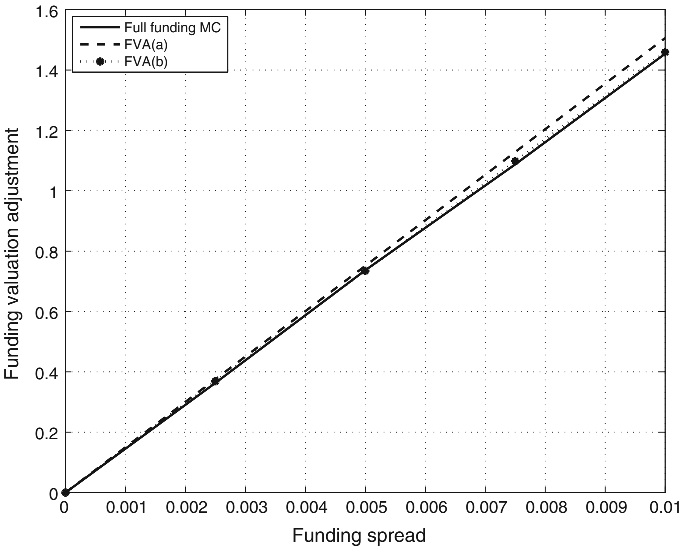

Fig. 1 Funding valuation adjustment of a long call position as a function of symmetric funding spreads $s_f := \hat{f} - r$ with $\hat{f} := f^+ = f^-$. The adjustments are computed under the assumption of no default risk nor collateralization

4.4 Complete Valuation Under Credit Risk, Collateral, and Asymmetric Funding

Let us now switch on credit risk and consider the impact of asymmetric funding rates. Due the presence of collateral as a control variate, the accuracy is quite good in our example even for relatively small numbers of sample paths. Based on the simulation of 1,000 paths, Tables 1 and 2 report the results of a ceteris paribus analysis of funding risk under counterparty credit risk and collateralization. Specifically, we investigate how the value of a deal changes for different values of the borrowing (lending) rate f^+ (f^-) while keeping the lending (borrowing) rate fixed to 100 bps. When both funding rates are equal to 100 bps, the deal is funded at the risk-free rate and we are in the classical derivatives valuation setting.

Remark 4 (*Potential Arbitrage*).
Note that if $f^+ < f^-$ arbitrage opportunities might be present, unless certain constraints are imposed on the funding policy of the treasury. Such constraints may look unrealistic and may be debated themselves from the point of view of arbitrageability, but since our point here is strictly to explore the impact of asymmetries in the funding equations, we will still apply our framework to a few examples where $f^+ < f^-$.

Table 1 reports the impact of changing funding rates for a call position when the posted collateral may not be used for funding the deal, i.e. rehypothecation is not allowed. First, we note that increasing the lending rate for a long position has a much

Table 1 Price impact of funding with default risk and collateralization

Funding[a] (bps)	Default risk, low[b]		Default risk, high[c]	
	Long	Short	Long	Short
Borrowing rate f^+				
100	28.70 (0.15)	−28.72 (0.15)	29.06 (0.21)	−29.07 (0.21)
125	28.53 (0.17)	−29.37 (0.18)	28.91 (0.21)	−29.70 (0.20)
150	28.37 (0.18)	−30.02 (0.22)	28.75 (0.22)	−30.34 (0.20)
175	28.21 (0.20)	−30.69 (0.27)	28.60 (0.22)	−30.99 (0.21)
200	28.05 (0.21)	−31.37 (0.31)	28.45 (0.22)	−31.66 (0.25)
Lending rate f^-				
100	28.70 (0.15)	−28.72 (0.15)	29.06 (0.21)	−29.07 (0.21)
125	29.35 (0.18)	−28.56 (0.17)	29.69 (0.20)	−28.92 (0.21)
150	30.01 (0.22)	−28.40 (0.18)	30.34 (0.20)	−28.76 (0.22)
175	30.68 (0.27)	−28.23 (0.20)	31.00 (0.21)	−28.61 (0.22)
200	31.37 (0.32)	−28.07 (0.39)	31.67 (0.25)	−28.46 (0.22)

Standard errors of the price estimates are given in parentheses
[a]Ceteris paribus changes in one funding rate while keeping the other fixed to 100 bps
[b]Based on the joint default distribution D_{low} with low dependence
[c]Based on the joint default distribution D_{high} with high dependence

Table 2 Price impact of funding with default risk, collateralization, and rehypothecation

Funding[a] (bps)	Default risk, low[b]		Default risk, high[c]	
	Long	Short	Long	Short
Borrowing rate f^+				
100	28.70 (0.15)	−28.73 (0.15)	29.07 (0.22)	−29.08 (0.22)
125	28.55 (0.17)	−29.56 (0.19)	28.92 (0.22)	−29.89 (0.20)
150	28.39 (0.18)	−30.40 (0.24)	28.77 (0.22)	−30.72 (0.20)
175	28.23 (0.20)	−31.26 (0.30)	28.63 (0.22)	−31.56 (0.23)
200	28.07 (0.22)	−32.14 (0.36)	28.48 (0.22)	−32.43 (0.29)
Lending rate f^-				
100	28.70 (0.15)	−28.73 (0.15)	29.07 (0.22)	−29.08 (0.22)
125	29.53 (0.19)	−28.57 (0.17)	29.07 (0.22)	−28.93 (0.22)
150	30.38 (0.24)	−28.42 (0.18)	32.44 (0.29)	−28.78 (0.22)
175	31.25 (0.30)	−28.26 (0.20)	36.19 (0.61)	−28.64 (0.22)
200	32.14 (0.37)	−28.10 (0.22)	32.44 (0.29)	−28.49 (0.22)

Standard errors of the price estimates are given in parentheses
[a]Ceteris paribus changes in one funding rate while keeping the other fixed to 100 bps
[b]Based on the joint default distribution D_{low} with low dependence
[c]Based on the joint default distribution D_{high} with high dependence

larger impact than increasing the borrowing rate. This is due to the fact that a call option is just a one-sided contract. Recall that F is defined as the cash account needed as part of the derivative replication strategy or, analogously, the cash account required to fund the hedged derivative position. To hedge a long call, the investor goes short in a delta position of the underlying asset and invests excess cash in the treasury at f^-. Correspondingly, to hedge the short position, the investor enters a long delta position in the stock and finances it by borrowing cash from the treasury at f^+, so changing the lending rate only has a small effect on the deal value. Finally, due to the presence of collateral, we observe an almost similar price impact of funding under the two different default distributions D_{low} and D_{high}.

Finally, assuming cash collateral, we consider the case of rehypothecation and allow the investor and counterparty to use any posted collateral as a funding source. If the collateral is posted to the investor, this means it effectively reduces his costs of funding the delta-hedging strategy. As the payoff of the call is one-sided, the investor only receives collateral when he holds a long position in the call option. But as he hedges this position by short-selling the underlying stock and lending the excess cash proceeds, the collateral adds to his cash lending position and increases the funding benefit of the deal. Analogously, if the investor has a short position, he posts collateral to the counterparty and a higher borrowing rate would increase his costs of funding the collateral he has to post as well as his delta-hedge position. Table 2 reports the results for the short and long positions in the call option when rehypothecation is allowed. Figures 2 and 3 plot the values of collateralized long and short positions in the call option as a function of asymmetric funding spreads. In addition, Fig. 4

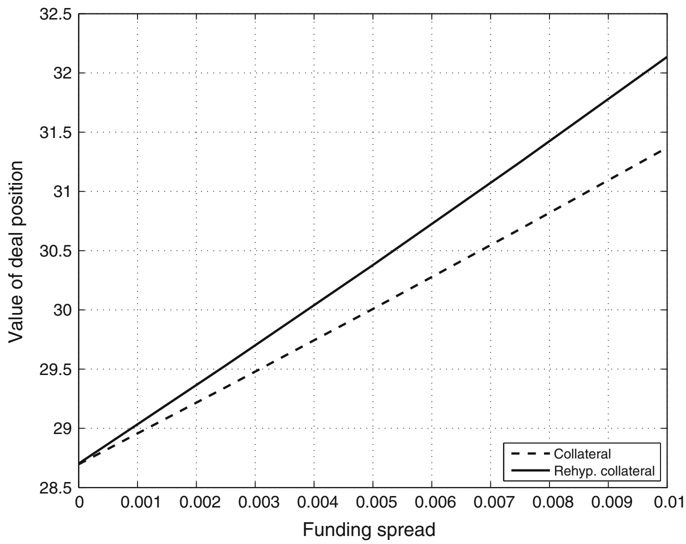

Fig. 2 The value of a long call position for asymmetric funding spreads $s_f^- = f^- - r$, i.e. fixing $f^+ = r = 0.01$ and varying $f^- \in (0.01, 0.0125, 0.015, 0.0175, 0.02)$

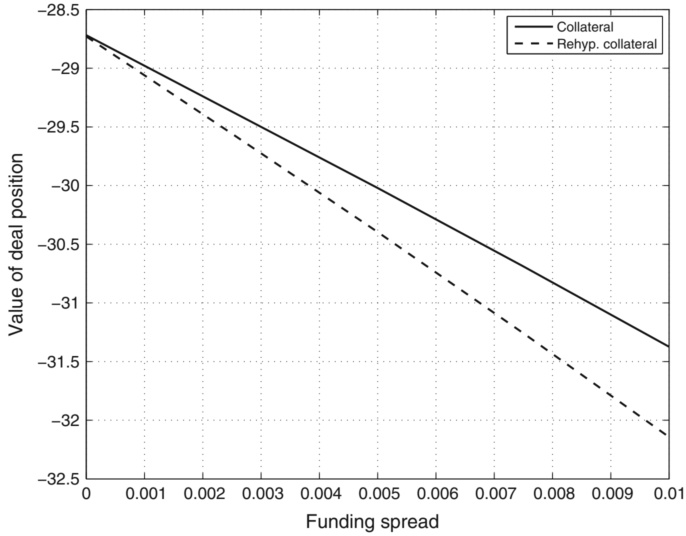

Fig. 3 The value of a short call position for asymmetric funding spreads $s_f^+ = f^+ - r$, i.e. fixing $f^- = r = 0.01$ and varying $f^+ \in (0.01, 0.0125, 0.015, 0.0175, 0.02)$

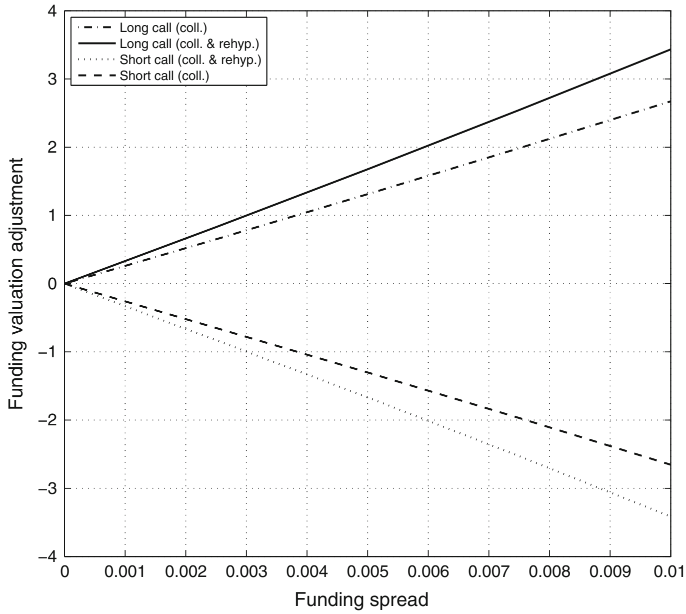

Fig. 4 Funding valuation adjustment as a function of asymmetric funding spreads. The adjustments are computed under the presence of default risk and collateralization

reports the FVA with respect to the magnitude of the funding spreads, where the FVA is defined as the difference between the full funding-inclusive deal price and the full deal price, but symmetric funding rates equal to the risk-free rate. Recall that the collateral rates are equal to the risk-free rate, so the LVA collapses to zero in these examples.

This shows that funding asymmetry matters even under full collateralization when there is no repo market for the underlying stock. In practice, however, the dealer cannot hedge a long call by shorting a stock he does not own. Instead, he would first borrow the stock in a repo transaction and then sell it in the spot market. Similarly, to enter the long delta position needed to hedge a short call, the dealer could finance the purchase by lending the stock in a reverse repo transaction. Effectively, the delta position in the underlying stock would be funded at the prevailing repo rate. Thus, once the delta hedge has to be executed through the repo market, there is no funding valuation adjustment (meaning any dependence on the funding rate $\tilde{f}$ drops out) given the deal is fully collateralized, but the underlying asset still grows at the repo rate. If there is no credit risk, this would leave us with the result of Piterbarg [36]. However, if the deal is not fully collateralized or the collateral cannot be rehypothecated, funding costs enter the picture even when there is a repo market for the underlying stock.

4.5 Nonlinearity Valuation Adjustment

In this last section we introduce a nonlinearity valuation adjustment, and to stay within the usual jargon of the business, we abbreviate it NVA. The NVA is defined by the difference between the true price $\bar{V}$ and a version of $\bar{V}$ where nonlinearities have been approximated away through blunt symmetrization of rates and possibly a change in the close-out convention from replacement close-out to risk-free close-out. This entails a degree of double counting (both positive and negative interest). In some situations the positive and negative double counting will offset each other, but in other cases this may not happen. Moreover, as pointed out by Brigo et al. [10], a further source of double counting might be neglecting the first-to-default time in bilateral CVA/DVA valuation. This is done in a number of industry approximations.

Let $\hat{V}$ be the resulting price when we replace both f^+ and f^- by $\hat{f} := (f^+ + f^-)/2$ and adopt a risk-free close-out at default in our valuation framework. A further simplification in $\hat{V}$ could be to neglect the first-to-default check in the close-out. We have the following definition

Definition 1 (*Nonlinearity Valuation Adjustment, NVA*)
NVA is defined as

$$\mathrm{NVA}_t \triangleq \bar{V}_t - \hat{V}_t$$

where $\bar{V}$ denotes the full nonlinear deal value while $\hat{V}$ denotes an approximate linearized price of the deal.

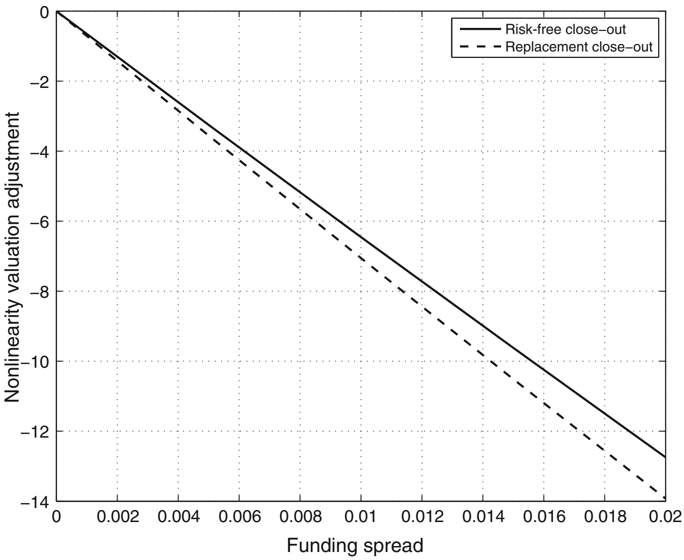

Fig. 5 Nonlinearity valuation adjustment (in percentage of $\hat{V}$) for different funding spreads $s_f^+ = f^+ - f^- \in (0, 0.005, 0.01, 0.015, 0.02)$ and fixed $\hat{f} = (f^+ + f^-)/2 = 0.01$

As an illustration, we revisit the above example of an equity call option and analyze the NVA in a number of cases. The results are reported in Figs. 5 and 6.

In both figures, we compare NVA under risk-free close-out and under replacement close-out. We can see that, depending on the direction of the symmetrization, NVA may be either positive or negative. As the funding spread increases, NVA grows in absolute value. In addition, adopting the replacement close-out amplifies the presence of double counting. The NVA accounts for up to 15 % of the full deal price $\bar{V}$ depending on the funding spread—a relevant figure in a valuation context.

Table 3 reports (a) $\%\widehat{\text{NVA}}$ denoting the fraction of the approximated deal price $\hat{V}$ explained by NVA, and (b) %NVA denoting the fraction of the full deal price $\bar{V}$ (with symmetric funding rates equal to the risk-free rate r) explained by NVA. Notice that for those cases where we adopt a risk-free close-out at default, the results primarily highlight the double-counting error due to symmetrization of borrowing and lending rates. We should point out that close-out nonlinearities play a limited role here, due to absence of wrong way risk. An analysis of close-out nonlinearity under wrong way risk is under development.

Finally, it should be noted that linearization may in fact be done in arbitrarily many ways by playing with the discount factor, hence taking the average of two funding rates as in our definition of NVA is not necessarily the best one. However, we postpone further investigations into this interesting topic for future research.

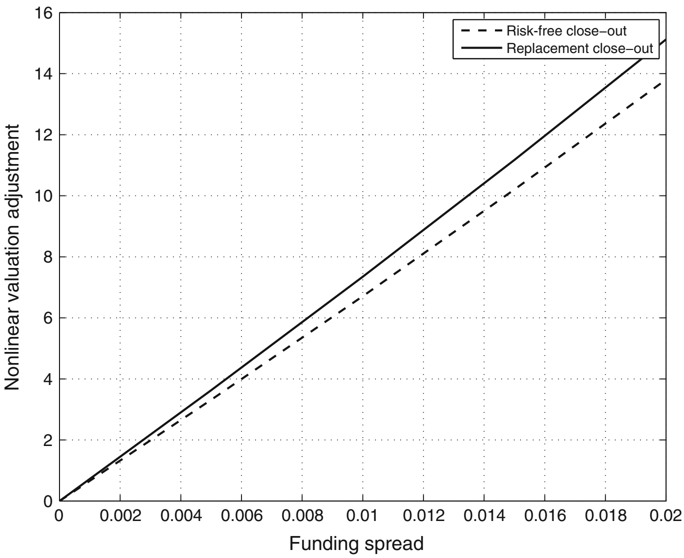

Fig. 6 Nonlinearity valuation adjustment (in percentage of $\hat{V}$) for different funding spreads $s_f^- = f^- - f^+ \in (0, 0.005, 0.01, 0.015, 0.02)$ and fixed $\hat{f} = (f^+ + f^-)/2 = 0.01$

Table 3 %NVA with default risk, collateralization and rehypothecation

Funding rates		Risk free		Replacement	
		$\%\widehat{\text{NVA}}$	%NVA	$\%\widehat{\text{NVA}}$	%NVA
s_f^{b} (bps)	$\hat{f}$ (bps)				
0	100	0 %	0 %	0 %	0 %
25	112.5	1.65 %	1.67 %	1.79 %	1.81 %
50	125	3.31 %	3.39 %	3.58 %	3.68 %
75	137.5	5.02 %	5.19 %	5.39 %	5.61 %
100	150	6.70 %	7.01 %	7.24 %	7.62 %

[a]Funding spread $s_f = f^- - f^+$
[b]The prices of the call option are based on the joint default distribution D_{high} with high dependence

5 Conclusions and Financial Implications

We have developed a consistent framework for valuation of derivative trades under collateralization, counterparty credit risk, and funding costs. Based on no arbitrage, we derived a generalized pricing equation where CVA, DVA, LVA, and FVA are introduced by simply modifying the payout cash flows of the trade. The framework is flexible enough to accommodate actual trading complexities such as asymmetric collateral and funding rates, replacement close-out, and rehypothecation of posted collateral. Moreover, we presented an invariance theorem showing that the valuation framework does not depend on any theoretical risk-free rate, but is purely based on observable market rates.

The generalized valuation equation under credit, collateral, and funding takes the form of a forward–backward SDE or semi-linear PDE. Nevertheless, it can be recast as a set of iterative equations which can be efficiently solved by a proposed least-squares Monte Carlo algorithm. Our numerical results confirm that funding risk as well as asymmetries in borrowing and lending rates have a significant impact on the ultimate value of a derivatives transaction.

Introducing funding costs into the pricing equation makes the valuation problem recursive and nonlinear. The price of the deal depends on the trader's funding strategy, while to determine the funding strategy we need to know the deal price itself. Credit and funding risks are in general non-separable; this means that FVA is not an additive adjustment, let alone a discounting spread. Thus, despite being common practice among market participants, treating it as such comes at the cost of double counting. We introduce the "nonlinearity valuation adjustment" (NVA) to quantify the effect of double counting and we show that its magnitude can be significant under asymmetric funding rates and replacement close-out at default.

Furthermore, valuation under funding costs is no longer bilateral as the particular funding policy chosen by the dealer is not known to the client, and vice versa. As a result, the value of the trade will generally be different to the two counterparties.

Finally, valuation depends on the level of aggregation; asset portfolios cannot simply be priced separately and added up. Theoretically, valuation is conducted under deal or portfolio-dependent risk-neutral measures. This has clear operational consequences for financial institutions; it is difficult for banks to establish CVA and FVA desks with separate, clear-cut responsibilities. In theory, they should adopt a consistent valuation approach across all trading desks and asset classes. A trade should be priced on an appropriate aggregation-level to quantify the value it actually adds to the business. This, of course, prompts to the old distinction between price and value: Should funding costs be charged to the client or just included internally to determine the profitability of a particular trade? The relevance of this question is reinforced by the fact that the client has no direct control on the funding policy of the bank and therefore cannot influence any potential inefficiencies for which he or she would have to pay.

While holistic trading applications may be unrealistic with current technology, our valuation framework offers a unique understanding of the nature and presence of nonlinearities and paves the way for developing more suitable and practical linearizations. The latter topic we will leave for future research.

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Analysis of Nonlinear Valuation Equations Under Credit and Funding Effects

Damiano Brigo, Marco Francischello and Andrea Pallavicini

Abstract We study conditions for existence, uniqueness, and invariance of the comprehensive nonlinear valuation equations first introduced in Pallavicini et al. (Funding valuation adjustment: a consistent framework including CVA, DVA, collateral, netting rules and re-hypothecation, 2011, [11]). These equations take the form of semi-linear PDEs and Forward–Backward Stochastic Differential Equations (FBSDEs). After summarizing the cash flows definitions allowing us to extend valuation to credit risk and default closeout, including collateral margining with possible re-hypothecation, and treasury funding costs, we show how such cash flows, when present-valued in an arbitrage-free setting, lead to semi-linear PDEs or more generally to FBSDEs. We provide conditions for existence and uniqueness of such solutions in a classical sense, discussing the role of the hedging strategy. We show an invariance theorem stating that even though we start from a risk-neutral valuation approach based on a locally risk-free bank account growing at a risk-free rate, our final valuation equations do not depend on the risk-free rate. Indeed, our final semi-linear PDE or FBSDEs and their classical solutions depend only on contractual, market or treasury rates and we do not need to proxy the risk-free rate with a real market rate, since it acts as an instrumental variable. The equations' derivations, their numerical solutions, the related XVA valuation adjustments with their overlap, and the invariance result had been analyzed numerically and extended to central clearing and multiple discount curves in a number of previous works, including Brigo and Pallavicini (J. Financ. Eng. 1(1):1–60 (2014), [3]), Pallavicini and Brigo (Interest-rate modelling in collateralized markets: multiple curves, credit-liquidity effects, CCPs, 2011, [10]), Pallavicini et al. (Funding valuation adjustment: a consistent framework including cva, dva, collateral, netting rules and re-hypothecation, 2011, [11]), Pallavicini et al. (Funding, collateral and hedging: uncovering the mechanics and the subtleties of

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funding valuation adjustments, 2012, [12]), and Brigo et al. (Nonlinear valuation under collateral, credit risk and funding costs: a numerical case study extending Black–Scholes, [5]).

Keywords Counterparty credit risk · Funding valuation adjustment · Funding costs · Collateralization · Nonlinearity valuation adjustment · Nonlinear valuation · Derivatives valuation · Semi-linear PDE · FBSDE · BSDE · Existence and uniqueness of solutions

1 Introduction

This is a technical paper where we analyze in detail invariance, existence, and uniqueness of solutions for nonlinear valuation equations inclusive of credit risk, collateral margining with possible re-hypothecation, and funding costs. In particular, we study conditions for existence, uniqueness, and invariance of the comprehensive nonlinear valuation equations first introduced in Pallavicini et al. (2011) [11]. After briefly summarizing the cash flows definitions allowing us to extend valuation to default closeout, collateral margining with possible re-hypothecation and treasury funding costs, we show how such cash flows, when present-valued in an arbitrage-free setting, lead straightforwardly to semi-linear PDEs or more generally to FBSDEs. We study conditions for existence and uniqueness of such solutions.

We formalize an invariance theorem showing that even though we start from a risk-neutral valuation approach based on a locally risk-free bank account growing at a risk-free rate, our final valuation equations do not depend on the risk-free rate at all. In other words, we do not need to proxy the risk-free rate with any actual market rate, since it acts as an instrumental variable that does not manifest itself in our final valuation equations. Indeed, our final semi-linear PDEs or FBSDEs and their classical solutions depend only on contractual, market or treasury rates and contractual closeout specifications once we use a hedging strategy that is defined as a straightforward generalization of the natural delta hedging in the classical setting.

The equations' derivations, their numerical solutions, and the invariance result had been analyzed numerically and extended to central clearing and multiple discount curves in a number of previous works, including [3, 5, 10–12], and the monograph [6], which further summarizes earlier credit and debit valuation adjustment (CVA and DVA) results. We refer to such works and references therein for a general introduction to comprehensive nonlinear valuation and to the related issues with valuation adjustments related to credit (CVA), collateral (LVA), and funding costs (FVA). In this paper, given the technical nature of our investigation and the emphasis on nonlinear valuation, we refrain from decomposing the nonlinear value into valuation adjustments or XVAs. Moreover, in practice such separation is possible only under very specific assumptions, while in general all terms depend on all risks due to nonlinearity. Forcing separation may lead to double counting, as initially analyzed through the Nonlinearity Valuation Adjustment (NVA) in [5]. Separation is discussed in the CCP setting in [3].

The paper is structured as follows.

Section 2 introduces the probabilistic setting, the cash flows analysis, and derives a first valuation equation based on conditional expectations. Section 3 derives an FBSDE under the default-free filtration from the initial valuation equation under assumptions of conditional independence of default times and of default-free initial portfolio cash flows. Section 4 specifies the FBSDE obtained earlier to a Markovian setting and studies conditions for existence and uniqueness of solutions for the non-linear valuation FBSDE and classical solutions to the associated PDE. Finally, we present the invariance theorem: when adopting delta-hedging, the solution does not depend on the risk-free rate.

2 Cash Flows Analysis and First Valuation Equation

We fix a filtered probability space $(\Omega, \mathscr{A}, \mathbb{Q})$, with a filtration $(\mathscr{G}_u)_{u\geq 0}$ representing the evolution of all the available information on the market. With an abuse of notation, we will refer to $(\mathscr{G}_u)_{u\geq 0}$ by $\mathscr{G}$. The object of our investigation is a portfolio of contracts, or "contract" for brevity, typically a netting set, with final maturity T, between two financial entities, the investor I and the counterparty C. Both I and C are supposed to be subject to default risk. In particular we model their default times with two $\mathscr{G}$-stopping times τ_I, τ_C. We assume that the stopping times are generated by Cox processes of positive, stochastic intensities λ^I and λ^C. Furthermore, we describe the *default-free* information by means of a filtration $(\mathscr{F}_u)_{u\geq 0}$ generated by the price of the underlying S_t of our contract. This process has the following dynamic under the measure $\mathbb{Q}$:

$$dS_t = r_t S_t dt + \sigma(t, S_t) dW_t$$

where r_t is an $\mathscr{F}$-adapted process, called the *risk-free* rate. We then suppose the existence of a risk-free account B_t following the dynamics

$$dB_t = r_t B_t dt.$$

We denote $D(s, t, x) = e^{-\int_s^t x_u du}$, the discount factor associated to the rate x_u. In the case of the risk-free rate, we define $D(s, t) := D(s, t, r)$.

We further assume that for all t we have $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{H}_t^I \vee \mathscr{H}_t^C$ where

$$\mathscr{H}_t^I = \sigma(1_{\{\tau_I \leq s\}},\ s \leq t),$$
$$\mathscr{H}_t^C = \sigma(1_{\{\tau_C \leq s\}},\ s \leq t).$$

Again we indicate $(\mathscr{F}_u)_{u\geq 0}$ by $\mathscr{F}$ and we will write $\mathbb{E}_t^{\mathscr{G}}[\cdot] := \mathbb{E}[\cdot|\mathscr{G}_t]$ and similarly for $\mathscr{F}$. As in the classic framework of Duffie and Huang [8], we postulate the default times to be *conditionally independent* with respect to $\mathscr{F}$, i.e. for any $t > 0$ and $t_1, t_2 \in [0, t]$, we assume $\mathbb{Q}\{\tau_I > t_1, \tau_C > t_2 | \mathscr{F}_t\} = \mathbb{Q}\{\tau_I > t_1 | \mathscr{F}_t\}\mathbb{Q}\{\tau_C > t_2 | \mathscr{F}_t\}$. Moreover, we indicate $\tau = \tau_I \wedge \tau_C$ and with these assumptions we have that τ has intensity $\lambda_u = \lambda_u^I + \lambda_u^C$. For convenience of notation we use the symbol $\bar{\tau}$ to indicate the minimum between τ and T.

Remark 1 We suppose that the measure $\mathbb{Q}$ is the so-called *risk-neutral* measure, i.e. a measure under which the prices of the traded non-dividend-paying assets discounted at the risk-free rate are martingales or, in equivalent terms, the measure associated with the numeraire B_t.

2.1 The Cash Flows

To price this portfolio we take the conditional expectation of all the cash flows of the portfolio and discount them at the risk-free rate. An alternative to the explicit cash flows approach adopted here is discussed in [4].

To begin with, we consider a collateralized hedged contract, so the cash flows generated by the contract are:

- The payments due to the contract itself: modeled by an $\mathscr{F}$-predictable process π_t and a final cash flow $\Phi(S_T)$ payed at maturity modeled by a Lipschitz function Φ. At time t the cumulated discounted flows due to these components amount to

$$1_{\{\tau > T\}} D(0, T)\Phi(S_T) + \int_t^{\bar{\tau}} D(t, u)\pi_u du.$$

- The payments due to default: in particular we suppose that at time τ we have a cash flow due to the default event (if it happened) modeled by a $\mathscr{G}_\tau$-measurable random variable θ_τ. So the flows due to this component are

$$1_{\{t < \tau < T\}} D(t, \tau)\theta_\tau = 1_{\{t < \tau < T\}} \int_t^T D(t, u)\theta_u d1_{\{\tau \le u\}}.$$

- The payments due to the collateral account: more precisely we model this account by an $\mathscr{F}$-predictable process C_t. We postulate that $C_t > 0$ if the investor is the collateral taker, and $C_t < 0$ if the investor is the collateral provider. Moreover, we assume that the collateral taker remunerates the account at a certain interest rate (written on the CSA); in particular we may have different rates depending on who the collateral taker is, so we introduce the rate

$$c_t = 1_{\{C_t > 0\}} c_t^+ + 1_{\{C_t \le 0\}} c_t^- , \tag{1}$$

where c_t^+, c_t^- are two $\mathscr{F}$-predictable processes. We also suppose that the collateral can be re-hypothecated, i.e. the collateral taker can use the collateral for funding purposes. Since the collateral taker has to remunerate the account at the rate c_t, the discounted flows due to the collateral can be expressed as a cost of carry and sum up to

$$\int_t^{\bar{\tau}} D(t,u)(r_u - c_u)C_u du.$$

- We suppose that the deal we are considering is to be hedged by a position in cash and risky assets, represented respectively by the $\mathcal{G}$-adapted processes F_t and H_t, with the convention that $F_t > 0$ means that the investor is borrowing money (from the bank's treasury for example), while $F < 0$ means that I is investing money. Also in this case to take into account different rates in the borrowing or lending case we introduce the rate

$$f_t = 1_{\{V_t - C_t > 0\}} f_t^+ + 1_{\{V_t - C_t \leq 0\}} f_t^- . \tag{2}$$

The flows due to the funding part are

$$\int_t^{\bar{\tau}} D(t,u)(r_u - f_u)F_u du.$$

For the flows related to the risky assets account H_t we assume that we are hedging by means of repo contracts. We have that $H_t > 0$ means that we need some risky asset, so we borrow it, while if $H < 0$ we lend. So, for example, if we need to borrow the risky asset we need cash from the treasury, hence we borrow cash at a rate f_t and as soon as we have the asset we can repo lend it at a rate h_t. In general h_t is defined as

$$h_t = 1_{\{H_t > 0\}} h_t^+ + 1_{\{H_t \leq 0\}} h_t^- . \tag{3}$$

Thus we have that the total discounted cash flows for the risky part of the hedge are equal to

$$\int_t^{\bar{\tau}} D(t,u)(h_u - f_u)H_u du.$$

The last expression could also be seen as resulting from $(r - f) - (r - h)$, in line with the previous definitions. If we add all the cash flows mentioned above we obtain that the value of the contract V_t must satisfy

$$\begin{aligned} V_t =& \mathbb{E}_t^{\mathcal{G}} \left[\int_t^{\bar{\tau}} D(t,u)(\pi_u + (r_u - c_u)C_u + (r_u - f_u)F_u - (f_u - h_u)H_u)du \right] \\ &+ \mathbb{E}_t^{\mathcal{G}} \left[1_{\{\tau > T\}} D(t,T)\Phi(S_T) + D(t,\tau)1_{\{t < \tau < T\}}\theta_\tau \right]. \end{aligned} \tag{4}$$

If we further suppose that we are able to replicate the value of our contract using the funding, the collateral (assuming re-hypothecation, otherwise C is to be omitted from the following equation) and the risky asset accounts, i.e.

$$V_u = F_u + H_u + C_u, \tag{5}$$

we have, substituting for F_u:

$$\begin{aligned} V_t =&\mathbb{E}_t^{\mathscr{G}}\left[\int_t^{\bar{\tau}} D(t,u)(\pi_u + (f_u - c_u)C_u + (r_u - f_u)V_u - (r_u - h_u)H_u)du\right] \\ &+ \mathbb{E}_t^{\mathscr{G}}\left[1_{\{\tau > T\}}D(t,T)\Phi(S_T) + D(t,\tau)1_{\{t<\tau<T\}}\theta_\tau\right]. \end{aligned} \tag{6}$$

Remark 2 In the classic no-arbitrage theory and in a complete market setting, without credit risk, the hedging process H would correspond to a delta hedging strategy account. Here we do not enforce this interpretation yet. However, we will see that a delta-hedging interpretation emerges from the combined effect of working under the default-free filtration $\mathscr{F}$ (valuation under partial information) and of identifying part of the solution of the resulting BSDE, under reasonable regularity assumptions, as a sensitivity of the value to the underlying asset price S.

2.2 Adjusted Cash Flows Under a Simple Trading Model

We now show how the adjusted cash flows originate assuming we buy a call option on an equity asset S_T with strike K. We analyze the operations a trader would enact with the treasury and the repo market in order to fund the trade, and we map these operations to the related cash flows. We go through the following steps in each small interval $[t, t + dt]$, seen from the point of view of the trader/investor buying the option. This is written in first person for clarity and is based on conversations with traders working with their bank treasuries.

Time t:

1. I wish to buy a call option with maturity T whose current price is $V_t = V(t, S_t)$. I need V_t cash to do that. So I borrow V_t cash from my bank treasury and buy the call.
2. I receive the collateral amount C_t for the call, that I give to the treasury.
3. Now I wish to hedge the call option I bought. To do this, I plan to repo-borrow Δ_t stock on the repo-market.
4. To do this, I borrow $H_t = \Delta_t S_t$ cash at time t from the treasury.
5. I repo-borrow an amount Δ_t of stock, posting cash H_t as a guarantee.
6. I sell the stock I just obtained from the repo to the market, getting back the price H_t in cash.
7. I give H_t back to treasury.
8. My outstanding debt to the treasury is $V_t - C_t$.

Time $t + dt$:

9. I need to close the repo. To do that I need to give back $\varDelta_t$ stock. I need to buy this stock from the market. To do that I need $\varDelta_t S_{t+dt}$ cash.
10. I thus borrow $\varDelta_t S_{t+dt}$ cash from the bank treasury.
11. I buy $\varDelta_t$ stock and I give it back to close the repo and I get back the cash H_t deposited at time t plus interest $h_t H_t$.
12. I give back to the treasury the cash H_t I just obtained, so that the net value of the repo operation has been

$$H_t(1 + h_t\, dt) - \varDelta_t S_{t+dt} = -\varDelta_t\, dS_t + h_t H_t\, dt$$

Notice that this $-\varDelta_t dS_t$ is the right amount I needed to hedge V in a classic delta hedging setting.

13. I close the derivative position, the call option, and get V_{t+dt} cash.
14. I have to pay back the collateral plus interest, so I ask the treasury the amount $C_t(1 + c_t\, dt)$ that I give back to the counterparty.
15. My outstanding debt plus interest (at rate f) to the treasury is $V_t - C_t + C_t(1 + c_t\, dt) + (V_t - C_t)f_t\, dt = V_t(1 + f_t\, dt) + C_t(c_t - f_t\, dt)$. I then give to the treasury the cash V_{t+dt} I just obtained, the net effect being

$$V_{t+dt} - V_t(1 + f_t\, dt) - C_t(c_t - f_t)\, dt = dV_t - f_t V_t\, dt - C_t(c_t - f_t)\, dt$$

16. I now have that the total amount of flows is:

$$-\varDelta_t\, dS_t + h_t H_t\, dt + dV_t - f_t V_t\, dt - C_t(c_t - f_t)\, dt$$

17. Now I present-value the above flows in t in a risk-neutral setting.

$$\begin{aligned}
&\mathbb{E}_t[-\varDelta_t\, dS_t + h_t H_t\, dt + dV_t - f_t V_t\, dt - C_t(c_t - f_t)\, dt] \\
&\quad = -\varDelta_t(r_t - h_t)S_t\, dt + (r_t - f_t)V_t\, dt - C_t(c_t - f_t)\, dt - d\varphi(t) \\
&\quad = -H_t(r_t - h_t)\, dt + (r_t - f_t)(H_t + F_t + C_t)\, dt - C_t(c_t - f_t)\, dt - d\varphi(t) \\
&\quad = (h_t - f_t)H_t\, dt + (r_t - f_t)F_t\, dt + (r_t - c_t)C_t\, dt - d\varphi(t)
\end{aligned}$$

This derivation holds assuming that $\mathbb{E}_t[dS_t] = r_t S_t\, dt$ and $\mathbb{E}_t[dV_t] = r_t V_t\, dt - d\varphi(t)$, where $d\varphi$ is a dividend of V in $[t, t + dt)$ expressing the funding costs. Setting the above expression to zero we obtain

$$d\varphi(t) = (h_t - f_t)H_t\, dt + (r_t - f_t)F_t\, dt + (r_t - c_t)C_t\, dt$$

which coincides with the definition given earlier in (6).

3 An FBSDE Under $\mathscr{F}$

We aim to switch to the default free filtration $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$, and the following lemma (taken from Bielecki and Rutkowski [1] Sect. 5.1) is the key in understanding how the information expressed by $\mathscr{G}$ relates to the one expressed by $\mathscr{F}$.

Lemma 1 *For any $\mathscr{A}$-measurable random variable X and any $t \in \mathbb{R}_+$, we have:*

$$\mathbb{E}_t^{\mathscr{G}}[1_{\{t<\tau\leq s\}}X] = 1_{\{\tau>t\}}\frac{\mathbb{E}_t^{\mathscr{F}}[1_{\{t<\tau\leq s\}}X]}{\mathbb{E}_t^{\mathscr{F}}[1_{\{\tau>t\}}]}. \tag{7}$$

In particular we have that for any $\mathscr{G}_t$-measurable random variable Y there exists an $\mathscr{F}_t$-measurable random variable Z such that

$$1_{\{\tau>t\}}Y = 1_{\{\tau>t\}}Z.$$

What follows is an application of the previous lemma exploiting the fact that we have to deal with a stochastic process structure and not only a simple random variable. Similar results are illustrated in [2].

Lemma 2 *Suppose that ϕ_u is a $\mathscr{G}$-adapted process. We consider a default time τ with intensity λ_u. If we denote $\bar{\tau} = \tau \wedge T$ we have:*

$$\mathbb{E}_t^{\mathscr{G}}\left[\int_t^{\bar{\tau}} \phi_u du\right] = 1_{\{\tau>t\}}\mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,\lambda)\widetilde{\phi}_u du\right]$$

where $\widetilde{\phi}_u$ is an $\mathscr{F}_u$ measurable variable such that $1_{\{\tau>u\}}\widetilde{\phi}_u = 1_{\{\tau>u\}}\phi_u$.

Proof

$$\mathbb{E}_t^{\mathscr{G}}\left[\int_t^{\bar{\tau}} \phi_u du\right] = \mathbb{E}_t^{\mathscr{G}}\left[\int_t^T 1_{\{\tau>t\}}1_{\{\tau>u\}}\phi_u du\right] = \int_t^T \mathbb{E}_t^{\mathscr{G}}\left[1_{\{\tau>t\}}1_{\{\tau>u\}}\phi_u\right] du$$

then by using Lemma 1 we have

$$= \int_t^T 1_{\{\tau>t\}}\frac{\mathbb{E}_t^{\mathscr{F}}\left[1_{\{\tau>t\}}1_{\{\tau>u\}}\phi_u\right]}{\mathbb{Q}[\tau > t\,|\mathscr{F}_t]}du = 1_{\{\tau>t\}}\int_t^T \mathbb{E}_t^{\mathscr{F}}\left[1_{\{\tau>u\}}\phi_u\right]D(0,t,\lambda)^{-1}du$$

now we choose an $\mathscr{F}_u$ measurable variable such that $1_{\{\tau>u\}}\widetilde{\phi}_u = 1_{\{\tau>u\}}\phi_u$ and obtain

$$\begin{aligned}
&= 1_{\{\tau>t\}}\int_t^T \mathbb{E}_t^{\mathscr{F}}\left[\mathbb{E}_u^{\mathscr{F}}\left[1_{\{\tau>u\}}\right]\widetilde{\phi}_u\right]D(0,t,\lambda)^{-1}du \\
&= 1_{\{\tau>t\}}\int_t^T \mathbb{E}_t^{\mathscr{F}}\left[D(0,u,\lambda)\widetilde{\phi}_u\right]D(0,t,\lambda)^{-1}du = 1_{\{\tau>t\}}\mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,\lambda)\widetilde{\phi}_u du\right]
\end{aligned}$$

where the penultimate equality comes from the fact that the default times are conditionally independent and if we define $\Lambda_X(u) = \int_0^u \lambda_s^X ds$ with $X \in \{I, C\}$ we have that $\tau_X = \Lambda_X^{-1}(\xi_X)$ with ξ_X mutually independent exponential random variables independent from λ^X.[1] A similar result will enable us to deal with the default cash flow term. In fact we have the following (Lemma 3.8.1 in [2])

Lemma 3 *Suppose that ϕ_u is an $\mathscr{F}$-predictable process. We consider two conditionally independent default times τ_I, τ_C generated by Cox processes with $\mathscr{F}$-intensity rates λ_t^I, λ_t^C. If we denote $\tau = \tau_C \wedge \tau_I$ we have:*

$$\mathbb{E}_t^{\mathscr{G}}\left[1_{\{t<\tau<T\}}1_{\{\tau_I<\tau_C\}}\phi_\tau\right] = 1_{\{\tau>t\}}\mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,\lambda^I+\lambda^C)\lambda_u^I\phi_u du\right].$$

Now we postulate a particular form for the default cash flow, more precisely if we indicate $\widetilde{V}_t$ the $\mathscr{F}$-adapted process such that

$$1_{\{\tau>t\}}\widetilde{V}_t = 1_{\{\tau>t\}}V_t$$

then we define

$$\theta_t = \epsilon_t - 1_{\{\tau_C<\tau_I\}}LGD_C(\epsilon_t - C_t)^+ + 1_{\{\tau_I<\tau_C\}}LGD_I(\epsilon_t - C_t)^-.$$

Where LGD indicates the loss given default, typically defined as $1 - REC$, where REC is the corresponding recovery rate and $(x)^+$ indicates the positive part of x and $(x)^- = -(-x)^+$. The meaning of these flows is the following, consider θ_τ:

- at first to default time τ we compute the close-out value ϵ_τ;
- if the counterparty defaults and we are net debtor, i.e. $\epsilon_\tau - C_\tau \le 0$ then we have to pay the whole close-out value ε_τ to the counterparty;
- if the counterparty defaults and we are net creditor, i.e. $\epsilon_\tau - C_\tau > 0$ then we are able to recover just a fraction of our credits, namely $C_\tau + REC_C(\varepsilon_\tau - C_\tau) = REC_C\varepsilon_\tau + LGD_C C_\tau = \varepsilon_\tau - LGD_C(\varepsilon_\tau - C_\tau)$ where LGD_C indicates the loss given default and is equal to one minus the recovery rate REC_C.

A similar reasoning applies to the case when the Investor defaults.

If we now change filtration, we obtain the following expression for V_t (where we omitted the tilde sign over the rates, see Remark 3):

[1] See for example Sect. 8.2.1 and Lemma 9.1.1 of [1].

$$\begin{aligned} V_t =& 1_{\{\tau>t\}}\mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,r+\lambda)((f_u-c_u)C_u+(r_u-f_u)\widetilde{V}_u-(r_u-h_u)\widetilde{H}_u)du\right] \\ &+1_{\{\tau>t\}}\mathbb{E}_t^{\mathscr{F}}\left[D(t,T,r+\lambda)\Phi(S_T)+\int_t^T D(t,u,r+\lambda)\pi_u du\right] \\ &+1_{\{\tau>t\}}\mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,r+\lambda)\widetilde{\theta}_u du\right], \end{aligned} \tag{8}$$

where, if we suppose ϵ_t to be $\mathscr{F}$-predictable, we have (using Lemma 3):

$$\widetilde{\theta}_u = \epsilon_u\lambda_u - LGD_C(\epsilon_u - C_u)^+\lambda_u^C + LGD_I(\epsilon_u - C_u)^-\lambda_u^I. \tag{9}$$

Remark 3 From now on we will omit the tilde sign over the rates f_u, h_u. Moreover, we note that if a rate is of the form

$$x_t = x^+ 1_{\{g(V_t,H_t,C_t)>0\}} + x^- 1_{\{g(V_t,H_t,C_t)\le 0\}}$$

then on the set $\{\tau > t\}$ it coincides with the rate

$$\widetilde{x}_t = \widetilde{x}^+ 1_{\{g(\widetilde{V}_t,\widetilde{H}_t,C_t)>0\}} + \widetilde{x}^- 1_{\{g(\widetilde{V}_t,\widetilde{H}_t,C_t)\le 0\}}$$

because $1_{\{\tau>t\}}x^+1_{\{g(V_t,H_t,C_t)>0\}} = \widetilde{x}^+1_{\{\tau>t\}}1_{\{g(V_t,H_t,C_t)>0\}}$, and on $\{\tau > t\}$ we have $V_t = \widetilde{V}_t$ and $H_t = \widetilde{H}_t$, and hence $g(V_t, H_t, C_t) > 0 \iff g(\widetilde{V}_t, \widetilde{H}_t, C_t) > 0$.

We note that this expression is of the form $V_t = 1_{\{\tau>t\}}\Upsilon$ meaning that V_t is zero on $\{\tau \le t\}$ and that on the set $\{\tau > t\}$ it coincides with the $\mathscr{F}$-measurable random variable Υ. But we already know a variable that coincides with V_t on $\{\tau > t\}$, i.e. $\widetilde{V}_t$. Hence we can write the following:

$$\begin{aligned} \widetilde{V}_t =& \mathbb{E}_t^{\mathscr{F}}\left[\int_t^T D(t,u,r+\lambda)(\pi_u+(f_u-c_u)C_u+(r_u-f_u)\widetilde{V}_u-(r_u-h_u)\widetilde{H}_u)du\right] \\ &+\mathbb{E}_t^{\mathscr{F}}\left[D(t,T,r+\lambda)\Phi(S_T)+\int_t^T D(t,u,r+\lambda)\widetilde{\theta}_u du\right]. \end{aligned} \tag{10}$$

We now show a way to obtain a BSDE from Eq. (10), another possible approach (without default risk) is shown for example in [9]. We introduce the process

$$\begin{aligned} X_t =& \int_0^t D(0,u,r+\lambda)\pi_u du + \int_0^t D(0,u,r+\lambda)\widetilde{\theta}_u du \\ &+\int_0^t D(0,u,r+\lambda)\left[(f_u-c_u)C_u+(r_u-f_u)\widetilde{V}_u-(r_u-h_u)\widetilde{H}_u\right]du. \end{aligned} \tag{11}$$

Now we can construct a martingale summing up X_t and the discounted value of the deal as in the following:

$$D(0,t,r+\lambda)\widetilde{V}_t + X_t = \mathbb{E}_t^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)].$$

So differentiating both sides we obtain:

$$\begin{aligned}
&-(r_u+\lambda_u)D(0,u,r+\lambda)\widetilde{V}_u du + D(0,u,r+\lambda)d\widetilde{V}_u + dX_u \\
&= d\mathbb{E}_u^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)].
\end{aligned}$$

If we substitute for X_t we have that the expression:

$$d\widetilde{V}_u + \left[\pi_u - (r_u+\lambda_u)\widetilde{V}_u + \widetilde{\theta}_u + (f_u - c_u)C_u + (r_u - f_u)\widetilde{V}_u - (r_u - h_u)\widetilde{H}_u\right] du$$

is equal to;

$$\frac{d\mathbb{E}_u^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)]}{D(0,u,r+\lambda)}.$$

The process $(\mathbb{E}_t^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)])_{t\geq 0}$ is clearly a closed $\mathscr{F}$-martingale, and hence

$$\int_0^t D(0,u,r+\lambda)^{-1} d\mathbb{E}_u^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)]$$

is a local $\mathscr{F}$-martingale. Then, being

$$\int_0^t D(0,u,r+\lambda)^{-1} d\mathbb{E}_u^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)]$$

adapted to the Brownian-driven filtration $\mathscr{F}$, by the martingale representation theorem we have

$$\int_0^t D(0,u,r+\lambda)^{-1} d\mathbb{E}_u^{\mathscr{F}}[X_T + D(0,T,r+\lambda)\Phi(S_T)] = \int_0^t Z_u dW_u$$

for some $\mathscr{F}$-predictable process Z_u. Hence we can write:

$$d\widetilde{V}_u + \left[\pi_u - (f_u+\lambda_u)\widetilde{V}_u + \widetilde{\theta}_u + (f_u - c_u)C_u - (r_u - h_u)\widetilde{H}_u\right] du = Z_u dW_u. \quad (12)$$

4 Markovian FBSDE and PDE for $\widetilde{V}_t$ and the Invariance Theorem

As it is, Eq. (12) is way too general, thus we will make some simplifying assumptions in order to guarantee existence and uniqueness of a solution. First we assume a Markovian setting, and hence we suppose that all the processes appearing in (12) are deterministic functions of S_u, $\widetilde{V}_u$ or Z_u and time. More precisely we assume that:

- the dividend process π_u is a deterministic function $\pi(u, S_u)$ of u and S_u, Lipschitz continuous in S_u;
- the rates $r, f^{\pm}, c^{\pm}, \lambda^I, \lambda^C$ are deterministic bounded functions of time;
- the rate h_t is a deterministic function of time, and does not depend on the sign of H, namely $h^+ = h^-$, hence there is only one rate relative to the repo market of assets;
- the collateral process is a fraction of the process $\widetilde{V}_u$, namely $C_u = \alpha_u \widetilde{V}_u$, where $0 \le \alpha_u \le 1$ is a function of time;
- the close-out value ϵ_t is equal to $\widetilde{V}_t$ (this adds a source of nonlinearity with respect to choosing a risk-free closeout, see for example [6] and [5]);
- the diffusion coefficient $\sigma(t, S_t)$ of the underlying dynamic is Lipschitz continuous, uniformly in time, in S_t;
- we consider a delta-hedging strategy, and to this extent we choose $\widetilde{H}_t = S_t \frac{Z_t}{\sigma(t,S_t)}$; this reasoning derives from the fact that if we suppose $\widetilde{V}_t = V(t, S_t)$ with $V(\cdot,\cdot) \in C^{1,2}$ applying Ito's formula and comparing it with Eq. (12), we have that $\sigma(t, S_t)\,\partial_S V(t, S_t) = Z_t$.[2]

Under our assumptions, Eq. (12) becomes the following FBSDE:

$$
\begin{aligned}
dS_t =& r_t S_t dt + \sigma(t, S_t) dW_t \\
S_0 =& s \\
d\widetilde{V}_t =& -\underbrace{\left[\pi_t + \widetilde{\theta}_t - \lambda_t \widetilde{V}_t + f_t \widetilde{V}_t(\alpha_t - 1) - c_t(\alpha_t \widetilde{V}_t) - (r_t - h_t) S_t \frac{Z_t}{\sigma(t, S_t)}\right]}_{B(t,S_t,\widetilde{V}_t,Z_t)} dt + Z_t dW_t \\
V_T =& \Phi(S_T)
\end{aligned}
\tag{13}
$$

We want to obtain existence and uniqueness of the solution to the above-mentioned FBSDE and a related PDE. A possible choice is the following (see J. Zhang [15] Theorem 2.4.1 on page 41):

[2] At this stage the assumption we made on V is not properly justified, see Theorem 3 and Remark 4 for details.

Theorem 1 *Consider the following FBSDE on* $[0, T]$*:*

$$
\begin{aligned}
dX_t^{q,x} &= \mu(t, X_t^{q,x})dt + \sigma(t, X_t^{q,x})dW_t \quad q < t \leq T \\
X_t &= x \quad 0 \leq t \leq q \\
dY_t^{q,x} &= -f(t, X_t^{q,x}, Y_t^{q,x}, Z_t^{q,x})dt + Z_t^{q,x}dW_t \\
Y_T^{q,x} &= g(X_T^{q,x})
\end{aligned}
\tag{14}
$$

If we assume that there exists a positive constant K such that

- $\sigma(t, x)^2 \geq \frac{1}{K}$*;*
- $|f(t, x, y, z) - f(t, x', y', z')| + |g(x) - g(x')| \leq K(|x - x'| + |y - y'| + |z - z'|)$*;*
- $|f(t, 0, 0, 0)| + |g(0)| \leq K$*;*

and moreover the functions $\mu(t, x)$ and $\sigma(t, x)$ are C^2 with bounded derivatives, then Eq. (14) *has a unique solution $(X_t^{q,x}, Y_t^{q,x}, Z_t^{q,x})$ and $u(t, x) = Y_t^{t,x}$ is the unique* classical *(i.e. $C^{1,2}$) solution to the following semilinear PDE*

$$
\begin{aligned}
\partial_t u(t, x) + \frac{1}{2}\sigma(t, x)^2 \partial_x^2 u(t, x) + \mu(t, x)\partial_x u(t, x) + f(t, x, u(t, x), \sigma(t, x)\partial_x u(t, x)) &= 0 \\
u(T, x) &= g(x)
\end{aligned}
\tag{15}
$$

We cannot directly apply Theorem 1 to our FBSDE because $B(t, s, v, z)$ is not Lipschitz continuous in s because of the hedging term. But, since the hedging term is linear in Z_t we can move it from the drift of the backward equation to the drift of the forward one. More precisely consider the following:

$$
\begin{aligned}
dS_t^{q,s} &= h_t S_t^{q,s} dt + \sigma(t, S_t^{q,s})dW_t \quad q < t \leq T \\
S_q &= s_q \quad 0 \leq t \leq q \\
dV_t^{q,s} &= -\underbrace{\left[\pi_t + \theta_t - \lambda_t V_t^{q,s} + f_t V_t^{q,s}(\alpha_t - 1) - c_t(\alpha_t V_t^{q,s})\right]}_{B'(t, S_t^{q,s}, V_t^{q,s})} dt + Z_t^{q,s} dW_t \\
V_T^{q,s} &= \Phi(S_T^{q,s}).
\end{aligned}
\tag{16}
$$

Indeed, one can check that the assumptions of Theorem 1 are satisfied for this equation:

Theorem 2 *If the rates λ_t, f_t, c_t, h_t, r_t are bounded, then $|B'(t, s, v) - B'(t, s', v')| \leq K(|s - s'| + |v - v'|)$ and $|B'(t, 0, 0)| + \Phi(0) \leq K$. Hence if $\sigma(t, s)$ is a positive C^2 function with bounded derivatives, then the assumptions of Theorem 1 are satisfied and so Eq.* (16) *has a unique solution, and moreover $V_t^{t,s} = u(t, s) \in C^{1,2}$ and satisfies the following semilinear PDE:*

$$\begin{aligned} \partial_t u(t,s) + \frac{1}{2}\sigma(t,s)^2\partial_s^2 u(t,s) + h_t s \partial_s u(t,s) + B'(t,s,u(t,s)) = 0 \\ u(T,s) = \Phi(s) \end{aligned} \tag{17}$$

Proof We start by rewriting the term

$$B'(t,s,v) = \pi_t(s) + \theta_t(v) + (f_t(\alpha_t - 1) - \lambda_t - c_t\alpha_t)v.$$

Since the sum of two Lipschitz functions is itself a Lipschitz function we can restrict ourselves to analyzing the summands that appear in the previous formula. The term π_t is Lipschitz continuous in s by assumption. The θ term and the $(f_t(\alpha_t - 1) - \lambda_t - c_t\alpha_t)v$ term are continuous and piece-wise linear, hence Lipschitz continuous and this concludes the proof.

Note that the S-dynamics in (16) has the repo rate h as drift. Since in general h will depend on the future values of the deal, this is a source of nonlinearity and is at times represented informally with an expected value $\mathbb{E}^h$ or a pricing measure $\mathbb{Q}^h$, see for example [5] and the related discussion on operational implications for the case $h = f$.

We now show that a solution to Eq. (13) can be obtained by means of the classical solution to the PDE (17). We start considering the following forward equation which is known to have a unique solution under our assumptions about $\sigma(t,s)$.

$$dS_t = r_t S_t dt + \sigma(t,S_t)dW_t \quad S_0 = s. \tag{18}$$

We define $V_t = u(t,S_t)$ and $Z_t = \sigma(t,S_t)\partial_s u(t,S_t)$. By Theorem 2 we know that $u(t,s) \in C^{1,2}$ and by applying Ito's formula and (17) we obtain:

$$\begin{aligned} dV_t &= du(t,S_t) \\ &= \left(\partial_t u(t,S_t) + r_t S_t \partial_s u(t,S_t) + \frac{1}{2}\sigma(t,S_t)^2\partial_s^2 u(t,S_t)\right)dt + \sigma(t,S_t)\partial_s u(t,S_t)dW_t \\ &= \big((r_t - h_t)S_t\partial_s u(t,S_t) - B'(t,S_t,u(t,S_t))\big)\,dt + \sigma(t,S_t)\partial_s u(t,S_t)dW_t \\ &= \left((r_t - h_t)S_t\frac{Z_t}{\sigma(t,S_t)} - \pi_t(S_t) - \theta_t(V_t) - (f_t(\alpha_t - 1) - \lambda_t - c_t\alpha_t)V_t)\right)dt + Z_t dW_t \end{aligned}$$

Hence we found the following:

Theorem 3 (Solution to the Valuation Equation) *Let S_t be the solution to Eq. (18) and $u(t,s)$ the classical solution to Eq. (17). Then the process $(S_t, u(t,S_t), \sigma(t,S_t)\partial_s u(t,S_t))$ is the unique solution to Eq. (13).*

Proof From the reasoning above we found that $(S_t, u(t,S_t), \sigma(t,S_t)\partial_s u(t,S_t))$ solves Eq. (13). Finally from the seminal result of [14] we know that if there exist $K > 0$ and $p \geq \frac{1}{2}$ such that:

- $|\mu(t,x) - \mu(t,x')| + |\sigma(t,x) - \sigma(t,x')| \leq K|x - x'|$

- $|\mu(t,x)| + |\sigma(t,x)| \leq K(1+|x|)$
- $|f(t,x,y,z) - f(t,x,y',z')| \leq K(|y-y'| + |z-z'|)$
- $|g(x)| + |f(t,x,0,0)| \leq K(1+|x|^p)$

then the FBSDE (14) has a unique solution. Since we have to check the Lipschitz continuity just for y and z we can verify that Eq. (13) satisfies the above-mentioned assumptions and hence has a unique solution.

Remark 4 Since we proved that $V_t = u(t, S_t)$ with $u(t,s) \in C^{1,2}$, the reasoning we used, when saying that $\widetilde{H}_t = S_t \frac{Z_t}{\sigma(t,S_t)}$ represented choosing a delta-hedge, it is actually more than a heuristic argument.

Moreover, since (17) does not depend on the risk-free rate r_t so we can state the following:

Theorem 4 (Invariance Theorem) *If we are under the assumptions at the beginning of Sect. 4 and we assume that we are backing our deal with a delta hedging strategy, then the price V_t can be calculated via the semilinear PDE (17) and does* not *depend on the risk-free rate $r(t)$.*

This invariance result shows that even when starting from a risk-neutral valuation theory, the risk-free rate disappears from the nonlinear valuation equations. A discussion on consequences of nonlinearity and invariance on valuation in general, on the operational procedures of a bank, on the legitimacy of fully charging the nonlinear value to a client, and on the related dangers of overlapping valuation adjustments is presented elsewhere, see for example [3, 5] and references therein.

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Nonlinear Monte Carlo Schemes for Counterparty Risk on Credit Derivatives

Stéphane Crépey and Tuyet Mai Nguyen

Abstract Two nonlinear Monte Carlo schemes, namely, the linear Monte Carlo expansion with randomization of Fujii and Takahashi (Int J Theor Appl Financ 15(5):1250034(24), 2012 [9], Q J Financ 2(3), 1250015(24), 2012, [10]) and the marked branching diffusion scheme of Henry-Labordère (Risk Mag 25(7), 67–73, 2012, [13]), are compared in terms of applicability and numerical behavior regarding counterparty risk computations on credit derivatives. This is done in two dynamic copula models of portfolio credit risk: the dynamic Gaussian copula model and the model in which default dependence stems from joint defaults. For such high-dimensional and nonlinear pricing problems, more standard deterministic or simulation/regression schemes are ruled out by Bellman's "curse of dimensionality" and only purely forward Monte Carlo schemes can be used.

Keywords Counterparty risk · Funding · BSDE · Gaussian copula · Marshall–Olkin copula · Particles

1 Introduction

Counterparty risk is a major issue since the global credit crisis and the ongoing European sovereign debt crisis. In a bilateral counterparty risk setup, counterparty risk is valued as the so-called credit valuation adjustment (CVA), for the risk of default of the counterparty, and debt valuation adjustment (DVA), for own default risk. In such a setup, the classical assumption of a locally risk-free funding asset used for both investing and unsecured borrowing is no longer sustainable. The proper accounting of the funding costs of a position leads to the funding valuation adjustment (FVA). Moreover, these adjustments are interdependent and must be computed jointly

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through a global correction dubbed total valuation adjustment (TVA). The pricing equation for the TVA is nonlinear due to the funding costs. It is posed over a random time interval determined by the first default time of the two counterparties. To deal with the corresponding backward stochastic differential equation (BSDE), a first reduced-form modeling approach has been proposed in Crépey [3], under a rather standard immersion hypothesis between a reference (or market) filtration and the full model filtration progressively enlarged by the default times of the counterparties. This basic immersion setup is fine for standard applications, such as counterparty risk on interest rate derivatives. But it is too restrictive for situations of strong dependence between the underlying exposure and the default risk of the two counterparties, such as counterparty risk on credit derivatives, which involves strong adverse dependence, called wrong-way risk (for some insights of related financial contexts, see Fujii and Takahashi [11], Brigo et al. [2]). For this reason, an extended reduced-form modeling approach has been recently developed in Crépey and Song [4–6]. With credit derivatives, the problem is also very high-dimensional. From a numerical point of view, for high-dimensional nonlinear problems, only purely forward simulation schemes can be used. In Crépey and Song [6], the problem is addressed by the linear Monte Carlo expansion with randomization of Fujii and Takahashi [9, 10].

In the present work, we assess another scheme, namely the marked branching diffusion approach of Henry-Labordère [13], which we compare with the previous one in terms of applicability and numerical behavior. This is done in two dynamic copula models of portfolio credit risk: the dynamic Gaussian copula model and the dynamic Marshall–Olkin model in which default dependence stems from joint defaults.

The paper is organized as follows. Sections 2 and 3 provide a summary of the main pricing and TVA BSDEs that are derived in Crépey and Song [4–6]. Section 4 exposes two nonlinear Monte Carlo schemes that can be considered for solving these in high-dimensional models, such as the portfolio credit models of Sect. 5. Comparative numerics in these models are presented in Sect. 6. Section 7 concludes.

2 Prices

2.1 Setup

We consider a netted portfolio of OTC derivatives between two defaultable counterparties, generally referred to as the contract between a bank, the perspective of which is taken, and its counterparty. After having bought the contract from its counterparty at time 0, the bank sets up a hedging, collateralization (or margining), and funding portfolio. We call the funder of the bank a third party, possibly composed in practice of several entities or devices, insuring funding of the bank's strategy. The funder, assumed default-free for simplicity, plays the role of lender/borrower of last resort after the exhaustion of the internal sources of funding provided to the bank through its hedge and collateral.

For notational simplicity we assume no collateralization. All the numerical considerations, our main focus in this work, can be readily extended to the case of collateralized portfolios using the corresponding developments in Crépey and Song [6]. Likewise, we assume hedging in the simplest sense of replication by the bank and we consider the case of a fully securely funded hedge, so that the cost of the hedge of the bank is exactly reflected by the wealth of its hedging and funding portfolio.

We consider a stochastic basis $(\Omega, \mathscr{G}_T, \mathscr{G}, \mathbb{Q})$, where $\mathscr{G} = (\mathscr{G}_t)_{t\in[0,T]}$ is interpreted as a risk-neutral pricing model on the primary market of the instruments that are used by the bank for hedging its TVA. The reference filtration $\mathscr{F}$ is a subfiltration of $\mathscr{G}$ representing the counterparty risk-free filtration, not carrying any direct information about the defaults of the two counterparties. The relation between these two filtrations will be pointed out in the condition (C) introduced later. We denote by:

- $\mathbb{E}_t$, the conditional expectation under $\mathbb{Q}$ given $\mathscr{G}_t$,
- r, the risk-free short rate process, with related discount factor $\beta_t = e^{-\int_0^t r_s ds}$,
- T, the maturity of the contract,
- τ_b and τ_c, the default time of the bank and of the counterparty, modeled as $\mathscr{G}$ stopping times with $(\mathscr{G}, \mathbb{Q})$ intensities γ^b and γ^c,
- $\tau = \tau_b \wedge \tau_c$, the first-to-default time of the two counterparties, also a $\mathscr{G}$ stopping time, with intensity γ such that $\max(\gamma^b, \gamma^c) \le \gamma \le \gamma^b + \gamma^c$,
- $\bar{\tau} = \tau \wedge T$, the effective time horizon of our problem (there is no cashflow after $\bar{\tau}$),
- D, the contractual dividend process,
- $\Delta = D - D_-$, the jump process of D.

2.2 Clean Price

We denote by P the reference (or clean) price of the contract ignoring counterparty risk and assuming the position of the bank financed at the risk-free rate r, i.e. the $\mathscr{G}$ conditional expectation of the future contractual cash-flows discounted at the risk-free rate r. In particular,

$$\beta_t P_t = \mathbb{E}_t \left[\int_t^{\bar{\tau}} \beta_s dD_s + \beta_{\bar{\tau}} P_{\bar{\tau}} \right], \ \forall t \in [0, \bar{\tau}]. \tag{1}$$

We also define $Q_t = P_t + \mathbb{1}_{\{t=\tau<T\}} \Delta_\tau$, so that Q_τ represents the clean value of the contract inclusive of the promised dividend at default (if any) Δ_τ, which also belongs to the "debt" of the counterparty to the bank (or vice versa depending on the sign of Q_τ) in case of default of a party. Accordingly, at time τ (if $< T$), the close-out cash-flow of the counterparty to the bank is modeled as

$$\mathscr{R} = \mathbb{1}_{\{\tau=\tau_c\}} \big(R_c Q_\tau^+ - Q_\tau^- \big) - \mathbb{1}_{\{\tau=\tau_b\}} \big(R_b Q_\tau^- - Q_\tau^+ \big) - \mathbb{1}_{\{\tau_b=\tau_c\}} Q_\tau, \tag{2}$$

where R_b and R_c are the recovery rates of the bank and of the counterparty to each other.

2.3 All-Inclusive Price

Let Π be the all-inclusive price of the contract for the bank, including the cost of counterparty risk and funding costs. Since we assume a securely funded hedge (in the sense of replication) and no collateralization, the amounts invested and funded by the bank at time t are respectively given by Π_t^- and Π_t^+. The all-inclusive price Π is the discounted conditional expectation of all effective future cash flows including the contractual dividends before τ, the cost of funding the position prior to time τ and the terminal cash flow at time τ. Hence,

$$\beta_t \Pi_t = \mathbb{E}_t \left[\int_t^{\bar{\tau}} \beta_s \mathbb{1}_{s<\tau} dD_s - \int_t^{\bar{\tau}} \beta_s \bar{\lambda}_s \Pi_s^+ ds + \beta_{\bar{\tau}} \mathbb{1}_{\tau<T} \mathscr{R} \right], \tag{3}$$

where $\bar{\lambda}$ is the funding spread over r of the bank toward the external funder, i.e. the bank borrows cash from its funder at rate $r + \bar{\lambda}$ (and invests cash at the risk-free rate r). Since the right hand side in (3) depends also on Π, (3) is in fact a backward stochastic differential equation (BSDE). Consistent with the no arbitrage principle, the gain process on the hedge is a $\mathbb{Q}$ martingale, which explains why it does not appear in (3).

3 TVA BSDEs

The total valuation adjustment (TVA) process Θ is defined as

$$\Theta = Q - \Pi. \tag{4}$$

In this section we review the main TVA BSDEs that are derived in Crépey and Song [4–6]. Three BSDEs are presented. These three equations are essentially equivalent mathematically. However, depending on the underlying model, they are not always amenable to the same numerical schemes or the numerical performance of a given scheme may differ between them.

3.1 Full TVA BSDE

By taking the difference between (1) and (3), we obtain

$$\beta_t \Theta_t = \mathbb{E}_t \left[\int_t^{\bar{\tau}} \beta_s fva_s(\Theta_s) ds + \beta_{\bar{\tau}} \mathbb{1}_{\tau<T} \xi \right], \ \forall t \in [0, \bar{\tau}], \tag{5}$$

where $fva_t(\vartheta) = \bar{\lambda}_t(P_t - \vartheta)^+$ is the funding coefficient and where

$$\xi = Q_\tau - \mathscr{R} = \mathbb{1}_{\{\tau=\tau_c\}}(1 - R_c)(P_\tau + \Delta_\tau)^+ - \mathbb{1}_{\{\tau=\tau_b\}}(1 - R_b)(P_\tau + \Delta_\tau)^- \quad (6)$$

is the exposure at default of the bank. Equivalent to (5), the "full TVA BSDE" is written as

$$\Theta_t = \mathbb{E}_t\left[\int_t^{\bar{\tau}} f_s(\Theta_s)ds + \mathbb{1}_{\tau<T}\xi\right],\ 0 \le t \le \bar{\tau}, \quad \text{(I)}$$

for the coefficient $f_t(\vartheta) = fva_t(\vartheta) - r_t\vartheta$.

3.2 Partially Reduced TVA BSDE

Let $\hat{\xi}$ be a $\mathscr{G}$-predictable process, which exists by Corollary 3.23 2 in He et al. [12], such that $\hat{\xi}_\tau = \mathbb{E}[\xi|\mathscr{G}_{\tau-}]$ on $\tau < \infty$ and let $\bar{f}$ be the modified coefficient such that

$$\bar{f}_t(\vartheta) + r_t\vartheta = \underbrace{\gamma_t\hat{\xi}_t}_{cdva_t} + \underbrace{\bar{\lambda}_t(P_t - \vartheta)^+}_{fva_t(\vartheta)}. \quad (7)$$

As easily shown (cf. [4, Lemma 2.2]), the full TVA BSDE (I) can be simplified into the "partially reduced BSDE"

$$\bar{\Theta}_t = \mathbb{E}_t\left[\int_t^{\bar{\tau}} \bar{f}_s(\bar{\Theta}_s)ds\right],\ 0 \le t \le \bar{\tau}, \quad \text{(II)}$$

in the sense that if Θ solves (I), then $\bar{\Theta} = \Theta\mathbb{1}_{[0,\tau)}$ solves (II), while if $\bar{\Theta}$ solves (II), then the process Θ defined as $\bar{\Theta}$ before $\bar{\tau}$ and $\Theta_{\bar{\tau}} = \mathbb{1}_{\tau<T}\xi$ solves (I). Note that both BSDEs (I) and (II) are $(\mathscr{G}, \mathbb{Q})$ BSDEs posed over the random time interval $[0, \bar{\tau}]$, but with the terminal condition ξ for (I) as opposed to a null terminal condition (and a modified coefficient) for (II).

3.3 Fully Reduced TVA BSDE

Let

$$\hat{f}_t(\vartheta) = \bar{f}_t(\vartheta) - \gamma_t\vartheta = cdva_t + fva_t(\vartheta) - (r_t + \gamma_t)\vartheta.$$

Assume the following conditions, which are studied in Crépey and Song [4–6]:

Condition (C). There exist:

(C.1) a subfiltration $\mathscr{F}$ of $\mathscr{G}$ satisfying the usual conditions and such that $\mathscr{F}$ semimartingales stopped at τ are $\mathscr{G}$ semimartingales,
(C.2) a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathscr{F}_T$ such that any $(\mathscr{F}, \mathbb{P})$ local martingale stopped at $(\tau-)$ is a $(\mathscr{G}, \mathbb{Q})$ local martingale on $[0, T]$,
(C.3) an $\mathscr{F}$ progressive "reduction" $\widetilde{f}_t(\vartheta)$ of $\hat{f}_t(\vartheta)$ such that $\int_0^{\cdot} \hat{f}_t(\vartheta)dt = \int_0^{\cdot} \widetilde{f}_t(\vartheta)dt$ on $[0, \bar{\tau}]$.

Let $\widetilde{\mathbb{E}}_t$ denote the conditional expectation under $\mathbb{P}$ given $\mathscr{F}_t$. It is shown in Crépey and Song [4–6]) that the full TVA BSDE (I) is equivalent to the following "fully reduced BSDE":

$$\widetilde{\Theta}_t = \widetilde{\mathbb{E}}_t \left[\int_t^T \widetilde{f}_s(\widetilde{\Theta}_s) ds \right], \quad t \in [0, T], \tag{III}$$

equivalent in the sense that if Θ solves (I), then the "$\mathscr{F}$ optional reduction" $\widetilde{\Theta}$ of Θ ($\mathscr{F}$ optional process that coincides with Θ before τ) solves (III), while if $\widetilde{\Theta}$ solves (III), then $\Theta = \widetilde{\Theta}\mathbb{1}_{[0,\tau)} + \mathbb{1}_{[\tau]}\mathbb{1}_{\tau<T}\xi$ solves (I).

Moreover, under mild assumptions (see e.g. Crépey and Song [6, Theorem 4.1]), one can easily check that $\bar{f}_t(\vartheta)$ in (7) (resp. $\widetilde{f}_t(\vartheta)$) satisfies the classical BSDE monotonicity assumption

$$\big(\bar{f}_t(\vartheta) - \bar{f}_t(\vartheta')\big)(\vartheta - \vartheta') \le C(\vartheta - \vartheta')^2$$

(and likewise for $\widetilde{f}$), for some constant C. Hence, by classical BSDE results nicely surveyed in Kruse and Popier [14, Sect. 2 (resp. 3)], the partially reduced TVA BSDE (II), hence the equivalent full TVA BSDE (I) (resp. the fully reduced BSDE (III)), is well-posed in the space of $(\mathscr{G}, \mathbb{Q})$ (resp. $(\mathscr{F}, \mathbb{P})$) square integrable solutions, where well-posedness includes existence, uniqueness, comparison and BSDE standard estimates.

3.4 Marked Default Time Setup

In order to be able to compute $\gamma\hat{\xi}$ in $\bar{f}$, we assume that τ is endowed with a mark e in a finite set E, in the sense that

$$\tau = \min_{e \in E} \tau_e, \tag{8}$$

where each τ_e is a stopping time with intensity γ_t^e such that $\mathbb{Q}(\tau_e \neq \tau_{e'}) = 1$, $e \neq e'$, and

$$\mathcal{G}_\tau = \mathcal{G}_{\tau-} \vee \sigma(\varepsilon),$$

where $\varepsilon = \mathrm{argmin}_{e \in E} \tau_e$ yields the "identity" of the mark. The role of the mark is to convey some additional information about the default, e.g. to encode wrong-way and gap risk features. The assumption of a finite set E in (8) ensures tractability of the setup. In fact, by Lemma 5.1 in Crépey and Song [6], there exists $\mathcal{G}$-predictable processes $\widetilde{P}_t^e$ and $\widetilde{\Delta}_t^e$ such that

$$P_\tau = \widetilde{P}_\tau^e \text{ and } \Delta_\tau = \widetilde{\Delta}_\tau^e \text{ on the event } \{\tau = \tau_e\}.$$

Assuming further that $\tau_b = \min_{e \in E_b} \tau_e$ and $\tau_c = \min_{e \in E_c} \tau_e$, where $E = E_b \cup E_c$ (not necessarily a disjoint union), one can then take on $[0, \bar{\tau}]$:

$$\gamma_t \hat{\xi}_t = (1 - R_c) \sum_{e \in E_c} \gamma_t^e \left(\widetilde{P}_t^e + \widetilde{\Delta}_t^e\right)^+ - (1 - R_b) \sum_{e \in E_b} \gamma_t^e \left(\widetilde{P}_t^e + \widetilde{\Delta}_t^e\right)^-,$$

where the two terms have clear respective CVA and DVA interpretation. Hence, (7) is rewritten, on $[0, \bar{\tau}]$, as

$$\begin{aligned} \bar{f}_t(\vartheta) + r_t \vartheta = & \underbrace{(1 - R_c) \sum_{e \in E_c} \gamma_t^e \left(\widetilde{P}_t^e + \widetilde{\Delta}_t^e\right)^+}_{\text{CVA coefficient } (cva_t)} - \underbrace{(1 - R_b) \sum_{e \in E_b} \gamma_t^e \left(\widetilde{P}_t^e + \widetilde{\Delta}_t^e\right)^-}_{\text{DVA coefficient } (dva_t)} \\ & + \underbrace{\bar{\lambda}_t (P_t - \vartheta)^+}_{\text{FVA coefficient } (fva_t(\vartheta))} . \end{aligned} \tag{9}$$

If the functions $\widetilde{P}_t^e$ and $\widetilde{\Delta}_t^e$ above not only exist, but can be computed explicitly (as will be the case in the concrete models of Sects. 5.1 and 5.2), once stated in a Markov setup where

$$\bar{f}_t(\vartheta) = \bar{f}(t, X_t, \vartheta), \ t \in [0, T], \tag{10}$$

for some $(\mathcal{G}, \mathbb{Q})$ jump diffusion X, then the partially reduced TVA BSDE (II) can be tackled numerically. Similarly, once stated in a Markov setup where

$$\widetilde{f}_t(\vartheta) = \widetilde{f}(t, \widetilde{X}_t, \vartheta), \ t \in [0, T], \tag{11}$$

for some $(\mathcal{F}, \mathbb{P})$ jump diffusion $\widetilde{X}$, then the fully reduced TVA BSDE (III) can be tackled numerically.

4 TVA Numerical Schemes

4.1 Linear Approximation

Our first TVA approximation is obtained replacing Θ_s by 0 in the right hand side of (I), i.e.

$$\Theta_0 \approx \mathbb{E}\left[\int_0^{\bar{\tau}} f_s(0)ds + \mathbb{1}_{\tau<T}\xi\right] = \mathbb{E}\left[\int_0^{\bar{\tau}} \bar{\lambda}_s P_s^+ ds + \mathbb{1}_{\tau<T}\xi\right]. \tag{12}$$

We then approximate the TVA by standard Monte-Carlo, with randomization of the integral to reduce the computation time (at the cost of a small increase in the variance). Hence, introducing an exponential time ζ of parameter μ, i.e. a random variable with density $\phi(s) = \mathbb{1}_{s\geq 0}\, \mu\, e^{-\mu s}$, we have

$$\mathbb{E}\left[\int_0^{\bar{\tau}} f_s(0)ds\right] = \mathbb{E}\left[\int_0^{\bar{\tau}} \phi(s)\frac{1}{\mu}e^{\mu s} f_s(0)ds\right] = \mathbb{E}\left[\mathbb{1}_{\zeta<\bar{\tau}}\frac{e^{\mu\zeta}}{\mu} f_\zeta(0)\right]. \tag{13}$$

We can use the same technic for (II) and (III), which yields:

$$\Theta_0 = \bar{\Theta}_0 \approx \mathbb{E}\left[\int_0^{\bar{\tau}} \bar{f}_s(0)ds\right] = \mathbb{E}\left[\mathbb{1}_{\zeta<\bar{\tau}}\frac{e^{\mu\zeta}}{\mu} \bar{f}_\zeta(0)\right], \tag{14}$$

$$\Theta_0 = \widetilde{\Theta}_0 \approx \widetilde{\mathbb{E}}\left[\int_0^{T} \widetilde{f}_s(0)ds\right] = \widetilde{\mathbb{E}}\left[\mathbb{1}_{\zeta<T}\frac{e^{\mu\zeta}}{\mu} \widetilde{f}_\zeta(0)\right]. \tag{15}$$

4.2 Linear Expansion and Interacting Particle Implementation

Following Fujii and Takahashi [9, 10], we can introduce a perturbation parameter ε and the following perturbed form of the fully reduced BSDE (III):

$$\widetilde{\Theta}_t^\varepsilon = \widetilde{\mathbb{E}}_t\left[\int_t^T \varepsilon \widetilde{f}_s(\widetilde{\Theta}_s^\varepsilon)ds\right], \quad t \in [0, T], \tag{16}$$

where $\varepsilon = 1$ corresponds to the original BSDE (III). Suppose that the solution of (16) can be expanded in a power series of ε:

$$\widetilde{\Theta}_t^\varepsilon = \widetilde{\Theta}_t^{(0)} + \varepsilon\widetilde{\Theta}_t^{(1)} + \varepsilon^2\widetilde{\Theta}_t^{(2)} + \varepsilon^3\widetilde{\Theta}_t^{(3)} + \cdots. \tag{17}$$

The Taylor expansion of f at $\widetilde{\Theta}^{(0)}$ reads

$$\begin{aligned}\widetilde{f}_t(\widetilde{\Theta}_t^{\varepsilon}) =& \widetilde{f}_t(\widetilde{\Theta}_t^{(0)}) + (\varepsilon\widetilde{\Theta}_t^{(1)} + \varepsilon^2\widetilde{\Theta}_t^{(2)} + \cdots)\partial_{\vartheta}\widetilde{f}_t(\widetilde{\Theta}_t^{(0)}) \\ &+ \frac{1}{2}(\varepsilon\widetilde{\Theta}_t^{(1)} + \varepsilon^2\widetilde{\Theta}_t^{(2)} + \cdots)^2\partial^2_{\vartheta^2}\widetilde{f}_t(\widetilde{\Theta}_t^{(0)}) + \cdots\end{aligned}$$

Collecting the terms of the same order with respect to ε in (16), we obtain $\widetilde{\Theta}_t^{(0)} = 0$, due to the null terminal condition of the fully reduced BSDE (III), and

$$\begin{aligned}\widetilde{\Theta}_t^{(1)} &= \widetilde{\mathbb{E}}_t\left[\int_t^T \widetilde{f}_s(\widetilde{\Theta}_s^{(0)})ds\right], \\ \widetilde{\Theta}_t^{(2)} &= \widetilde{\mathbb{E}}_t\left[\int_t^T \widetilde{\Theta}_s^{(1)}\partial_{\vartheta}\widetilde{f}_s(\widetilde{\Theta}_s^{(0)})ds\right], \\ \widetilde{\Theta}_t^{(3)} &= \widetilde{\mathbb{E}}_t\left[\int_t^T \widetilde{\Theta}_s^{(2)}\partial_{\vartheta}\widetilde{f}_s(\widetilde{\Theta}_s^{(0)})ds\right],\end{aligned} \tag{18}$$

where the third order term should contain another component based on $\partial^2_{\vartheta^2}\widetilde{f}$. But, in our case, $\partial^2_{\vartheta^2}\widetilde{f}$ involves a Dirac measure via the terms $(P_t - \vartheta)^+$ in $fva_t(\vartheta)$, so that we truncate the expansion to the term $\widetilde{\Theta}_t^{(3)}$ as above. If the nonlinearity in (III) is sub-dominant, one can expect to obtain a reasonable approximation of the original equation by setting $\varepsilon = 1$ at the end of the calculation, i.e.

$$\widetilde{\Theta}_0 \approx \widetilde{\Theta}_0^{(1)} + \widetilde{\Theta}_0^{(2)} + \widetilde{\Theta}_0^{(3)}.$$

Carrying out a Monte Carlo simulation by an Euler scheme for every time s in a time grid and integrating to obtain $\widetilde{\Theta}_0^{(1)}$ would be quite heavy. Moreover, this would become completely unpractical for the higher order terms that involve iterated (multivariate) time integrals. For these reasons, Fujii and Takahashi [10] have introduced a particle interpretation to randomize and compute numerically the integrals in (18), which we call the FT scheme. Let η_1 be the interaction time of a particle drawn independently as the first jump time of a Poisson process with an arbitrary intensity $\mu > 0$ starting from time $t \geq 0$, i.e., η_1 is a random variable with density

$$\phi(t,s) = \mathbb{1}_{s \geq t}\, \mu\, e^{-\mu(s-t)}. \tag{19}$$

From the first line in (18), we have

$$\widetilde{\Theta}_t^{(1)} = \widetilde{\mathbb{E}}_t\left[\int_t^T \phi(t,s)\frac{e^{\mu(s-t)}}{\mu}\widetilde{f}_s(\widetilde{\Theta}_s^{(0)})ds\right] = \widetilde{\mathbb{E}}_t\left[\mathbb{1}_{\eta_1 < T}\frac{e^{\mu(\eta_1 - t)}}{\mu}\widetilde{f}_{\eta_1}(\widetilde{\Theta}_{\eta_1}^{(0)})\right]. \tag{20}$$

Similarly, the particle representation is available for the higher order. By applying the same procedure as above, we obtain

$$\widetilde{\Theta}_t^{(2)} = \widetilde{\mathbb{E}}_t\left[\mathbb{1}_{\eta_1<T}\widetilde{\Theta}_{\eta_1}^{(1)}\frac{e^{\mu(\eta_1-t)}}{\mu}\partial_\vartheta\widetilde{f}_{\eta_1}(\widetilde{\Theta}_{\eta_1}^{(0)})\right],$$

where $\widetilde{\Theta}_{\eta_1}^{(1)}$ can be computed by (20). Therefore, by using the tower property of conditional expectations, we obtain

$$\widetilde{\Theta}_t^{(2)} = \widetilde{\mathbb{E}}_t\left[\mathbb{1}_{\eta_2<T}\frac{e^{\mu(\eta_2-\eta_1)}}{\mu}\widetilde{f}_{\eta_2}(\widetilde{\Theta}_{\eta_2}^{(0)})\frac{e^{\mu(\eta_1-t)}}{\mu}\partial_\vartheta\widetilde{f}_{\eta_1}(\widetilde{\Theta}_{\eta_1}^{(0)})\right], \tag{21}$$

where η_1, η_2 are the two consecutive interaction times of a particle randomly drawn with intensity μ starting from t. Similarly, for the third order, we get

$$\widetilde{\Theta}_t^{(3)} = \widetilde{\mathbb{E}}_t\left[\mathbb{1}_{\eta_3<T}\frac{e^{\mu(\eta_3-\eta_2)}}{\mu}\widetilde{f}_{\eta_3}(\widetilde{\Theta}_{\eta_3}^{(0)})\frac{e^{\mu(\eta_2-\eta_1)}}{\mu}\partial_\vartheta\widetilde{f}_{\eta_2}(\widetilde{\Theta}_{\eta_2}^{(0)})\frac{e^{\mu(\eta_1-t)}}{\mu}\partial_\vartheta\widetilde{f}_{\eta_1}(\widetilde{\Theta}_{\eta_1}^{(0)})\right], \tag{22}$$

where η_1, η_2, η_3 are consecutive interaction times of a particle randomly drawn with intensity μ starting from t. In case $t = 0$, (20), (21) and (22) can be simplified as

$$\begin{aligned}
\widetilde{\Theta}_0^{(1)} &= \widetilde{\mathbb{E}}\left[\mathbb{1}_{\zeta_1<T}\frac{e^{\mu\zeta_1}}{\mu}\widetilde{f}_{\zeta_1}(\widetilde{\Theta}_{\zeta_1}^{(0)})\right]\\
\widetilde{\Theta}_0^{(2)} &= \widetilde{\mathbb{E}}\left[\mathbb{1}_{\zeta_1+\zeta_2<T}\frac{e^{\mu\zeta_1}}{\mu}\partial_\vartheta\widetilde{f}_{\zeta_1}(\widetilde{\Theta}_{\zeta_1}^{(0)})\frac{e^{\mu\zeta_2}}{\mu}\widetilde{f}_{\zeta_1+\zeta_2}(\widetilde{\Theta}_{\zeta_1+\zeta_2}^{(0)})\right]\\
\widetilde{\Theta}_0^{(3)} &= \widetilde{\mathbb{E}}\left[\mathbb{1}_{\zeta_1+\zeta_2+\zeta_3<T}\frac{e^{\mu\zeta_1}}{\mu}\partial_\vartheta\widetilde{f}_{\zeta_1}(\widetilde{\Theta}_{\zeta_1}^{(0)})\frac{e^{\mu\zeta_2}}{\mu}\partial_\vartheta\widetilde{f}_{\zeta_1+\zeta_2}(\widetilde{\Theta}_{\zeta_1+\zeta_2}^{(0)})\frac{e^{\mu\zeta_3}}{\mu}\widetilde{f}_{\zeta_1+\zeta_2+\zeta_3}(\widetilde{\Theta}_{\zeta_1+\zeta_2+\zeta_3}^{(0)})\right]
\end{aligned} \tag{23}$$

where $\zeta_1, \zeta_2, \zeta_3$ are the elapsed time from the last interaction until the next interaction, which are independent exponential random variables with parameter μ.

Note that the pricing model is originally defined with respect to the full stochastic basis $(\mathscr{G}, \mathbb{Q})$. Even in the case where there exists a stochastic basis $(\mathscr{F}, \mathbb{Q})$ satisfying the condition (C), $(\mathscr{F}, \mathbb{Q})$ simulation may be nontrivial. Lemma 8.1 in Crépey and Song [6] allows us to reformulate the $\mathbb{Q}$ expectations in (23) as the following $\mathbb{Q}$ expectations, with $\bar{\Theta}^{(0)} = 0$:

$$\begin{aligned}
\widetilde{\Theta}_0^{(1)} &= \bar{\Theta}_0^{(1)} = \mathbb{E}\left[\mathbb{1}_{\zeta_1<\bar{\tau}}\frac{e^{\mu\zeta_1}}{\mu}\bar{f}_{\zeta_1}(\bar{\Theta}_{\zeta_1}^{(0)})\right]\\
\widetilde{\Theta}_0^{(2)} &= \bar{\Theta}_0^{(2)} = \mathbb{E}\left[\mathbb{1}_{\zeta_1+\zeta_2<\bar{\tau}}\frac{e^{\mu\zeta_1}}{\mu}\partial_\vartheta\bar{f}_{\zeta_1}(\bar{\Theta}_{\zeta_1}^{(0)})\frac{e^{\mu\zeta_2}}{\mu}\bar{f}_{\zeta_1+\zeta_2}(\bar{\Theta}_{\zeta_1+\zeta_2}^{(0)})\right]\\
\widetilde{\Theta}_0^{(3)} &= \bar{\Theta}_0^{(3)} = \mathbb{E}\Big[\mathbb{1}_{\zeta_1+\zeta_2+\zeta_3<\bar{\tau}}\frac{e^{\mu\zeta_1}}{\mu}\partial_\vartheta\bar{f}_{\zeta_1}(\bar{\Theta}_{\zeta_1}^{(0)})\frac{e^{\mu\zeta_2}}{\mu}\partial_\vartheta\bar{f}_{\zeta_1+\zeta_2}(\bar{\Theta}_{\zeta_1+\zeta_2}^{(0)})\\
&\qquad\times\ \frac{e^{\mu\zeta_3}}{\mu}\bar{f}_{\zeta_1+\zeta_2+\zeta_3}(\bar{\Theta}_{\zeta_1+\zeta_2+\zeta_3}^{(0)})\Big],
\end{aligned} \tag{24}$$

which is nothing but the FT scheme applied to the partially reduced BSDE (II). The tractability of the FT schemes (23) and (24) relies on the nullity of the terminal condition of the related BSDEs (III) and (II), which implies that $\bar{\Theta}^{(0)} = \widetilde{\Theta}^{(0)} = 0$. By contrast, an FT scheme would not be practical for the full TVA BSDE (5) with terminal condition $\xi \neq 0$. Also note that the first order in the FT scheme (23) (resp. (24)) is nothing but the linear approximation (15) (resp. (14)).

4.3 Marked Branching Diffusion Approach

Based on an old idea of McKean [16], the solution $u(t_0, x_0)$ to a PDE

$$\partial_t u + \mathscr{L} u + \mu(F(u) - u) = 0, \quad u(T, x) = \Psi(x), \tag{25}$$

where $\mathscr{L}$ is the infinitesimal generator of a strong Markov process X and $F(y) = \sum_{k=0}^{d} a_k y^k$ is a polynomial of order d, admits a probabilistic representation in terms of a random tree $\mathscr{T}$ (branching diffusion). The tree starts from a single particle ("trunk") born from (t_0, x_0). Subsequently, every particle born from a node (t, x) evolves independently according to the generator $\mathscr{L}$ of X until it dies at time $t' = (t + \zeta)$ in a state x', where ζ is an independent μ-exponential time (one for each particle). Moreover, in dying, a particle gives birth to an independent number of k' new particles starting from the node (t', x'), where k' is drawn in the finite set $\{0, 1, \ldots, d\}$ with some fixed probabilities $p_0, p_1, \ldots, p_d$. The marked branching diffusion probabilistic representation reads

$$\begin{aligned} u(t_0, x_0) &= \mathbb{E}_{t_0, x_0}\left[\prod_{\{\text{inner nodes } (t,x,k) \text{ of } \mathscr{T}\}} \frac{a_k}{p_k} \prod_{\{\text{states } x \text{ of particles alive at } T\}} \Psi(x)\right] \\ &= \mathbb{E}_{t_0, x_0}\left[\prod_{k=0}^{d}\left(\frac{a_k}{p_k}\right)^{n_k} \prod_{l=1}^{\nu} \Psi(x_l)\right], \end{aligned} \tag{26}$$

where n_k is the number of branching with k descendants up on $(0, T)$ and ν is the number of particles alive at T, with corresponding locations $x_1, \ldots, x_\nu$.

The marked branching diffusion method of Henry-Labordère [13] for CVA computations, dubbed PHL scheme henceforth, is based on the idea that, by approximating y^+ by a well-chosen polynomial $F(y)$, the solution to the PDE

$$\partial_t u + \mathscr{L} u + \mu(u^+ - u) = 0, \quad u(T, x) = \Psi(x), \tag{27}$$

can be approximated by the solution to the PDE (25), hence by (26). We want to apply this approach to solve the TVA BSDEs (I), (II) or (III) for which, instead of fixing the approximating polynomial $F(y)$ once for all in the simulations, we need a state-dependent polynomial approximation to $g_t(y) = (P_t - y)^+$ (cf. (7)) in a suitable range for y. Moreover, (I) and (II) are BSDEs with random terminal time $\bar{\tau}$, equivalently written in a Markov setup as Cauchy–Dirichlet PDE problems, as opposed to the pure Cauchy problem (27). Hence, some adaptation of the method is required. We show how to do it for (II), after which we directly give the algorithm in the similar case of (I) and in the more classical (pure Cauchy) case of (III). Assuming τ given in terms of a $(\mathscr{G}, \mathbb{Q})$ Markov factor process X as $\tau = \inf\{t > 0 : X_t \notin \mathscr{D}\}$ for some domain $\mathscr{D}$, the Cauchy–Dirichlet PDE used for approximating the partially reduced BSDE (II) reads:

$$(\partial_t + \mathscr{A})\bar{u} + \mu\left(\bar{F}(\bar{u}) - \bar{u}\right) = 0 \text{ on } [0, T] \times \mathscr{D}, \quad \bar{u}(t, x) = 0 \text{ for } t = T \text{ or } x \notin \mathscr{D}, \tag{28}$$

where $\mathscr{A}$ is the generator of X and $\bar{F}_{t,x}(y) = \sum_{k=0}^{d} \bar{a}_k(t, x)y^k$ is such that

$$\mu(\bar{F}_{t,x}(y) - y) \approx \bar{f}(t, x, y), \text{ i.e. } \bar{F}_{t,x}(y) \approx \frac{\bar{f}(t, x, y)}{\mu} + y. \tag{29}$$

Specifically, in view of (9), one can set

$$\bar{F}_{t,x}(y) = \frac{1}{\mu}\left(cdva(t, x) + \bar{\lambda} pol\big(P(t, x) - y\big) - ry\right) + y = \sum_{k=0}^{d} \bar{a}_k(t, x)y^k, \tag{30}$$

where $pol(r)$ is a d-order polynomial approximation of r^+ in a suitable range for r. The marked branching diffusion probabilistic representation of $\bar{u}(t_0, x_0) \in \mathscr{D}$ involves a random tree $\overline{\mathscr{T}}$ made of nodes and "particles" between consecutive nodes as follows. The tree starts from a single particle (trunk) born from the root (t_0, x_0). Subsequently, every particle born from a node (t, x) evolves independently according to the generator $\mathscr{L}$ of X until it dies at time $t' = (t + \zeta)$ in a state x', where ζ is an independent μ-exponential time. Moreover, in dying, if its position x' at time t' lies in $\mathscr{D}$, the particle gives birth to an independent number of k' new particles starting from the node (t', x'), where k' is drawn in the finite set $\{0, 1, \ldots, d\}$ with some fixed probabilities $p_0, p_1, \ldots, p_d$. Figure 1 describes such a random tree in case $d = 2$. The first particle starts from the root (t_0, x_0) and dies at time t_1, generating two new particles. The first one dies at time t_{11} and generates a new particle, who dies at time $t_{111} > T$ without descendant. The second one dies at time t_{12} and generates two new particles, where the first one dies at time t_{121} without descendant and the second one dies at time t_{122} outside the domain $\mathscr{D}$, hence also without descendant. The blue points represent the inner nodes, the red points the outer nodes and the green points the exit points of the tree out of the time–space domain $[0, T] \times \mathscr{D}$.

The marked branching diffusion probabilistic representation of $\bar{u}$ is written as

$$\bar{u}(t_0, x_0) = \mathbb{E}_{t_0,x_0}\left[\mathbb{1}_{\overline{\mathscr{T}} \subset [0,T]\times\mathscr{D}} \prod_{\{\text{inner nodes } (t,x,k) \text{ of } \overline{\mathscr{T}}\}} \frac{\bar{a}_k(t, x)}{p_k}\right], \quad (t_0, x_0) \in [0, T] \times \mathscr{D}. \tag{31}$$

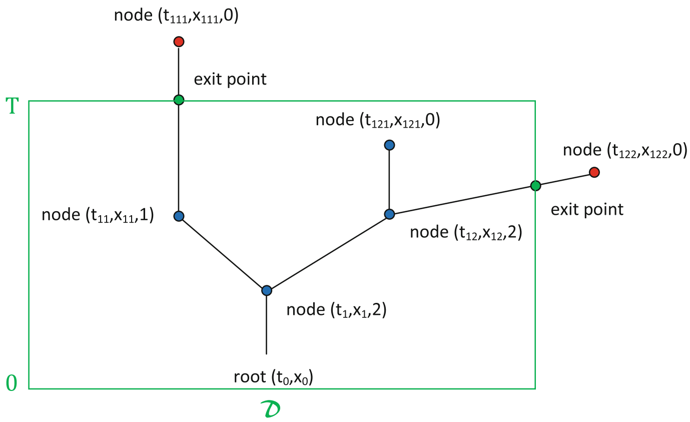

Fig. 1 PHL random tree

Note that (31) is unformal at that stage, where we did not justify whether the PDE (28) has a solution $\bar{u}$ and in which sense. In fact, the following result could be used for proving that the function $\bar{u}$ defined in the first line is a viscosity solution to (28).

Proposition 1 *Denoting by $\bar{u}$ the function defined by the right hand side in (31) (assuming integrability of the integrand on the domain $[0, T] \times \mathcal{D}$), the process $Y_t = \bar{u}(t, X_t)$, $0 \le t \le \bar{\tau}$, solves the BSDE associated with the Cauchy–Dirichlet PDE (28), namely*

$$Y_t = \mathbb{E}_t \left[\int_t^{\bar{\tau}} \mu\Big(\bar{F}_{s,X_s}(Y_s) - Y_s\Big) ds \right], \quad t \in [0, \bar{\tau}] \tag{32}$$

(which, in view of (29), approximates the partially reduced BSDE (II), so that $Y \approx \bar{\Theta}$ provided Y is square integrable).

Proof Let (t_1, x_1, k_1) be the first branching point in the tree rooted at $(0, X_0)$ and let $\overline{\mathcal{T}}_j$ denote k_1 independent trees of the same kind rooted at (t_1, x_1). By using the independence and the strong Markov property postulated for X, we obtain

$$\begin{aligned}
\bar{u}(t, X_t) &= \sum_{k_1=0}^{d} \mathbb{E}_{t,X_t} \Bigg[\mathbb{1}_{t_1<T} p_{k_1} \frac{a_{k_1}(t_1, x_1)}{p_{k_1}} \\
&\quad \times \prod_{j=1}^{k_1} \mathbb{E}_{t_1,x_1} \Bigg[\mathbb{1}_{\overline{\mathcal{T}}_j \subset [0,T]\times\mathcal{D}\}} \prod_{\{\text{inner node } (s,x,k) \text{ of } \overline{\mathcal{T}}_j\}} \frac{a_k(s,x)}{p_k} \Bigg]\Bigg] \\
&= \mathbb{E}_{t,X_t} \Bigg[\mathbb{1}_{t_1<T} \sum_{k_1=0}^{d} a_{k_1}(t_1, x_1) \prod_{j=1}^{k_1} \mathbb{E}_{t_1,x_1} \Bigg[\mathbb{1}_{\overline{\mathcal{T}}_j \subset [0,T]\times\mathcal{D}} \prod_{\{\text{inner node } (s,x,k) \text{ of } \overline{\mathcal{T}}_j\}} \frac{a_k(s,x)}{p_k} \Bigg]\Bigg] \\
&= \mathbb{E}_{t,X_t} \Bigg[\mathbb{1}_{t_1<T} \sum_{k_1=0}^{d} a_{k_1}(t_1, x_1) \prod_{j=1}^{k_1} \bar{u}(t_1, x_1) \Bigg]
\end{aligned}$$

$$= \mathbb{E}_{t,X_t}\left[\mathbb{1}_{t_1<T}\bar{F}_{t_1,x_1}(\bar{u}(t,X_t^{t_1,x_1}))\right]$$
$$= \mathbb{E}_{t,X_t}\left[\int_t^{\bar{\tau}} \mu(s)e^{-\int_t^s \mu(u)du}\bar{F}_{s,X_s^{t,x}}(\bar{u}(s,X_s^{t,x}))ds\right],\ 0\le t\le\bar{\tau},$$

i.e. $Y_t = \bar{u}(t,X_t)$ solves (32). □

If $\mathbb{1}_{\tau<T}\xi$ is given as a deterministic function $\Psi(\tau,X_\tau)$, then a similar approach (using the same tree $\overline{\mathscr{T}}$) can be applied to the full BSDE (I) in terms of the Cauchy–Dirichlet PDE

$$(\partial_t+\mathscr{A})u+\mu\,(F(u)-u)=0 \text{ on } [0,T]\times\mathscr{D},\quad u(t,x)=\Psi(t,x) \text{ for } t=T \text{ or } x\notin\mathscr{D}, \tag{33}$$

where $F_{t,x}(y)=\sum_{k=0}^d a_k(t,x)y^k$ is such that

$$\mu(F_{t,x}(y)-y)\approx f(t,x,y), \text{ i.e. } F_{t,x}(y)\approx\frac{f(t,x,y)}{\mu}+y.$$

This yields the approximation formula alternative to (31):

$$\Theta_0\approx\mathbb{E}\left[\prod_{\{\text{inner node }(t,x,k)\text{ of }\overline{\mathscr{T}}\}}\frac{a_k(t,x)}{p_k}\prod_{\{\text{exit point }(t,x)\text{ of }\overline{\mathscr{T}}\}}\Psi(t,x)\right], \tag{34}$$

where an exit point of $\overline{\mathscr{T}}$ means a point where a branch of the tree leaves for the first time the time–space domain $[0,T]\times\mathscr{D}$. Last, regarding the $(\mathscr{F},\mathbb{Q})$ reduced BSDE (III), assuming an $(\mathscr{F},\mathbb{Q})$ Markov factor process $\widetilde{X}$ with generator $\widetilde{\mathscr{A}}$ and domain $\mathscr{D}$, we can apply a similar approach in terms of the Cauchy PDE

$$(\partial_t+\widetilde{\mathscr{A}})\widetilde{u}+\mu\left(\widetilde{F}_{t,x}(\widetilde{u})-\widetilde{u}\right)=0 \text{ on } [0,T]\times\mathscr{D},\quad \widetilde{u}(t,x)=0 \text{ for } t=T \text{ or } x\notin\mathscr{D}, \tag{35}$$

where $\widetilde{F}_{t,x}(y)=\sum_{k=0}^d\widetilde{a}_k(t,x)y^k$ is such that

$$\mu(\widetilde{F}_{t,x}(y)-y)\approx\widetilde{f}(t,x,y), \text{ i.e. } \widetilde{F}_{t,x}(y)\approx\frac{\widetilde{f}(t,x,y)}{\mu}+y.$$

We obtain

$$\Theta_0=\widetilde{\Theta}_0\approx\widetilde{\mathbb{E}}\left[\mathbb{1}_{\widetilde{\mathscr{T}}\subset[0,T]\times\mathscr{D}}\prod_{\text{inner node }(t,x,k)\text{ of }\widetilde{\mathscr{T}}}\frac{\widetilde{a}_k(t,x)}{p_k}\right], \tag{36}$$

where $\widetilde{\mathscr{T}}$ is the branching tree associated with the Cauchy PDE (35) (similar to $\widetilde{\mathscr{T}}$ but for the generator $\widetilde{\mathscr{A}}$).

5 TVA Models for Credit Derivatives

Our goal is to apply the above approaches to TVA computations on credit derivatives referencing the names in $N^\star = \{1, \ldots, n\}$, for some positive integer n, traded between the bank and the counterparty respectively labeled as -1 and 0. In this section we briefly survey two models of the default times τ_i, $i \in N = \{-1, 0, 1, \ldots, n\}$, that will be used for that purpose with $\tau_b = \tau_{-1}$ and $\tau_c = \tau_0$, namely the dynamic Gaussian copula (DGC) model and the dynamic Marshall–Olkin copula (DMO) model. For more details the reader is referred to [8, Chaps. 7 and 8] and [6, Sects. 6 and 7].

5.1 *Dynamic Gaussian Copula TVA Model*

5.1.1 Model of Default Times

Let there be given a function $\varsigma(\cdot)$ with unit L^2 norm on $\mathbb{R}_+$ and a multivariate Brownian motion $\mathbf{B} = (B^i)_{i\in N}$ with pairwise constant correlation $\rho \geq 0$ in its own completed filtration $\mathscr{B} = (\mathscr{B}_t)_{t\geq 0}$. For each $i \in N$, let h_i be a continuously differentiable increasing function from $\mathbb{R}_+^*$ to $\mathbb{R}$, with $\lim_0 h_i(s) = -\infty$ and $\lim_{+\infty} h_i(s) = +\infty$, and let

$$\tau_i = h_i^{-1}\big(\varepsilon_i\big), \text{ where } \varepsilon_i = \int_0^{+\infty} \varsigma(u)dB_u^i. \tag{37}$$

Thus the $(\tau_i)_{i\in N}$ follow the standard Gaussian copula model of Li [15], with correlation parameter ρ and with marginal survival function $\Phi \circ h_i$ of τ_i, where Φ is the standard normal survival function. In particular, these τ_i do not intersect each other. In order to make the model dynamic as required by counterparty risk applications, the model filtration $\mathscr{G}$ is given as the Brownian filtration $\mathscr{B}$ progressively enlarged by the τ_i, i.e.

$$\mathscr{G}_t = \mathscr{B}_t \vee \bigvee_{i\in N} \big(\sigma(\tau_i \wedge t) \vee \sigma(\{\tau_i > t\})\big),\ \forall t \geq 0, \tag{38}$$

and the reference filtration $\mathscr{F}$ is given as $\mathscr{B}$ progressively enlarged by the default times of the reference names, i.e.

$$\mathscr{F}_t = \mathscr{B}_t \vee \bigvee_{i\in N^\star} \big(\sigma(\tau_i \wedge t) \vee \sigma(\{\tau_i > t\})\big),\ \forall t \geq 0. \tag{39}$$

As shown in Sect. 6.2 of Crépey and Song [6], for the filtrations $\mathscr{G}$ and $\mathscr{F}$ as above, there exists a (unique) probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ such that the condition (C) holds. For every $i \in N$, let

$$m_t^i = \int_0^t \varsigma(u)dB_u^i,\ k_t^i = \tau_i \mathbb{1}_{\{\tau_i \leq t\}},$$

and let $\mathbf{m}_t = (m^i_t)_{i \in N}$, $\mathbf{k}_t = (k^i_t)_{i \in N}$, $\widetilde{\mathbf{k}}_t = (\mathbb{1}_{i \in N^\star} k^i_t)_{i \in N}$. The couple $X_t = (\mathbf{m}_t, \mathbf{k}_t)$ (resp. $\widetilde{X}_t = (\mathbf{m}_t, \widetilde{\mathbf{k}}_t)$) plays the role of a $(\mathscr{G}, \mathbb{Q})$ (resp. $(\mathscr{F}, \mathbb{P})$) Markov factor process in the dynamic Gaussian copula (DGC) model.

5.1.2 TVA Model

A DGC setup can be used as a TVA model for credit derivatives, with mark $i = -1, 0$ and $E_b = \{-1\}$, $E_c = \{0\}$. Since there are no joint defaults in this model, it is harmless to assume that the contract promises no cash-flow at τ, i.e., $\Delta_\tau = 0$, so that $Q_\tau = P_\tau$. By [8, Propositions 7.3.1 p. 178 and 7.3.3 p. 181], in the case of vanilla credit derivatives on the reference names, namely CDS contracts and CDO tranches (cf. (47)), there exists a continuous, explicit function $\widetilde{P}_i$ such that

$$P_\tau = \widetilde{P}_i(\tau, \mathbf{m}_\tau, \mathbf{k}_{\tau-}), \tag{40}$$

or $\widetilde{P}^i_\tau$ in a shorthand notation, on the event $\{\tau = \tau_i\}$. Hence, (9) yields

$$\bar{f}_t(\vartheta) + r_t\vartheta = (1 - R_c)\gamma^0_t(\widetilde{P}^0_t)^+ - (1 - R_b)\gamma^{-1}_t(\widetilde{P}^{-1}_t)^- + \bar{\lambda}_t(P_t - \vartheta)^+, \quad \forall t \in [0, \bar{\tau}].$$

Assume that the processes r and $\bar{\lambda}$ are given before τ as continuous functions of (t, X_t), which also holds for P in the case of vanilla credit derivatives on names in N. Then the coefficients $\bar{f}$ and in turn $\widetilde{f}$ are deterministically given in terms of the corresponding factor processes as

$$\bar{f}_t(\vartheta) = \bar{f}(t, X_t, \vartheta), \ \widetilde{f}_t(\vartheta) = \widetilde{f}(t, \widetilde{X}_t, \vartheta),$$

so that we are in the Markovian setup where the FT and the PHL schemes are valid and, in principle, applicable.

5.2 *Dynamic Marshall–Olkin Copula TVA Model*

The above dynamic Gaussian copula model allows dealing with TVA on CDS contracts. But a Gaussian copula dependence structure is not rich enough for ensuring a proper calibration to CDS and CDO quotes at the same time. If CDO tranches are also present in a portfolio, a possible alternative is the following dynamic Marshall–Olkin (DMO) copula model, also known as the "common shock" model.

5.2.1 Model of Default Times

We define a family $\mathscr{Y}$ of "shocks", i.e. subsets $Y \subseteq N$ of obligors, usually consisting of the singletons $\{-1\}, \{0\}, \{1\}, \ldots, \{n\}$, and a few "common shocks" $I_1, I_2, \ldots, I_m$ representing simultaneous defaults. For $Y \in \mathscr{Y}$, the shock time η_Y is defined as an i.i.d. exponential random variable with parameter γ_Y. The default time of obligor i in the common shock model is then defined as

$$\tau_i = \min_{Y \in \mathscr{Y}, i \in Y} \eta_Y. \tag{41}$$

Example 1 Figure 2 shows one possible default path in a common-shock model with $n = 3$ and $\mathscr{Y} = \{\{-1\}, \{0\}, \{1\}, \{2\}, \{3\}, \{2, 3\}, \{0, 1, 2\}, \{-1, 0\}\}$. The inner oval shows which shocks happened and caused the observed default scenarios at successive default times.

The full model filtration $\mathscr{G}$ is defined as

$$\mathscr{G}_t = \bigvee_{Y \in \mathscr{Y}} \left(\sigma(\eta_Y \wedge t) \vee \sigma(\{\eta_Y > t\})\right), \; \forall t \geq 0.$$

Letting $\mathscr{Y}_\circ = \{Y \in \mathscr{Y};\ -1, 0 \notin Y\}$, the reference filtration $\mathscr{F}$ is given as

$$\mathscr{F}_t = \bigvee_{Y \in \mathscr{Y}_\circ} \left(\sigma(\eta_Y \wedge t) \vee \sigma(\{\eta_Y > t\})\right), \; t \geq 0.$$

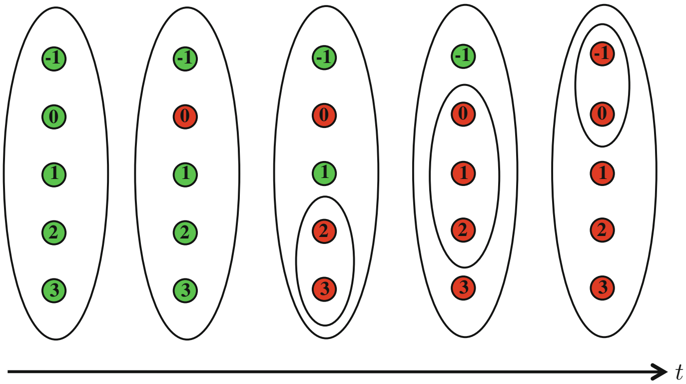

Fig. 2 One possible default path in the common-shock model with $n = 3$ and $\mathscr{Y} = \{\{-1\}, \{0\}, \{1\}, \{2\}, \{3\}, \{2, 3\}, \{0, 1, 2\}, \{-1, 0\}\}$

As shown in Sect. 7.2 of Crépey and Song [6], in the DMO model with $\mathscr{G}$ and $\mathscr{F}$ as above, the condition (C) holds for $\mathbb{P} = \mathbb{Q}$. Let $J^Y = \mathbb{1}_{[0,\eta_Y)}$. Similar to $(\mathbf{m}, \mathbf{k})$ (resp. $(\mathbf{m}, \widetilde{\mathbf{k}})$) in the DGC model, the process

$$X = (J^Y)_{Y\in\mathscr{Y}} \text{ (resp. } \widetilde{X} = (\mathbb{1}_{Y\in\mathscr{Y}_\circ} J^Y)_{Y\in\mathscr{Y}}) \tag{42}$$

plays the role of a $(\mathscr{G}, \mathbb{Q})$ (resp. $(\mathscr{F}, \mathbb{Q})$) Markov factor in the DMO model.

5.2.2 TVA Model

A DMO setup can be used as a TVA model for credit derivatives, with

$$E_b = \mathscr{Y}_b := \{Y \in \mathscr{Y};\ -1 \in Y\},\ E_c = \mathscr{Y}_c := \{Y \in \mathscr{Y};\ 0 \in Y\},\ E = \mathscr{Y}_\bullet := \mathscr{Y}_b \cup \mathscr{Y}_c$$

and

$$\tau_b = \tau_{-1} = \min_{Y\in\mathscr{Y}_b} \eta_Y,\ \tau_c = \tau_0 = \min_{Y\in\mathscr{Y}_c} \eta_Y,$$

hence

$$\tau = \min_{Y\in\mathscr{Y}_\bullet} \eta_Y,\ \gamma = \mathbb{1}_{[0,\tau)}\widetilde{\gamma} \text{ with } \widetilde{\gamma} = \sum_{Y\in\mathscr{Y}_\bullet} \gamma_Y. \tag{43}$$

By [8, Proposition 8.3.1 p. 205], in the case of CDS contracts and CDO tranches, for every shock $Y \in \mathscr{Y}$ and process $U = P$ or Δ, there exists a continuous, explicit function $\widetilde{U}_Y$ such that

$$U_\tau = \widetilde{U}_Y(\tau, X_{\tau-}), \tag{44}$$

or $\widetilde{U}^Y_\tau$ in a shorthand notation, on the event $\{\tau = \eta_Y\}$. The coefficient $\bar{f}_t(\vartheta)$ in (9) is then given, for $t \in [0, \bar{\tau}]$, by

$$\begin{aligned} \bar{f}_t(\vartheta) + r_t\vartheta = (1 - R_c) \sum_{Y\in\mathscr{Y}_c} \gamma^Y_t \big(\widetilde{P}^Y_t + \widetilde{\Delta}^Y_t\big)^+ - (1 - R_b) \sum_{Y\in\mathscr{Y}_b} \gamma^Y_t \big(\widetilde{P}^Y_t + \widetilde{\Delta}^Y_t\big)^- \\ + \bar{\lambda}_t(P_t - \vartheta)^+. \end{aligned} \tag{45}$$

Assuming that the processes r and $\bar{\lambda}$ are given before τ as continuous functions of (t, X_t), which also holds for P in case of vanilla credit derivatives on the reference names, then

$$\bar{f}_t(\vartheta) = \bar{f}(t, X_t, \vartheta), \widetilde{f}_t(\vartheta) = \bar{f}_t(\vartheta) - \widetilde{\gamma}\vartheta = \widetilde{f}(t, \widetilde{X}_t, \vartheta) \tag{46}$$

(cf. (43)), so that we are again in a Markovian setup where the FT and the PHL schemes are valid and, in principle, applicable.

5.3 *Strong Versus Weak Dynamic Copula Model*

However, one peculiarity of the TVA BSDEs in our credit portfolio models is that, even though full and reduced Markov structures have been identified, which is required for justifying the validity of the FT and/or PHL numerical schemes, and the corresponding generators $\mathscr{A}$ or $\widetilde{\mathscr{A}}$ can be written explicitly, the Markov structures are too heavy for being of any practical use in the numerics. Instead, fast and exact simulation and clean pricing schemes are available based on the dynamic copula structures.

Moreover, in the case of the DGC model, we lose the Gaussian copula structure after a branching point in the PHL scheme. In fact, as visible in [8, Formula (7.7) p. 175], the DGC conditional multivariate survival probability function is stated in terms of a ratio of Gaussian survival probability functions, which is explicit but does not simplify into a single Gaussian survival probability function. It is only in the DMO model that the conditional multivariate survival probability function, which arises as a ratio of exponential survival probability functions (see [8, Formula (8.11) p. 197 and Sect. 8.2.1.1]), simplifies into a genuine exponential survival probability function. Hence, the PHL scheme is not applicable in the DGC model.

The FT scheme based on (III) is not practical either because the Gaussian copula structure is only under $\mathbb{Q}$ and, again, the (full or reduced) Markov structures are not practical. In the end, the only practical scheme in the DGC model is the FT scheme based on the partially reduced BSDE (II). Eventually, it is only in the DMO model that the FT and the PHL schemes are both practical and can be compared numerically.

6 Numerics

For the numerical implementation, we consider stylized CDS contracts and protection legs of CDO tranches corresponding to dividend processes of the respective form, for $0 \le t \le T$:

$$
\begin{aligned}
D^i_t &= \big((1-R_i)\mathbb{1}_{t\geq\tau_i} - S_i(t\wedge\tau_i)\big)Nom_i \\
D_t &= \Big(\big((1-R)\sum_{j\in N}\mathbb{1}_{t\geq\tau_j} - (n+2)a\big)^+ \wedge (n+2)(b-a)\Big)Nom,
\end{aligned}
\tag{47}
$$

where all the recoveries R_i and R (resp. nominals Nom_i and Nom) are set to 40 % (resp. to 100). The contractual spreads S_i of the CDS contracts are set such that the corresponding prices are equal to 0 at time 0. Protection legs of CDO tranches, where the attachment and detachment points a and b are such that $0 \le a \le b \le 100\,\%$, can also be seen as CDO tranches with upfront payment. Note that credit derivatives traded as swaps or with upfront payment coexist since the crisis. Unless stated otherwise, the following numerical values are used:

$$r = 0, R_b = 1, R_c = 40\,\%, \bar{\lambda} = 100\text{ bp} = 0.01, \mu = \frac{2}{T}, m = 10^4.$$

6.1 Numerical Results in the DGC Model

First we consider DGC random times τ_i defined by (37), where the function h_i is chosen so that τ_i follows an exponential distribution with parameter γ_i (which in practice can be calibrated to a related CDS spread or a suitable proxy). More precisely, let Φ and Ψ_i be the survival functions of a standard normal distribution and an exponential distribution with intensity γ_i. We choose $h_i = \Phi^{-1} \circ \Psi_i$, so that (cf. (37))

$$\mathbb{Q}(\tau_i \geq t) = \mathbb{Q}\left(\Psi_i^{-1}\left(\Phi\left(\varepsilon_i\right)\right) \geq t\right) = \mathbb{Q}\Big(\Phi\left(\varepsilon_i\right) \leq \Psi_i(t)\Big) = \Psi_i(t),$$

for $\Phi\left(\varepsilon_i\right)$ has a standard uniform distribution. Moreover, we use a function $\varsigma(\cdot)$ in (37) constant before a time horizon $\bar{T} > T$ and null after $\bar{T}$, so that $\varsigma(0) = \frac{1}{\sqrt{\bar{T}}}$ (given the constraint that $\nu^2(0) = \int_0^\infty \varsigma^2(s)ds = 1$) and, for $t \leq \bar{T}$,

$$\nu^2(t) = \int_t^\infty \varsigma^2(s)ds = \frac{\bar{T}-t}{\bar{T}},\ m_t^i = \int_0^t \varsigma(u)dB_u^i = \frac{1}{\sqrt{\bar{T}}}B_t^i,\ \int_0^\infty \varsigma(u)dB_u^i = \frac{1}{\sqrt{\bar{T}}}B_{\bar{T}}^i.$$

In the case of the DGC model, the only practical TVA numerical scheme is the FT scheme (24) based on the partially reduced BSDE (II), which can be described by the following steps:

1. Draw a time ζ_1 following an exponential law of parameter μ. If $\zeta_1 < T$, then simulate $\mathbf{m}_{\zeta_1} = (\frac{1}{\sqrt{\bar{T}}}B^i_{\zeta_1})_{l\in N} \sim \mathcal{N}(0, \frac{\zeta_1}{\bar{T}}I_n(1,\rho))$, where $I_n(1,\rho)$ is a $n \times n$ matrix with diagonal equal to 1 and all off-diagonal entries equal to ρ, and go to Step 2. Otherwise, go to Step 4.
2. Draw a second time ζ_2, independent from ζ_1, following an exponential law of parameter μ. If $\zeta_1 + \zeta_2 < T$, then obtain the vector $\mathbf{m}_{\zeta_1+\zeta_2}$ as $\mathbf{m}_{\zeta_1} + (\mathbf{m}_{\zeta_1+\zeta_2} - \mathbf{m}_{\zeta_1})$, where $\mathbf{m}_{\zeta_1+\zeta_2} - \mathbf{m}_{\zeta_1} = (\frac{1}{\sqrt{\bar{T}}}(B^i_{\zeta_1+\zeta_2} - B^i_{\zeta_1}))_{l\in N} \sim \mathcal{N}(0, \frac{\zeta_2}{\bar{T}}I_n(1,\rho))$, and go to Step 3. Otherwise, go to Step 4.
3. Draw a third time ζ_3, independent from ζ_1 and ζ_2, following an exponential law of parameter μ. If $\zeta_1 + \zeta_2 + \zeta_3 < T$, then obtain the vector $\mathbf{m}_{\zeta_1+\zeta_2+\zeta_3}$ as $\mathbf{m}_{\zeta_1+\zeta_2} + (\mathbf{m}_{\zeta_1+\zeta_2+\zeta_3} - \mathbf{m}_{\zeta_1+\zeta_2})$, where $\mathbf{m}_{\zeta_1+\zeta_2+\zeta_3} - \mathbf{m}_{\zeta_1+\zeta_2} = (\frac{1}{\sqrt{\bar{T}}}(B^i_{\zeta_1+\zeta_2+\zeta_3} - B^i_{\zeta_1+\zeta_2}))_{l\in N} \sim \mathcal{N}(0, \frac{\zeta_3}{\bar{T}}I_n(1,\rho))$. Go to Step 4.

4. Simulate the vector $\mathbf{m}_{\bar{T}}$ from the last simulated vector $\mathbf{m}_t$ ($t = 0$ by default) as $\mathbf{m}_t + (\mathbf{m}_{\bar{T}} - \mathbf{m}_t)$, where $\mathbf{m}_{\bar{T}} - \mathbf{m}_t = (\frac{1}{\sqrt{\bar{T}}}(B^i_{\bar{T}} - B^i_t))_{i\in N} \sim \mathcal{N}(0, \frac{\bar{T}-t}{\bar{T}} I_n(1,\rho))$. Deduce $(B^i_{\bar{T}})_{i\in N}$, hence $\tau_i = \Psi_i^{-1} \circ \Phi\left(\frac{1}{\sqrt{\bar{T}}} B^i_{\bar{T}}\right)$, $i \in N$, and in turn the vectors $\mathbf{k}_{\zeta_1}$ (if $\zeta_1 + \zeta_2 + \zeta_3 < T$), $\mathbf{k}_{\zeta_1+\zeta_2}$ (if $\zeta_1 + \zeta_2 < T$) and $\mathbf{k}_{\zeta_1+\zeta_2+\zeta_3}$ (if $\zeta_1 + \zeta_2 + \zeta_3 < T$).
5. Compute $\bar{f}_{\zeta_1}, \bar{f}_{\zeta_1+\zeta_2}$, and $\bar{f}_{\zeta_1+\zeta_2+\zeta_3}$ for the three orders of the FT scheme.

We perform TVA computations on CDS contracts with maturity $T = 10$ years, choosing for that matter $\bar{T} = T + 1 = 11$ years, hence $\varsigma = \frac{\mathbb{1}_{[0,11]}}{\sqrt{11}}$, for $\rho = 0.6$ unless otherwise stated. Table 1 displays the contractual spreads of the CDS contracts used in these experiments. In Fig. 3, the left graph shows the TVA on a CDS on name 1, computed in a DGC model with $n = 1$ by FT scheme of order 1 to 3, for different levels of nonlinearity represented by the value of the unsecured borrowing spread $\bar{\lambda}$. The right graph shows similar results regarding a portfolio comprising one CDS contract per name $i = 1, \ldots, 10$. The time-0 clean value of the default leg of the CDS in case $n = 1$, respectively the sum of the ten default legs in case $n = 10$, is 4.52, respectively 40.78 (of course $P_0 = 0$ in both cases by definition of fair contractual spreads). Hence, in relative terms, the TVA numbers visible in Fig. 3 are quite high, much greater for instance than in the cases of counterparty risk on interest rate derivatives considered in Crépey et al. [7]. This is explained by the wrong-way risk feature of the DGC model, namely, the default intensities of the surviving names and the value of the CDS protection spike at defaults in this model. When $\bar{\lambda}$ increases (for $\bar{\lambda} = 0$ that's a case of linear TVA where FT higher order terms equal 0), the second (resp. third) FT term may represent in each case up to 5–10 % of the first

Table 1 Time-0 bp CDS spreads of names −1 (the bank), 0 (the counterparty) and of the reference names 1 to n used when $n = 1$ (*left*) and $n = 10$ (*right*)

i	-1	0	1
S_i	36	41	47

i	−1	0	1	2	3	4	5	6	7	8	9	10
S_i	39	40	47	36	41	48	54	54	27	30	36	50

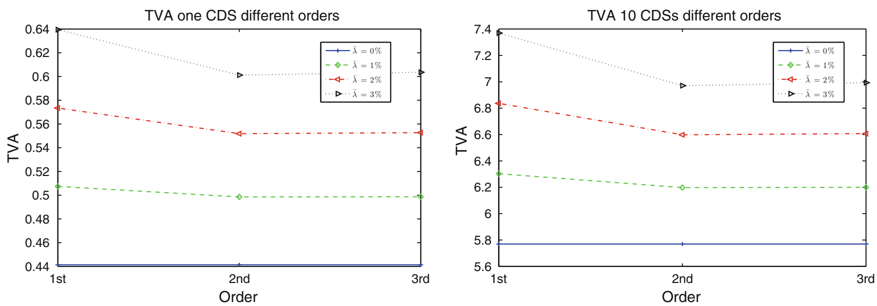

Fig. 3 *Left* DGC TVA on one CDS computed by FT scheme of order 1–3, for different levels of nonlinearity (unsecured borrowing spread $\bar{\lambda}$). *Right* similar results regarding the portfolio of CDS contracts on ten names

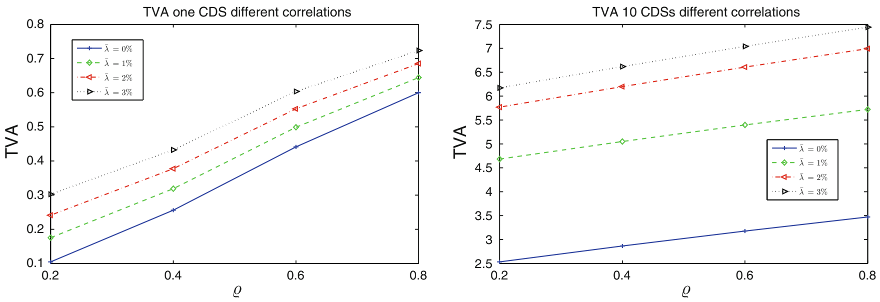

Fig. 4 *Left* TVA on one CDS computed by FT scheme of order 3 as a function of the DGC correlation parameter ρ. *Right* similar results regarding a portfolio of CDS contracts on ten different names

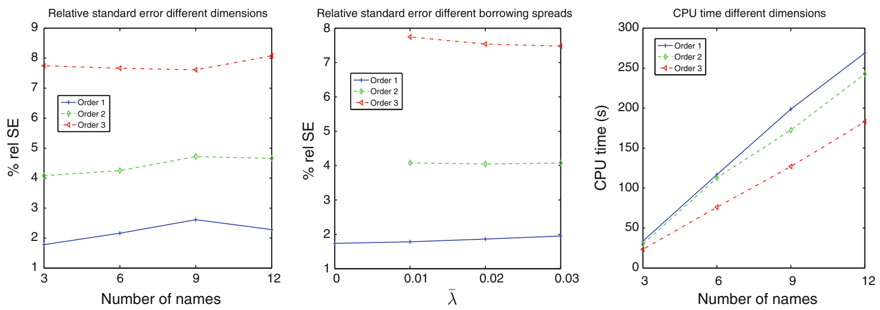

Fig. 5 *Left* the % relative standard errors of the different orders of the expansions do not explode with the number of names ($\bar{\lambda} = 100$ bp). *Middle* the % relative standard errors of the different orders of the expansions do not explode with the level of nonlinearity represented by the unsecured borrowing spread $\bar{\lambda}$ ($n = 1$). *Right* since FT terms are computed by purely forward Monte Carlo schemes, their computation times are linear in the number of names ($\bar{\lambda} = 100$ bp)

(resp. second) FT term, from which we conclude that the first FT term can be used as a first order linear estimate of the TVA, with a nonlinear correction that can be estimated by the second FT term.

In Fig. 4, the left graph shows the TVA on one CDS computed by FT scheme of order 3 as a function of the DGC correlation parameter ρ, with other parameters set as before. The right graph shows the analogous results regarding the portfolio of ten CDS contracts. In both cases, the TVA numbers increase (roughly linearly) with ρ, including for high values of ρ, as desirable from the financial interpretation point of view, whereas it has been noted in Brigo and Chourdakis [1] (see the blue curve in Fig. 1 of the ssrn version of the paper) that for high levels of the correlation between names, other models may show some pathological behaviors.

In Fig. 5, the left graph shows that the errors, in the sense of the relative standard errors (% rel. SE), of the different orders of the FT scheme do not explode with the dimension (number of credit names that underlie the CDS contracts). The middle graph, produced with $n = 1$, shows that the errors do not explode with the level of nonlinearity represented by the unsecured borrowing spread $\bar{\lambda}$. Consistent with the fact that the successive FT terms are computed by purely forward Monte Carlo schemes, their computation times are essentially linear in the number of names, as visible in the right graph.

To conclude this section, we compare the linear approximation (14) corresponding to the first FT term in (24) (FT1 in Table 2) with the linear approximations (12)–(13) (LA in Table 2). One can see from Table 2 that the LA and FT1 estimates are consistent (at least in the sense of their 95 % confidence intervals, which always intersect each other). But the LA standard errors are larger than the FT1 ones. In fact, using the formula for the intensity γ of τ in FT1 can be viewed as a form of variance reduction with respect to LA, where τ is simulated. Of course, for $\bar{\lambda} \neq 0$ (case of the right tables where $\bar{\lambda} = 3\,\%$), both linear approximations are biased as compared with the complete FT estimate (with nonlinear correction, also shown in Table 2), particularly in the high dimensional case with 10 CDS contracts (see the bottom panels in Table 2). Figure 6 completes these results by showing the LA, FT1

Table 2 LA, FT1 and FT estimates: 1 CDS (*top*) and 10 CDSs (*bottom*), with parameters $\bar{\lambda} = 0\,\%$, $\rho = 0.8$ (*left*) and $\bar{\lambda} = 3\,\%$, $\rho = 0.6$ (*right*)

Method	TVA	95% CI	Rel. SE	Method	TVA	95% CI	Rel. SE
LA	0.65	[0.57, 0.73]	6.08 %	LA	0.66	[0.60, 0.72]	4.39%
FT1	0.61	[0.59, 0.63]	1.66%	FT1	0.62	[0.59, 0.64]	1.96%
FT	0.60	[0.58, 0.62]	1.64 %	FT	0.60	[0.58, 0.63]	1.84%

Method	TVA	95% CI	Rel. SE	Method	TVA	95% CI	Rel. SE
LA	6.17	[5.43, 6.92]	6.03%	LA	6.81	[6.16, 7.45]	4.76%
FT1	6.24	[5.77, 6.72]	3.78%	FT1	7.82	[7.39, 8.25]	2.73%
FT	6.17	[5.66, 6.68]	4.15%	FT	6.99	[6.67, 7.31]	2.28%

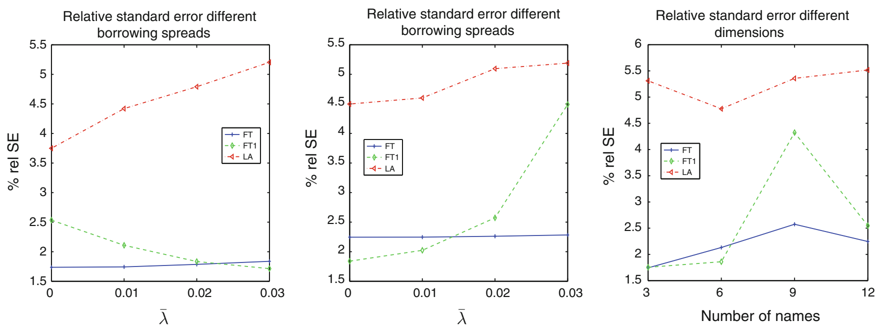

Fig. 6 The % relative standard errors of the different schemes do not explode with the level of nonlinearity represented by the unsecured borrowing spread $\bar{\lambda}$. *Left* 1 CDS. *Middle* 10 CDSs. *Right* the % relative standard errors of the different schemes (LA, FT1, FT in figures) do not explode with the number of names ($\bar{\lambda} = 100$ bp, $\rho = 0.6$)

and FT standard errors computed for different levels of nonlinearity and different dimensions.

Summarizing, in the DGC model, the PHL is not practical. The FT scheme based on the partially reduced TVA BSDE (II) gives an efficient way of estimating the TVA. The nonlinear correction with respect to the linear approximations (14) or (15) amounts up to 5 % in relative terms, depending on the unsecured borrowing spread $\bar{\lambda}$.

6.2 Numerical Results in the DMO Model

In the DMO model, the FT scheme (18) for the fully reduced BSDE (23) can be implemented through following steps:

1. Simulate the time η_Y of each (individual or joint) shock following an independent exponential law of parameter γ_Y, $Y \in \mathscr{Y}$, then retrieve the τ_i through the formula (41).
2. Draw a time ζ_1 following an exponential law of parameter μ. If $\zeta_1 < T$, compare the default time of each name with ζ_1 to obtain the reduced Markov factor $\widetilde{X}_{\zeta_1}$ as of (42) and in turn $\widetilde{f}_{\zeta_1}$ as of (45)–(46), then go to Step 3. Otherwise stop.
3. Draw a second time ζ_2 following an independent exponential law of parameter μ. If $\zeta_1 + \zeta_2 < T$, compare the default time τ_i of each name with $\zeta_1 + \zeta_2$ to obtain the Markov factor $\widetilde{X}_{\zeta_1+\zeta_2}$ and $\widetilde{f}_{\zeta_1+\zeta_2}$ then go to Step 4. Otherwise stop.
4. Draw a third time ζ_3 following an independent exponential law of parameter μ. If $\zeta_1 + \zeta_2 + \zeta_3 < T$, compare the default time of each name with $\zeta_1 + \zeta_2 + \zeta_3$ to obtain the Markov factor $\widetilde{X}_{\zeta_1+\zeta_2+\zeta_3}$ and $\widetilde{f}_{\zeta_1+\zeta_2+\zeta_3}$.

We can also consider the PHL scheme (31) based on the partially reduced BSDE (II) with

$$\mathscr{D} = \{x = (x^Y)_{Y\in\mathscr{Y}} \in \{0, 1\}^{\mathscr{Y}} \text{ such that } x^Y = 1 \text{ for } Y \in \mathscr{Y}_\bullet\}.$$

To simulate the random tree $\overline{\mathscr{T}}$ in (31), we follow the approach sketched before (31) where, in order to evolve X according to the DMO generator $\mathscr{A}$ during a time interval ζ, a particle born from a node $x = (j_Y)_{Y\in\mathscr{Y}} \in \{0, 1\}^{\mathscr{Y}}$ at time t, all one needs is, for each Y such that $j_Y = 1$, draw an independent exponential random variable η_Y of parameter γ_Y and then set $x' = (j_Y \mathbb{1}_{[0,\eta_Y)}(\zeta))_{Y\in\mathscr{Y}}$. Rephrasing in more algorithmic terms:

1. To simulate the random tree $\overline{\mathscr{T}}$ under the expectation in (31), we repeat the following step (generation of particles, or segments between consecutive nodes of the tree) until a generation of particles dies without children:

> For each node $(t, x = (j_Y)_{Y\in\mathscr{Y}}, k)$ issued from the previous generation of particles (starting with the root-node $(0, X_0, k = 1)$), for each of the k new particles, indexed by

l, issued from that node, simulate an independent exponential random variable ζ_l and set

$$(t'_l, x'_l, k'_l) = (t + \zeta_l, (j_Y \mathbb{1}_{[0,\eta^l_Y)}(\zeta_l))_{Y \in \mathscr{Y}}, \mathbb{1}_{x'_l \in \mathscr{D}} \nu_l),$$

where, for each l, the η^l_Y are independent exponential-γ_Y random draws and ν_l is an independent draw in the finite set $\{0, 1, \dots, d\}$ with some fixed probabilities $p_0, p_1, \dots, p_d$.

2. To compute the random variable Φ under the expectation in (31), we loop over the nodes of the tree $\overline{\mathscr{T}}$ thus constructed (if $\overline{\mathscr{T}} \subset [0, T] \times \mathscr{D}$, otherwise $\Phi = 0$ in the first place) and we form the product in (31), where the $\bar{a}_k(t, x)$ are retrieved as in (30).

The PHL schemes (34) based on the full BSDE (I) or (36) based on the fully reduced BSDE (III) can be implemented along similar lines.

We perform TVA computations in a DMO model with $n = 120$, for individual shock intensities taken as $\gamma_{\{i\}} = 10^{-4} \times (100 + i)$ (increasing from $\sim$100 bps to 220 bps as i increases from 1 to 120) and four nested groups of common shocks $I_1 \subset I_2 \subset I_3 \subset I_4$, respectively consisting of the riskiest 3, 9, 21 and 100 % (i.e. all) names, with respective shock intensities $\gamma_{I_1} = 20$ bp, $\gamma_{I_2} = 10$ bp, $\gamma_{I_3} = 6.67$ bp and $\gamma_{I_4} = 5$ bp. The counterparty (resp. the bank) is taken as the eleventh (resp. tenth) safest name in the portfolio. In the model thus specified, we consider CDO tranches with upfront payment, i.e. credit protection bought by the bank from the counterparty at time 0, with nominal 100 for each obligor, maturity $T = 2$ years and attachment (resp. detachment) points are 0, 3 and 14 % (resp. 3 %, 14 % and 100 %). The respective value of P_0 (upfront payment) for the equity, mezzanine and senior tranche is 229.65, 5.68, and 2.99. Accordingly, the ranges of approximation chosen for $pol(y) \approx y^+$ in the respective PHL schemes are 250, 200, and 10. We use polynomial approximation of order $d = 4$ with $(p_0, p_1, p_2, p_3, p_4) = (0.5, 0.3, 0.1, 0.09, 0.01)$. We set $\mu = 0.1$ in all PHL schemes and $\mu = 2/T = 0.2$ in all FT schemes.

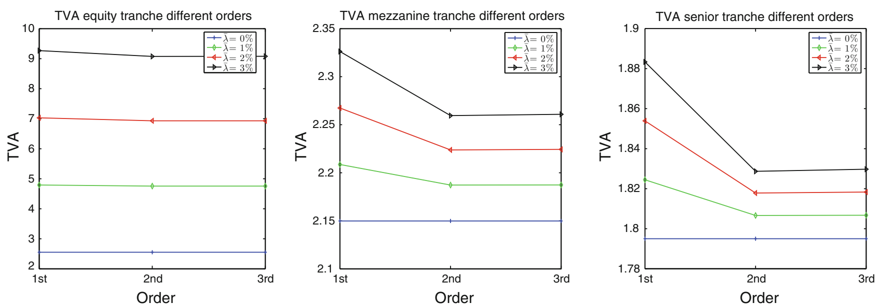

Fig. 7 TVA on CDO tranches with 120 underlying names computed by FT scheme of order 1–3 for different levels of nonlinearity (unsecured borrowing basis $\bar{\lambda}$). *Left* equity tranche. *Middle* mezzanine tranche. *Right* senior tranche. Originally published in Crépey and Song [6]. Published with kind permission of © Springer-Verlag Berlin Heidelberg 2016. All Rights Reserved. This figure is subject to copyright protection and is not covered by a Creative Commmons License

Table 3 FT, PHL, $\overline{\text{PHL}}$ and $\widetilde{\text{PHL}}$ schemes applied to the equity (*top*), mezzanine (*middle*), and senior (*bottom*) tranche, for the parameters $\bar{\lambda} = 0\,\%$, $\lambda_{I_j} = 60bp/j$ (*left*) or $\bar{\lambda} = 3\,\%$, $\lambda_{I_j} = 20bp/j$ (*right*)

Method	TVA	95% CI	Rel. SE	Method	TVA	95% CI	Rel. SE
FT	3.13	[3.10 , 3.16]	0.48 %	FT	9.08	[9.00 , 9.16]	0.46 %
$\widetilde{\text{PHL}}$	3.07	[2.87 , 3.28]	3.35 %	$\widetilde{\text{PHL}}$	9.05	[8.40 , 9.70]	3.58 %
$\overline{\text{PHL}}$	3.16	[2.94 , 3.37]	3.37 %	$\overline{\text{PHL}}$	9.28	[8.63 , 9.94]	3.51 %
PHL	2.53	[2.13 , 2.94]	8.02%	PHL	12.59	[6.92 , 18.27]	22.54%

Method	TVA	95% CI	Rel. SE	Method	TVA	95% CI	Rel. SE
FT	6.43	[6.33 , 6.53]	0.75 %	FT	2.29	[2.25 , 2.32]	0.77 %
$\widetilde{\text{PHL}}$	6.34	[5.93 , 6.75]	3.22 %	$\widetilde{\text{PHL}}$	2.51	[2.35 , 2.67]	3.17 %
$\overline{\text{PHL}}$	6.34	[5.93 , 6.75]	3.25 %	$\overline{\text{PHL}}$	2.68	[2.52 , 2.85]	3.12 %
PHL	4.86	[2.84 , 6.89]	20.82%	PHL	1.93	[0.79 , 3.08]	29.57%

Method	TVA	95% CI	Rel. SE	Method	TVA	95% CI	Rel. SE
FT	5.32	[5.24 , 5.40]	0.75 %	FT	1.83	[1.80 , 1.86]	0.78 %
$\widetilde{\text{PHL}}$	5.24	[4.90 , 5.58]	3.22 %	$\widetilde{\text{PHL}}$	1.80	[1.69 , 1.92]	3.13 %
$\overline{\text{PHL}}$	5.25	[4.90 , 5.58]	3.25 %	$\overline{\text{PHL}}$	1.87	[1.75 , 1.99]	3.11 %
PHL	4.01	[2.32 , 5.70]	21.03%	PHL	1.36	[0.41 , 2.31]	35.05%

Figure 7 shows the TVA computed by the FT scheme (23) based on the fully reduced BSDE (III), for different levels of nonlinearity (unsecured borrowing basis $\bar{\lambda}$). We observe that, in all cases, the third order term is negligible. Hence,

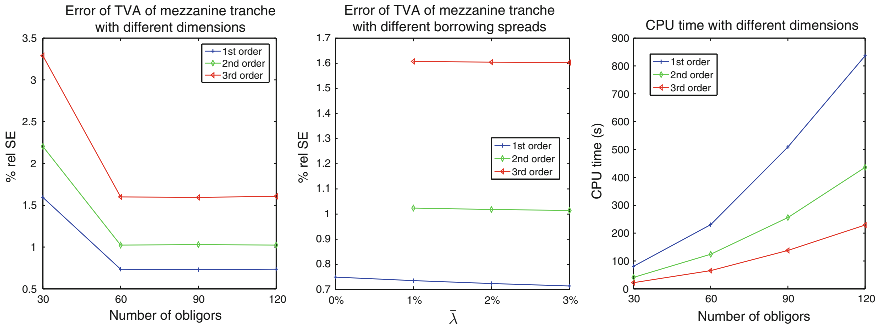

Fig. 8 Analog of Fig. 5 for the CDO tranche of Fig. 7 in the DMO model ($\bar{\lambda} = 0.01$). Originally published in Crépey and Song [6]. Published with kind permission of ©Springer-Verlag Berlin Heidelberg 2016. All Rights Reserved. This figure is subject to copyright protection and is not covered by a Creative Commmons License

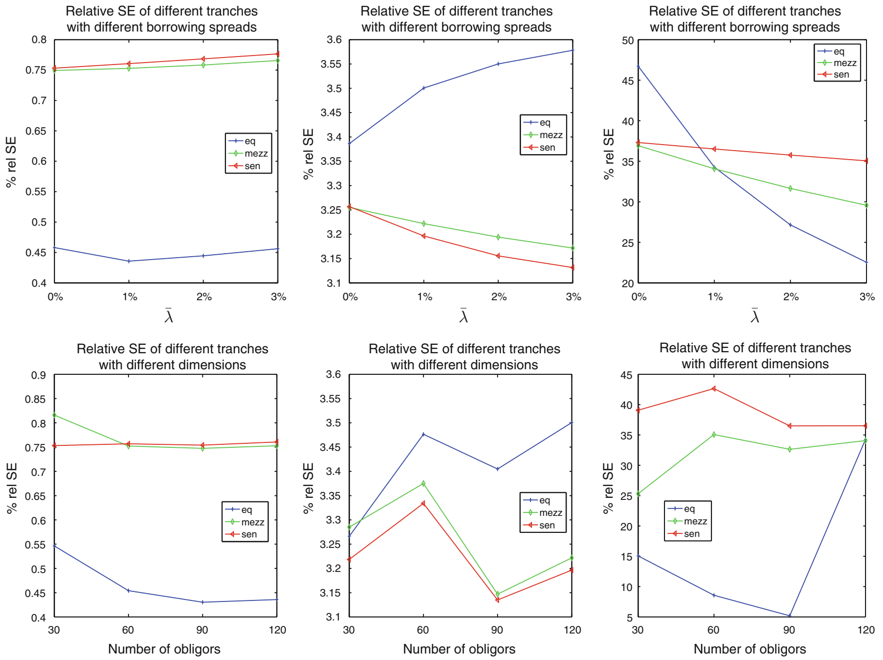

Fig. 9 *Bottom* the % relative standard errors do not explode with the number of names ($\bar{\lambda} = 100$ bp). *Top* the % relative standard errors do not explode with the level of nonlinearity represented by the unsecured borrowing spread $\bar{\lambda}$ ($n = 120$). *Left* FT scheme. *Middle* $\widetilde{\text{PHL}}$ scheme. *Right* PHL scheme

in further FT computations, we only compute the orders 1 (linear part) and 2 (nonlinear correction) (Fig. 8). Table 3 compares the results of the above FT scheme (23) based on the fully reduced BSDE (III) with those of the PHL schemes (36) based on (III) again ($\widetilde{\text{PHL}}$ in the tables), (31) based on the partially reduced BSDE (II) ($\overline{\text{PHL}}$ in the tables) and (34) based on the full BSDE (I) (PHL in the tables), for the three CDO tranches and two sets of parameters. The three PHL schemes are of course slightly biased, but the first two, based on the BSDEs with null terminal condition (III) or (II), exhibit much less variance than the third one, based on the full BSDE with terminal condition ξ. This is also visible in Fig. 9 (note the different scales of the y axes going from left to right in the picture), which also shows that, for any of these schemes, the relative standard errors do not explode with the level of nonlinearity or the number of reference names in the CDO (the results for the $\overline{\text{PHL}}$ scheme are not shown on the figure as very similar to those of the $\widetilde{\text{PHL}}$ scheme). In comparing the TVA values on the left and the right hand side of Table 3, we see that the intensities of the common shocks, which play a role similar to the correlation ρ in the DGC model, have a more important impact on the higher tranches (mezzanine and senior tranche), whereas the equity tranche is more sensitive to the level of the unsecured borrowing spread $\bar{\lambda}$.

7 Conclusion

Under mild assumptions, three equivalent TVA BSDEs are available. The original "full" BSDE (I) is stated with respect to the full model filtration $\mathscr{G}$ and the original pricing measure $\mathbb{Q}$. It does not involve the intensity γ of the counterparty first-to-default time τ. The partially reduced BSDE (II) is also stated with respect to $(\mathscr{G}, \mathbb{Q})$ but it involves both τ and γ. The fully reduced BSDE (III) is stated with respect to a smaller "reference filtration" $\mathscr{F}$ and it only involves γ. Hence, in principle, the full BSDE (I) should be preferred for models with a "simple" τ whereas the fully reduced BSDE (III) should be preferred for models with a "simple" γ. But, in nonimmersive setups, the fully reduced BSDE (III) is stated with respect to a modified probability measure $\mathbb{P}$. Even though switching from $(\mathscr{G}, \mathbb{Q})$ to $(\mathscr{F}, \mathbb{P})$ is transparent in terms of the generator of related Markov factor processes, this can be an issue in situations where the Markov structure is important in the theory to guarantee the validity of the numerical schemes, but is not really practical from an implementation point of view. This is for instance the case with the credit portfolio models that we use for illustrative purposes in our numerics, where the Markov structure that emerges from the dynamic copula model is too heavy and it is only the copula features that can be used in the numerics—copula features under the original stochastic basis $(\mathscr{G}, \mathbb{Q})$, which do not necessarily hold under a reduced basis $(\mathscr{F}, \mathbb{P})$ (especially when $\mathbb{P} \neq \mathbb{Q}$). As for the partially reduced BSDE (II), as compared with the full BSDE (I), its interest is its null terminal condition, which is key for the FT scheme as recalled below. But of course (II) can only be used when one has an explicit formula for γ.

For nonlinear and very high-dimensional problems such as counterparty risk on credit derivatives, the only feasible numerical schemes are purely forward simulation schemes, such as the linear Monte Carlo expansion of Fujii and Takahashi [9, 10] or the branching particles scheme of Henry–Labordère [13], respectively dubbed "FT scheme" and "PHL scheme" in the paper. In our setup, the PHL scheme involves a nontrivial and rather sensitive fine-tuning for finding a polynomial in ϑ that approximates the terms $(P_t - \vartheta)^{\pm}$ in $fva_t(\vartheta)$ in a suitable range for ϑ. This fine-tuning requires a preliminary knowledge on the solution obtained by running another approximation (linear approximation or FT scheme) in the first place. Another limitation of the PHL scheme in our case is that it is more demanding than the FT scheme in terms of the structural model properties that it requires. Namely, in our credit portfolio problems, both a Markov structure and a dynamic copula are required for the PHL scheme. But, whereas a "weak" dynamic copula structure in the sense of simulation and forward pricing by copula means is sufficient for the FT scheme, a dynamic copula in the stronger sense that the copula structure is preserved in the future is required in the case of the PHL scheme. This strong dynamic copula property is satisfied by our common-shock model but not in the Gaussian copula model. In conclusion, the FT schemes applied to the partially or fully reduced BSDEs (II) or (III) (a null terminal condition is required so that the full BSDE (I) is not eligible for this scheme) appear as the method of choice on these problems.

An important message of the numerics is that, even for realistically high levels of nonlinearity, i.e. an unsecured borrowing spread $\bar{\lambda} = 3\,\%$, the third order FT correction was always found negligible and the second order FT correction less than 5–10 % of the first order, linear FT term. In conclusion, a first order FT term can be used for obtaining "the best linear approximation" to our problem, whereas a nonlinear correction, if wished, can be computed by a second order FT term.

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Tight Semi-model-free Bounds on (Bilateral) CVA

Jördis Helmers, Jan-J. Rückmann and Ralf Werner

Abstract In the last decade, counterparty default risk has experienced an increased interest both by academics as well as practitioners. This was especially motivated by the market turbulences and the financial crises over the past decade which have highlighted the importance of counterparty default risk for uncollateralized derivatives. After a succinct introduction to the topic, it is demonstrated that standard models can be combined to derive semi-model-free tight lower and upper bounds on bilateral CVA (BCVA). It will be shown in detail how these bounds can be easily and efficiently calculated by the solution of two corresponding linear optimization problems.

Keywords Counterparty credit risk · CVA · Tight bounds · Mass transportation problem

1 Introduction

Events such as Lehman's default have drawn the attention to counterparty default risk. At the very latest after this default, it has become obvious to all market participants that the credit qualities of both counterparties—usually a client and an investment bank—need to be considered in the pricing of uncollateralized OTC derivatives.

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Over the past years, several authors have been investigating the pricing of derivatives based on a variety of models which take into account these default risks. Most of these results are covered by a variety of excellent books, for example Pykhtin [16], Gregory [12], or Brigo et al. [7] just to name a few. For a profound discussion on the pros and cons of unilateral versus bilateral counterparty risk let us refer to the two articles by Gregory [11, 13].

In the following exposition, we are concerned with the quantification of the smallest and largest BCVA which can be obtained by any given model with predetermined marginal laws. This takes considerations of Turnbull [21] much further, who first derived weak bounds on CVA for certain types of products. Our approach extends first ideas from Hull and White [15], where the hazard rate determining defaults is coupled to the exposure or other risk factors in either deterministic or stochastic way. Still, Hull and White rely on an explicit choice of the default model and on an explicit coupling. More related is the work by Rosen and Saunders et al. [8, 17], on which we prefer to comment later in Remark 8. As the most related work we note the paper by Cherubini [9] which provided the basis for this semi-model-free approach. There, only one particular two-dimensional copula was used to couple each individual forward swap par rate with the default time. Obviously, a more general approach couples each forward swap par rate with each other and the default time—which is in gist similar to Hull and White [15]. From there the final step to our approach is to observe that the most general approach directly links the whole stochastic evolution of the exposure with both random default times. We will illustrate in the following that these couplings can be readily derived by linear programming. For this purpose the BCVA will be decomposed into three main components: the first component is represented by the loss process, the second component consists of the default indicators of the two counterparties and the third component is comprised of the exposure-at-default of the OTC derivative, i.e. the risk-free present value of the outstanding amount[1] at time of default. This approach takes further early considerations of Haase and Werner [14], where comparable results were obtained from the point of view of generalized stopping problems.

In a very recent working paper by Scherer and Schulz [18], the above idea was analyzed in more detail. It was shown that the computational complexity of the problem is the same, no matter if only marginal distributions of defaults or the joint distribution of defaults are known.

After submission of this paper we became aware of related results by Glasserman and Yang, see [10]. Although the main idea of their exposition is similar in gist, Glasserman and Yang focus on the unilateral CVA instead of bilateral CVA. Besides an analysis of the convergence of finite samples to the continuous setup, their exposition is mainly focused on the penalization of deviation from some *base distribution*. In contrast, our focus is on bilateral CVA, with special attention to numerical solution and to the case that payoffs also depend on the credit quality.

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[1] In accordance with the *full two-way payment* rule under ISDA master contracts, see e.g. Bielecki and Rutkowski [2] (Sect. 14.4.4), we assume that the close-out value is determined by the then prevailing risk-free present value.

In summary, this exposition makes the following main contributions:

- First, the three main building blocks of such an adjustment are clearly identified and separated, and it is shown how any coupling of these blocks leads to a feasible adjustment. Unlike Cherubini, who only considered the very specific case of an interest rate swap, all kinds of derivatives (interest rate, FX, commodity, and even credit derivatives) are covered in a unified way—even if the payoff, and thus the present value of the derivative, is explicitly depending on the credit quality of any of the two counterparties.
- Second, by generalizing Cherubini's approach, upper and lower bounds on unilateral and bilateral counterparty value adjustments are derived. It will be demonstrated that these bounds can be efficiently obtained by the solution of linear optimization problems, more specifically, by the solution of balanced transportation problems. In contrast to the approaches of Turnbull [21] or Cherubini [9], both the upper and lower bound derived here are *tight bounds*, i.e. there exists some stochastic model which is consistent with all given market prices in which these bounds are attained.

The rest of the paper is organized as follows. In Sect. 2 a succinct introduction to bilateral counterparty risk is given, before the decomposition of the BCVA into its building blocks is carried out in Sect. 3. In Sect. 4 the two main approaches for the calculation of counterparty valuation adjustments are briefly reviewed. Finally, the tight bounds on CVA are derived in Sect. 5, before the paper concludes.

2 Counterparty Default Risk

As usual, to model financial transactions with default risk, let $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbf{Q})$ be a probability space where $\mathscr{G}_t$ models the flow of information and $\mathbf{Q}$ denotes the risk-neutral measure for a given risk-free numéraire process $N_t > 0$, see e.g. Bielecki and Rutkowski [2] for more details. Further, let the space be endowed with a right-continuous and complete sub-filtration $\mathscr{F}_t$ modeling the flow of information except default, such that $\mathscr{F}_t \subseteq \mathscr{G}_t := \mathscr{F}_t \vee \mathscr{H}_t$ with $\mathscr{H}_t$ being the right-continuous filtration generated by the default events.

Subsequently, we consider a transaction with maturity T between a client A and a counterparty B where both are subject to default. The respective random default times are denoted by τ_A and τ_B. In order to take into account counterparty default risk we distinguish three cases:

- neither A nor B defaults before T: $D_0 := \{\tau_A > T\} \cap \{\tau_B > T\}$,
- A defaults before B and before T: $D_A := \{\tau_A \leq T\} \cap \{\tau_A \leq \tau_B\}$,
- B defaults before A and before T: $D_B := \{\tau_B \leq T\} \cap \{\tau_B \leq \tau_A\}$.

For simplicity of presentation, we assume in the following that $\mathbf{Q}[\tau_A = T] = \mathbf{Q}[\tau_B = T] = \mathbf{Q}[\tau_A = \tau_B] = 0$. Under this assumption these sets[2] yield a decomposition of one, i.e. it holds

$$\mathbf{1}_{D_0} + \mathbf{1}_{D_A} + \mathbf{1}_{D_B} = 1 \quad \mathbf{Q}\text{-almost-surely.}$$

In the following, let us consider a transaction consisting of cash flows $C(B, A, T_i)$ paid by the counterparty B at times $T_i, i = 1, \ldots, m_B$, and cash flows $C(A, B, T_j)$ paid by the client A at times $T_j, j = 1, \ldots, m_A$. Taking into account default risk of both counterparties, the quantification of the bilateral CVA is summarized in the following well-known theorem, which in essence goes back to Sorensen and Bollier [19].

Theorem 1 *Conditional on the event* $\{t < \min(\tau_A, \tau_B)\}$, *i.e. no default has occurred until time t, the value* $V_A^D(t, T)$ *of the transaction under consideration of bilateral counterparty risk at time t is given by*

$$V_A^D(t, T) = V_A(t, T) - CVA_A(t, T) = -\big(V_B(t, T) - CVA_B(t, T)\big) = -V_B^D(t, T)$$

where the risk-free present value of the transaction is given as

$$\begin{aligned} V_A(t, T) &= \mathbb{E}\left[\sum_{i=1}^{m_B} \frac{N_t}{N_{T_i}} \cdot C(B, A, T_i) \middle| \mathscr{F}_t\right] - \mathbb{E}\left[\sum_{j=1}^{m_A} \frac{N_t}{N_{T_j}} \cdot C(A, B, T_j) \middle| \mathscr{F}_t\right] \\ &= -V_B(t, T) \end{aligned}$$

and where the bilateral counterparty value adjustment $CVA_A(t, T)$ *is defined as*

$$\begin{aligned} CVA_A(t, T) :=& \mathbb{E}\left[\mathbf{1}_{D_B} \cdot \frac{N_t}{N_{\tau_B}} \cdot L^B_{\tau_B} \cdot \max(0, V_A(\tau_B, T)) \,|\, \mathscr{G}_t\right] \\ &- \mathbb{E}\left[\mathbf{1}_{D_A} \cdot \frac{N_t}{N_{\tau_A}} \cdot L^A_{\tau_A} \cdot \max(0, V_B(\tau_A, T)) \,|\, \mathscr{G}_t\right] \\ =& - CVA_B(t, T). \end{aligned} \tag{1}$$

Here L_t^i denotes the random loss (between 0 and 1) of counterparty i at time t.

Proof A proof of Theorem 1 can be found in Bielecki and Rutkowski [2], Formula (14.25) or Brigo and Capponi [4], Proposition 2.1 and Appendix A, respectively.

Based on Theorem 1, the general approach for the calculation of the counterparty risk adjusted value $V_A^D(t, T)$ is to determine first the risk-free value $V_A(t, T)$ of the transaction. This can be done by any common valuation method for this kind of transaction. In a second step the counterparty value adjustment $CVA_A(t, T)$ needs to be determined. So far, two main approaches have emerged in the academic literature, which will be briefly reviewed in Sect. 4.

[2] We note that Brigo et al. (in [4, 6]) use different sets to order the default times, which are in essence reducible to the above three events.

3 The Main Building Blocks of CVA

Subsequently, let us assume that the default times τ_i with $i \in \{A, B\}$ can only take a finite number of values $\{\bar{t}_1, \dots, \bar{t}_K\}$ in the interval $]0, T[$. For continuous time models this assumption can be justified by the *default bucketing* approach, which can, for example, be found in Brigo and Chourdakis [5], if K is chosen sufficiently large. To be able to separate the default dynamics from the market value dynamics, let us introduce the auxiliary time s, $s \in [t, T]$ and the discounted market value

$$\widetilde{V}_i^+(t, s, T) := \frac{N_t}{N_s} \cdot \max(0, V_i(s, T)).$$

Then we can rewrite Eq. (1) as:

$$\begin{aligned} CVA_A(t,T) = \mathbb{E}\left[\sum_{k=1}^{K} L^B_{\bar{t}_k} \cdot \mathbf{1}_{D_B} \cdot \mathbf{1}_{\bar{t}_k}(\tau_B) \cdot \widetilde{V}_A^+(t, \bar{t}_k, T) \mid \mathcal{G}_t\right] \quad (2) \\ - \mathbb{E}\left[\sum_{k=1}^{K} L^A_{\bar{t}_k} \cdot \mathbf{1}_{D_A} \cdot \mathbf{1}_{\bar{t}_k}(\tau_A) \cdot \widetilde{V}_B^+(t, \bar{t}_k, T) \mid \mathcal{G}_t\right]. \end{aligned}$$

Here, $\mathbf{1}_M$ is the indicator function of the set M; if $M = \{m\}$ we simply write $\mathbf{1}_m$ instead. Now, collecting all terms relating to the default in the default indicator process δ,

$$\delta_k^i := \mathbf{1}_{D_i} \cdot \mathbf{1}_{\bar{t}_k}(\tau_i),$$

we can rewrite the BCVA in a more compact manner as

$$\begin{aligned} CVA_A(t,T) = \mathbb{E}\left[\sum_{k=1}^{K} L^B_{\bar{t}_k} \cdot \delta_k^B \cdot \widetilde{V}_A^+(t, \bar{t}_k, T) \mid \mathcal{G}_t\right] \quad (3) \\ - \mathbb{E}\left[\sum_{k=1}^{K} L^A_{\bar{t}_k} \cdot \delta_k^A \cdot \widetilde{V}_B^+(t, \bar{t}_k, T) \mid \mathcal{G}_t\right]. \end{aligned}$$

From Eq. (3) we immediately see that the BCVA at time t is composed of six discrete time[3] processes:

- two *default indicator processes* δ_s^A and δ_s^B,
- two *loss processes* L_s^A and L_s^B, and
- two *discounted exposure processes* $\widetilde{V}_A^+(t, s, T)$ and $\widetilde{V}_B^+(t, s, T)$.

In this way, we are able to separate the default dynamics δ from the loss process L and the exposure process $\widetilde{V}$. From this decomposition, it becomes obvious that the BCVA is completely determined by the joint distribution of these six processes.

[3] In the following, we replace the time index $\bar{t}_k$ with k for notational convenience.

Remark 1 We note that in general it is even sufficient to model four processes (loss dynamics and market value dynamics) plus a two-dimensional random variable (τ_A, τ_B). However, in the case of finitely many default times, it is more convenient to work with the default indicator process instead.

Remark 2 For simplicity of the subsequent exposition, we assume that the loss process is actually constant and equals 1: $L_t^i = l^i = 1$. The theory of the remainder of this exposition is not affected by this simplifying assumption, with one notable exception: the resulting two-dimensional transportation problems will become a multi-dimensional transportation problem which renders its numerical solution more complex, but still feasible.

Remark 3 As we have noted, the default indicator process can only take a finite number of values in the bucketing approach. More exactly, it holds that the joint (i.e. two-dimensional) default indicator process $\delta = (\delta_k)_{k=1,\ldots,K} \in \mathbb{R}^{2\times K}$, defined by

$$\delta_k := \begin{pmatrix} \delta_k^A \\ \delta_k^B \end{pmatrix}, \quad k = 1, \ldots, K,$$

takes only values in the finite set

$$\mathscr{Y} := \left\{ \gamma \in \mathbb{R}^{2\times K} \mid \gamma_{i,k} \in \{0, 1\}, \sum_{i,k} \gamma_{i,k} \leq 1 \right\}$$

which has exactly $2K + 1$ elements. Therefore, the discrete time default indicator process is also a process with a finite state space.

Let us further introduce the joint exposure process in analogy to the above,

$$X_k := \begin{pmatrix} \widetilde{V}_A^+(t, \bar{t}_k, T) \\ \widetilde{V}_B^+(t, \bar{t}_k, T) \end{pmatrix}, \quad k = 1, \ldots, K.$$

Then it holds

$$CVA_A(t, T) = \sum_{k=1}^{K} \left(\mathbb{E}\left[\delta_k^B \cdot X_k^A \mid \mathscr{G}_t \right] - \mathbb{E}\left[\delta_k^A \cdot X_k^B \mid \mathscr{G}_t \right] \right). \tag{4}$$

To avoid technical considerations for brevity of presentation, we prefer to work with discrete processes (i.e. discrete state space) in discrete time. Thus, it may be necessary to discretize the state space of the remaining discounted exposure process. In general, there exist (at least) two different approaches how a suitable discrete state space version of the process X could be obtained:

- In the first approach—completely similar to the default bucketing approach—the state space $\mathbb{R}^{2\times K}$ for the joint exposure process X is divided into N disjoint

components. Then X is replaced by some representative value on this component (usually an average value) on each of the components, and the probabilities of the discretized process are set in accordance with the original probabilities of each component (cf. the default bucketing approach).
- From a computational and practical point of view, a much more convenient approach relies on Monte Carlo simulation: N different scenarios (i.e. realizations) of the process X are used instead of the original process. Each realization is assumed to have probability $1/N$.

For both approaches it is known that they converge at least[4] in distribution to the original process, which is sufficient for our purposes. For more details on the convergence, let us refer to the recent working paper by Glasserman and Yang [10].

4 Models for Counterparty Risk

In the last decade two main approaches have emerged in the literature how to model the individual, resp. joint distribution of the processes δ and X:

- The most popular approach is based on the rather strong assumption of independence between exposure and default. Based on this independence assumption, only individual models for δ and X need to be specified for the CVA calculation. This kind of independence assumption is quite standard in the market, see for example the Bloomberg CVA function (for more details on the Bloomberg model let us refer to Stein and Lee [20]).
- Alternatively, and more recently, a more general approach is based on a joint model (also called *hybrid model*) for the building blocks δ and X of the CVA calculation, see Sect. 4.3.

4.1 Independence of CVA Components

Let us assume that the exposure process X is independent of the default process δ. Then the expectation inside the summation can be split into two parts:

$$\sum_{k=1}^{K} \mathbb{E}\left[\,\delta_k^B \cdot X_k^A \mid \mathscr{G}_t\right] = \sum_{k=1}^{K} \mathbb{E}\left[\,\delta_k^B \mid \mathscr{G}_t\right] \cdot \mathbb{E}\left[X_k^A \mid \mathscr{G}_t\right]. \tag{5}$$

[4]The Monte Carlo approach converges in distribution due to the Theorem of Glivenko–Cantelli. For state space discretization, if for example conditional expectations are used on each bucket, then convergence is in fact almost surely and in L^1 due Lévy's 0–1 law.

It is well known that the expected value

$$\mathbb{E}\left[X_k^A \,|\, \mathscr{G}_t\right] = \mathbb{E}\left[\widetilde{V}_A^+(t, \bar{t}_k, T) \middle| \mathscr{G}_t\right] = \mathbb{E}\left[\frac{N_t}{N_{\bar{t}_k}} \cdot \max(V_A(t, \bar{t}_k, T), 0) \middle| \mathscr{F}_t\right] \tag{6}$$

matches exactly the price of a call option on the basis transaction at time t with strike 0 and exercise time $\bar{t}_k$. The CVA equation can hence be rewritten as

$$CVA_A(t, T) = \sum_{k=1}^{K} \left(\mathbb{E}\left[\delta_k^B \,|\, \mathscr{G}_t\right] \cdot \mathbb{E}\left[X_k^A \,|\, \mathscr{G}_t\right] - \mathbb{E}\left[\delta_k^A \,|\, \mathscr{G}_t\right] \cdot \mathbb{E}\left[X_k^B \,|\, \mathscr{G}_t\right]\right), \tag{7}$$

and thus the BCVA can be calculated without any further problems as the corresponding default probabilities[5] $\mathbb{E}\left[\delta_k^B \,|\, \mathscr{G}_t\right] = \mathbf{Q}\left[\tau_B \in \Delta_k, \tau_B \leq \tau_A \,|\, \mathscr{G}_t\right]$ can be easily computed from any given credit risk model: in order to calculate the probability $\mathbf{Q}\left[\tau_B \in \Delta_k, \tau_B \leq \tau_A \,|\, \mathscr{G}_t\right]$, the default times τ_A and τ_B together with their dependence structure have to be modeled. One of the most popular models for default times in general are intensity models, as for example described in Bielecki and Rutkowsi [2], Part III.

Remark 4 It has to be noted that a model with deterministic default intensities plus a suitable copula is sufficient for the arbitrary specification of the joint distribution of default times. Stochastic intensities do not add any value in this context. This is true as long as the default risk-free discounted present value is independent of the credit quality of each counterpart. This means that the payoff itself is not allowed to be linked explicitly to the credit quality of any counterparty.

Remark 5 Let us point out that the intensity model is just one specific example how default times could be modeled. The big advantage of our approach is that any arbitrary credit risk model can be used instead, as only the distribution of the default indicator δ finally matters. In case only marginal default models are available, we can also take into account the remaining unknown dependence between the default times, however, at the price of a higher dimensional transportation problem.

4.2 Modeling Options on the Basis Transaction

Since it could be observed in Eq. (6) that options on the basis transaction need to be priced, a suitable model for this option pricing task needs to be available. Depending on the type of derivative, any model which can be reasonably well calibrated to the market data is sufficient. For instance, for interest rate derivatives, any model ranging from a simple Vasicek or CIR model to sophisticated Libor market models or two-factor Hull–White models could be applied. In case of a credit default swap,

[5] With $\Delta_k :=]\bar{t}_{k-1}, \bar{t}_k]$ if the default bucketing approach has been used, otherwise $\Delta_k := \{\bar{t}_k\}$.

any model which allows to price CDS options, i.e. any model with stochastic credit spread would be feasible. However, for CVA calculations, usually a trade-off between accuracy of the model and efficiency of calculations needs to be made. For this reason, usually simpler models are applied for CVA calculations than for other pricing applications. It needs to be noted that since the financial market usually provides sufficiently many prices of liquid derivatives, any reasonable model can be calibrated to these market prices, and therefore, we can assume in the following that the market implied distribution of the discounted exposure process is fully known and available.

4.3 Hybrid Models—An Example

Another way to calculate the CVA is to use a so-called *hybrid approach* which models all the involved underlying risk factors. Instances of such models can for example be found in Brigo and Capponi [4] for the case of a credit default swap, or Brigo et al. [6] for interest rate derivatives. In Brigo et al. [6], an integrated framework is introduced, where a two-factor Gaussian interest-rate model is set up for a variety of interest rate derivatives[6] in order to deal with the option inherent in the CVA. Further, to model the possible default of the client and its counterparty their stochastic default intensities are given as CIR processes with exponentially distributed positive jumps. The Brownian motions driving those risk factors are assumed to be correlated. Additionally, the defaults of the client and the counterparty are linked by a Gaussian copula.

In summary, the amount of wrong-way risk which can be modeled within such a framework strongly depends on the model choice. If solely correlations between default intensities (i.e. credit spreads) and interest rates are taken into account, only a rather weak relation will emerge between default and the exposure of interest rate derivatives, cf. Brigo et al. [6]. Figure 5 in Scherer and Schulz [18] provides an overview of potential CVA values for different models which illustrates that models can differ quite significantly.

5 Tight Bounds on CVA

From the previous section it becomes obvious that hybrid models yield different CVAs depending on the (model and parameter implied) degree of dependence between default and exposure. However, it remains unclear how large the impact of this dependence can be. In other words: *Is it possible to quantify, how small or large the CVA can get for any model, given that the marginal distributions for expo-*

[6]Although this modeling approach is a rather general one, it has to be noted that it links the dependence on tenors of swaption volatilities to the form of the initial yield curve. Therefore, the limits of such an approach became apparent as the yield curve steepened in conjunction with a movement of the volatility surface in the aftermath of the beginning of financial crisis in 2008, when these effects could not be reproduced by such a model.

sure and default are already given? In the following, we want to address this question based on our initially given decomposition of the CVA in building blocks.

As mentioned in Sect. 4.2, we can reasonably assume that the distribution of the exposure process X is already completely determined by the available market information. In a similar manner, we have argued that also the distribution of the default indicator process δ can be assumed to be given by the market. Nevertheless, let us point out that the following ideas and concepts could indeed be generalized to the case that only the marginal distributions of the default times are known. Further, we can even consider the case that the dependence structure between different market risk factors is not known but remains uncertain. However, all these generalizations come at the price that the resulting two-dimensional transportation problem will become multi-dimensional.

For the above reasons, we argue that the following approach is indeed *semi-model-free* in the sense that no model needs to be specified which links the default indicator process with the discounted exposure processes.

5.1 Tight Bounds on CVA by Mass Transportation

Let us reconsider Eq. (4) and let us highlight the dependence of the BCVA on the measure $\mathbf{P}$.

$$CVA_A^{\mathbf{P}}(t,T) = \sum_{k=1}^{K} \left(\mathbb{E}_{\mathbf{P}} \left[\delta_k^B \cdot X_k^A \,|\, \mathscr{G}_t \right] - \mathbb{E}_{\mathbf{P}} \left[\delta_k^A \cdot X_k^B \,|\, \mathscr{G}_t \right] \right).$$

With some abuse of notation, the measure $\mathbf{P}$ denotes the joint distribution of the default process δ and the exposure process X. Since both processes have finite support, $\mathbf{P}$ can be represented as a $(2K+1) \times N$ matrix with entries in $[0, 1]$. We note that the marginals of $\mathbf{P}$, i.e. the distributions of δ and X (denoted by the probability vectors $\mathbf{p}^{(X)} \in \mathbb{R}^N$ and $\mathbf{p}^{(\delta)} \in \mathbb{R}^{2K+1}$) are already predetermined from the market. Therefore, $\mathbf{P}$ has to satisfy

$$\mathbf{1}^\top \mathbf{P} = \mathbf{p}^{(X)}, \quad \text{and} \quad \mathbf{P}\mathbf{1} = \mathbf{p}^{(\delta)}.$$

Remark 6 In case of independence between δ and X, $\mathbf{P}$ is given by the product distribution of δ and X, whereas in hybrid models the joint distribution $\mathbf{P}$ is determined by the specification and parametrization of the hybrid model. In the independent case, $\mathbf{P}$ is hence given by the dyadic product

$$\mathbf{P} = \mathbf{p}^{(\delta)} \mathbf{p}^{(X)\top}.$$

Obviously, the smallest and largest CVA which can be obtained by any $\mathbf{P}$ which is consistent with the given marginals, is given by

$$CVA_A^l(t,T) := \min_{\mathbf{P}\in\mathscr{P}} \; CVA_A^{\mathbf{P}}(t,T),$$
$$CVA_A^u(t,T) := \max_{\mathbf{P}\in\mathscr{P}} \; CVA_A^{\mathbf{P}}(t,T),$$

where

$$\mathscr{P} := \{\mathbf{P} \in [0,1]^{(2K+1)\times N} \mid \mathbf{1}^\top\mathbf{P} = \mathbf{p}^{(X)},\ \mathbf{P1} = \mathbf{p}^{(\delta)}\}.$$

It can be easily noted that the set $\mathscr{P}$ is a convex polytope. Thus, the computation of $CVA_A^l(t,T)$ and $CVA_A^u(t,T)$ essentially requires the solution of a linear program, as the objective functions are linear in $\mathbf{P}$.

Remark 7 The structure of the above LPs coincides with the structure of so-called *balanced linear transportation problems*. Transportation problems constitute a very important subclass of linear programming problems, see for example Bazaraa et al. [1], Chap. 10, for more details. There exist several very efficient algorithms for the numerical solution of such transportation problems, see also Bazaraa et al. [1], Chaps. 10, 11 and 12.

Let us summarize our results in the following theorem:

Theorem 2 *Under the given prerequisites, it holds:*

1. $CVA_A^l(t,T) \leq CVA_A(t,T) \leq CVA_A^u(t,T)$.
2. *These bounds are tight, i.e. they represent the lowest and the highest CVA which can be obtained by any (hybrid) model which is consistent with the market data and there exists at least one model which reaches these bounds.*

The tightness of our bounds is in contrast to Turnbull [21], where only weak bounds were derived. Of course, bounds always represent a best-case and a worst-case estimate only, which may strongly under- and overestimate the true CVA.

Remark 8 We note that a related approach of coupling default and exposure via copulas was presented by Rosen and Saunders [17] and Crepedes et al. [8]. However, their approach differs from ours in some significant aspects. First, exposure scenarios are sorted by a single number (e.g. effective exposure) to be able to couple exposure scenarios with risk factors of defaults by copulas. Second, risk factors of some credit risk model are employed instead of working with the default indicator directly. Third, their approach is restricted to the real-world setting and does not consider restrictions on the marginal distributions in the coupling process, which is e.g. necessary if stochastic credit spreads should be considered.

5.2 An Alternative Formulation as Assignment Problem

For the above setup we have assumed that the probabilities for all possible realizations of the default indicator process could be precomputed from a suitable default model. If for some default model this should not be the case, but only scenarios (with repeated outcomes for the default indicator) could be obtained by a simulation, an alternative LP formulation could be obtained. In such a scenario setting, it is advisable that for both Monte Carlo simulations, the same number N of scenarios is chosen. Then for both given marginal distributions we have $\mathbf{p}_j^{(\delta)} = \mathbf{p}_i^{(X)} = 1/N$. If we apply the same arguments as above we obtain again a transportation problem, however, with probabilites $1/N$ each. If we have a closer look at this problem, we see that the optimization actually runs over all $N \times N$ permutation matrices—since each default scenario is mapped onto exactly one exposure scenario. This means that this problem eventually belongs to the class of assignment problems, for which very efficient algorithms are available, cf. Bazaraa et al. [1]. Nevertheless, please note that although assignment problems can be solved more efficiently than transportation problems, it is still advisable to solve the transportation problem due to its lower dimensionality, as usually $2K + 1 \ll N$ (i.e. time discretization is usually much coarser than exposure discretization). However, if stochastic credit spreads have to be considered, they have to be part of the default simulation and thus assignment problems (with additional linear constraints to guarantee consistency of exposure paths and spreads) become unavoidable.

6 Example

6.1 Setup

To illustrate these semi-model-free CVA bounds let us give a brief example. For this purpose let us consider a standard payer swap with a remaining lifetime of $T = 4$ years analyzed within a Cox–Ingersoll–Ross (CIR) model at time $t = 0$. The time interval $]0, 4[$ is split up into $K = 8$ disjoint time intervals each covering half a year. For simplicity, the loss process is again assumed to be 1.

6.1.1 Counterparty's Default Modeling

To model the defaults we have chosen the well-known copula approach with constant intensities using the Gaussian copula. For further analyses in this example we will focus on the case of uncorrelated counterparties ($\rho = 0$) and highly correlated counterparties ($\rho = 0.9$). Furthermore, the counterpartys' default intensities are assumed to be deterministic. We will distinguish between symmetric counterparties with identical default intensities and asymmetric counterparties. Thus, four different settings

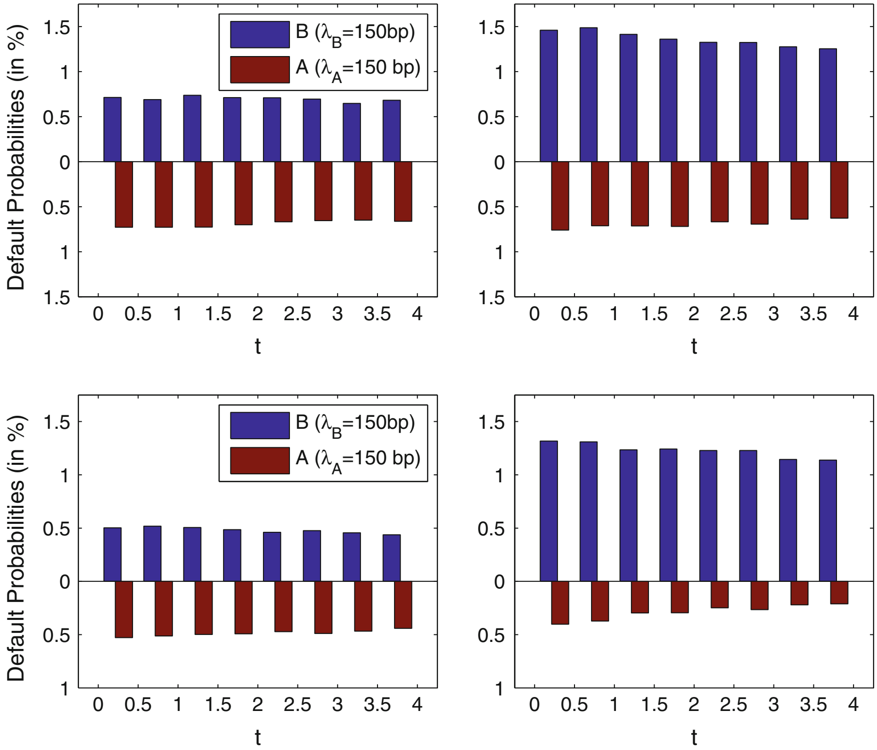

Fig. 1 Probabilities $\mathbb{E}_{\mathbf{Q}}[\,\delta_k^i]$ in % for Case 1 to Case 4

result: Fig. 1 shows the probabilities $\mathbf{Q}[\,\delta_k^i = 1] = \mathbb{E}_{\mathbf{Q}}[\,\delta_k^i]$ in each of the four cases under the risk-neutral measure $\mathbf{Q}$ implied from the market. To be in line with the following figures, the probabilities for a default of counterparty B in Δ_k, i.e. $\mathbb{E}_{\mathbf{Q}}[\,\delta_k^B]$, correspond to the positive bars and defaults of counterparty A to the negative bars. The left plots show identical counterparties (cases 1 and 2) and the right ones the cases, where counterparty B has a higher default intensity (cases 3 and 4). Furthermore, the upper plots correspond to uncorrelated defaults and for the ones below we have $\rho = 0.9$.

Case 1: symmetric, uncorrelated	$\lambda_A = 150$ bps	$\lambda_B = 150$ bps	$\rho = 0$
Case 2: symmetric, correlated	$\lambda_A = 150$ bps	$\lambda_B = 150$ bps	$\rho = 0.9$
Case 3: asymmetric, uncorrelated	$\lambda_A = 150$ bps	$\lambda_B = 300$ bps	$\rho = 0$
Case 4: asymmetric, correlated	$\lambda_A = 150$ bps	$\lambda_B = 300$ bps	$\rho = 0.9$

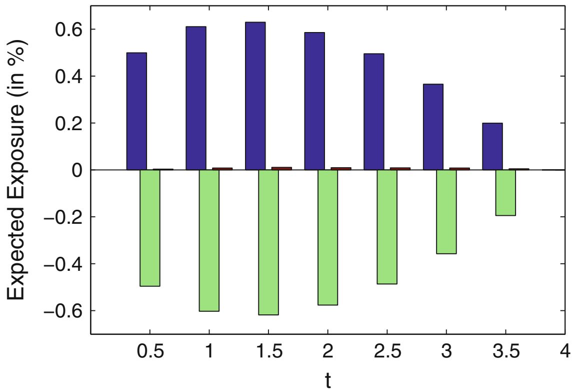

Fig. 2 Expected exposures $\mathbb{E}_{\mathbf{Q}}\left[X_k^A\right]$, $\mathbb{E}_{\mathbf{Q}}\left[X_k^B\right]$ and $\mathbb{E}_{\mathbf{Q}}[\tilde{V}_A(t_k, T)]$

Table 1 $\mathbb{E}_{\mathbf{Q}}\left[X_k^A\right]$ and $\mathbb{E}_{\mathbf{Q}}\left[X_k^B\right]$ in basis points

k	1	2	3	4	5	6	7	8
$\mathbb{E}_{\mathbf{Q}}[X_k^A]$ in bp	49.2	59.2	60.1	55.2	45.9	33.4	17.9	0
$\mathbb{E}_{\mathbf{Q}}[X_k^B]$ in bp	48.9	58.5	59.1	54.2	45.1	32.6	17.5	0

6.1.2 Counterparty Exposure Modeling

As already mentioned, a simple CIR model is applied for the valuation of the payer swap. Since our focus is on the coupling of the default and the exposure model, we have opted for such a simple model for ease of presentation. In the CIR model, the short rate r_t follows the stochastic differential equation

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

where $(W_t)_{t\geq 0}$ denotes a standard Brownian motion. Instead of calibrating the parameters to market data (yield curve plus selected swaption prices) on one specific day, we have set the parameters in the following way

$$\kappa = 0.0156, \quad \theta = 0.0311, \quad \sigma = 0.0313, \quad r_0 = 0.030$$

to obtain an interest rate market which is typical for the last years. Considering now the discounted exposure of each counterparty within the discrete time framework of our example, we can easily compute $\mathbb{E}_{\mathbf{Q}}\left[X_k^i\right]$ as the average of all generated scenarios from a Monte Carlo simulation. Figure 2 illustrates the results of a simulation, which are also given in Table 1. Positive bars correspond to $\mathbb{E}_{\mathbf{Q}}\left[X_k^A\right]$, negative bars to $\mathbb{E}_{\mathbf{Q}}\left[X_k^B\right]$, and the small bars correspond to $\mathbb{E}_{\mathbf{Q}}[\tilde{V}_A(t_k, T)]$. Since payer and receiver swap are not completely symmetric instruments, there remains a residual expectation, as can be observed from Fig. 2.

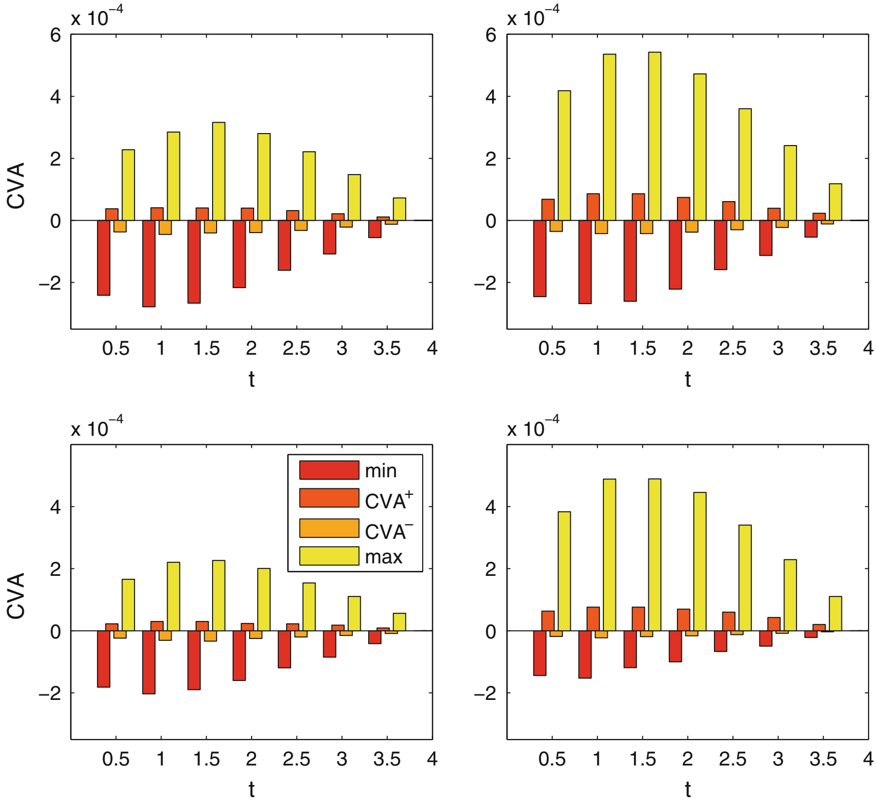

Fig. 3 Minimal and maximal CVA_A, $\mathbb{E}_{\mathbf{Q}_i}\left[\delta_k^B \cdot X_k^A \,|\, \mathcal{G}_t\right]$ and $-\mathbb{E}_{\mathbf{Q}_i}\left[\delta_k^A \cdot X_k^B \,|\, \mathcal{G}_t\right]$ in bps

6.2 Results

In case of independence between default and exposure, the bilateral CVA is easily obtained by multiplying the default probabilities (as shown in Fig. 1) with the corresponding exposures (as shown in Fig. 2) and summation. Besides the independent CVA^i, the minimal and maximal CVA^l and CVA^u have been calculated as well.

The results of these calculations are illustrated in Fig. 3 and Table 2 for each time interval Δ_k. Analogously to Fig. 1 we have for each of the four cases a separate subplot and the left plots belong again to cases 1 and 2. The positive bars now correspond to $\mathbb{E}_{\mathbf{Q}_i}\left[\delta_k^B \cdot X_k^A\right]$ and the negative ones to $\mathbb{E}_{\mathbf{Q}_i}\left[\delta_k^A \cdot X_k^B\right]$. In the case of the minimal *CVA*, $\mathbb{E}_{\mathbf{Q}_l}\left[\delta_k^B \cdot X_k^A\right]$ vanishes, meaning that for counterparty *A* in case of a default of counterparty *B* the exposure is zero, as the present value of the swap at that time is negative from counterparty *A*'s point of view. Contrarily, for the maximal *CVA*, $\mathbb{E}_{\mathbf{Q}_u}\left[\delta_k^A \cdot X_k^B\right]$ is zero. Here, $\mathbf{Q}_u$, $\mathbf{Q}_l$, and $\mathbf{Q}_i$ denote the optimal measures for the maximal, the minimal, and the independent CVA, respectively. As expected there

Table 2 Minimal and maximal CVA_A, $\mathbb{E}_{\mathbf{Q}_i}\left[\,\delta_k^B \cdot X_k^A \mid \mathscr{G}_t\right]$ and $-\mathbb{E}_{\mathbf{Q}_i}\left[\,\delta_k^A \cdot X_k^B \mid \mathscr{G}_t\right]$ in bps

	k	1	2	3	4	5	6	7	8	$\sum$
Case 1	min	−2.41	−2.78	−2.67	−2.16	−1.61	−1.08	−0.55	0.00	−13.26
	$\mathbb{E}_{\mathbf{Q}_i}\left[\,\delta_k^B \cdot X_k^A \mid \mathscr{G}_t\right]$	0.37	0.41	0.41	0.40	0.32	0.22	0.12	0.00	2.24
	$\mathbb{E}_{\mathbf{Q}_i}\left[\,\delta_k^A \cdot X_k^B \mid \mathscr{G}_t\right]$	−0.37	−0.45	−0.40	−0.39	−0.32	−0.21	−0.12	0.00	−2.27
	max	2.28	2.85	3.16	2.80	2.21	1.48	0.73	0.00	15.51
Case 2	min	−1.82	−2.03	−1.90	−1.60	−1.19	−0.85	−0.42	0.00	−9.80
	$\mathbb{E}_{\mathbf{Q}_i}\left[\,\delta_k^B \cdot X_k^A \mid \mathscr{G}_t\right]$	0.23	0.30	0.30	0.24	0.22	0.18	0.09	0.00	1.56
	$\mathbb{E}_{\mathbf{Q}_i}\left[\,\delta_k^A \cdot X_k^B \mid \mathscr{G}_t\right]$	−0.24	−0.31	−0.34	−0.25	−0.20	−0.15	−0.09	0.00	−1.57
	max	1.66	2.21	2.26	2.01	1.54	1.10	0.56	0.00	11.34
Case 3	min	−2.45	−2.68	−2.60	−2.21	−1.58	−1.13	−0.54	0.00	−13.19
	$\mathbb{E}_{\mathbf{Q}_i}\left[\,\delta_k^B \cdot X_k^A \mid \mathscr{G}_t\right]$	0.68	0.86	0.86	0.74	0.61	0.39	0.23	0.00	4.38
	$\mathbb{E}_{\mathbf{Q}_i}\left[\,\delta_k^A \cdot X_k^B \mid \mathscr{G}_t\right]$	−0.35	−0.43	−0.42	−0.37	−0.30	−0.22	−0.11	0.00	−2.21
	max	4.18	5.36	5.42	4.72	3.60	2.41	1.18	0.00	26.88
Case 4	min	−1.44	−1.53	−1.19	−1	−0.67	−0.49	−0.21	0.00	−6.53
	$\mathbb{E}_{\mathbf{Q}_i}\left[\,\delta_k^B \cdot X_k^A \mid \mathscr{G}_t\right]$	0.63	0.76	0.76	0.69	0.60	0.43	0.20	0.00	4.08
	$\mathbb{E}_{\mathbf{Q}_i}\left[\,\delta_k^A \cdot X_k^B \mid \mathscr{G}_t\right]$	−0.18	−0.23	−0.19	−0.16	−0.12	−0.08	−0.03	0.00	−1.00
	max	3.83	4.88	4.89	4.46	3.40	2.29	1.11	0.00	24.87

Table 3 Computation times for the two-dimensional transportation problem

K	10	20	20	20
N	1024	1024	2048	4096
Time in seconds	0.2	0.5	1.5	6

are large gaps between the lower and the independent CVA, as well as between the independent CVA and the upper bound. This means that wrong-way risk (i.e. higher exposure comes with higher default rates) can have a significant impact on the bilateral CVA. Interestingly, this observation holds true for all four cases, of course, with different significance depending on the specific setup. Although it is clear that our analysis naturally shows more extreme gaps than any hybrid model, it has to be mentioned that these bounds are indeed tight.

6.3 Computation Time, Choice of Algorithm, and Impact of Assumptions

Theoretically, the computation of the bounds boils down to the solution of a linear programming problem. From this it can be expected that state-of-the-art solvers like CPLEX or Gurobi will yield the optimal solution within reasonable computation time. Using CPLEX, we have obtained the following computation times on a standard workstation (Table 3).

It can be observed that the problem can be solved for reasonable discretization levels within decent time. Rather similar computation times have been obtained with an individual implementation of the standard network simplex based on Fibonacci heaps. However, for larger sizes, the performance of standard solvers begins to deteriorate. To dampen the explosion of computation time, we have resorted to a special purpose solver for min cost network flows (which are a general case of the transportation problem) for highly asymmetric problems, as in our case $2K + 1 \ll N$. Based on Brenner's min cost flow algorithm, see Brenner [3], we could still solve problems with $K = 40$ and $N = 8192$ beneath a minute.

If one has to resort to the assignment formulation (to consider credit spreads accordingly), computation times increase due to the fact that now assignment problems have to be solved. Here, a factor 100 compared to the above computation times cannot be avoided.

If the coupling of the two default times is left flexible, the problem becomes a transportation problem with three margins, i.e. of size $K + 1 \times K + 1 \times N$. For these types of problems, no special purpose solver is available and one has to resort to CPLEX. Scherer and Schultz [18] have exploited the structure of this three-dimensional transportation problem to reduce computational complexity. They were able to reduce the problem to a standard two-dimensional transportation problem, hence rendering the computation of bounds similarly easy, no matter if default times are already coupled or not.

7 Conclusion and Outlook

In this paper we have shown how tight bounds on unilateral and bilateral counterparty valuation adjustment can be derived by a linear programming approach. This approach has the advantage that simulations of the uncertain loss, of the default times and of the uncertain value of a transaction during her remaining life can be completely separated. Although we have restricted the exposition to the case of two counterparties and one derivative transaction, the model can easily be extended to more counterparties and a whole netting node of trades. Further, as exposure is simulated separately from default, all risk-mitigating components like CSAs, rating triggers, and netting agreements can be easily included in a such a framework.

Interesting open questions for future research include the analogous treatment in continuous time, which requires much more technically involved arguments. Further, this approach yields a new motivation to consider efficient algorithms for transportation or assignment problems with more than two marginals, which did not yet get much attention so far.

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CVA with Wrong-Way Risk in the Presence of Early Exercise

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Abstract Hull–White approach of CVA with embedded WWR (Hull and White, Financ. Anal. J. 68:58-69, 2012, [11]) can be easily applied also to portfolios of derivatives with early termination features. The tree-based approach described in Baviera et al. (Int. J. Financ. Eng. 2015, [1]) allows to deal with American or Bermudan options in a straightforward way. Extensive numerical results highlight the non-trivial impact of early exercise on CVA.

Keywords American and Bermudan options · Wrong-way risk · Credit value adjustment

1 Introduction

As a direct consequence of the 2008 financial turmoil, counterparty credit risk has become substantial in OTC derivatives transactions. In particular, the credit value adjustment (CVA) is meant to measure the impact of counterparty riskiness on a derivative portfolio value as requested by the current Basel III regulatory framework. Accounting standards (IFRS 13, FAS 157), moreover, require a CVA[1] adjustment as part of a consistent fair value measurement of financial instruments.

CVA is strongly affected by derivative transaction arrangements: exposure depends on collateral and netting agreement between the two counterparties that have written

[1] Even if in this paper we focus on CVA pricing, it is worthwhile to note that accounting standards ask also for a debt value adjustment (DVA) to take into account the own credit risk.

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the derivative contracts of interest. Despite the increased use of collateral, however, a significant portion of OTC derivatives remains uncollateralized. This is mainly due to the nature of the counterparties involved, such as corporates and sovereigns, without the liquidity and operational capacity to adhere to daily collateral calls. In such cases, an institution must consider the impact of counterparty risk on the overall portfolio value and a correct CVA quantification acquires even more importance. Extensive literature has been produced on the topic in recent years, as for example [5] and [9] that give a comprehensive overview of CVA computation and the more general topic of counterparty credit risk management. It seems, however, that attention has been mainly paid to CVA with respect to portfolios of European-style derivatives. Dealing with derivatives with early exercise features is even more delicate. Indeed, as pointed out in [3], for American- and Bermudan-style derivatives CVA computation becomes path-dependent since we need to take into account the exercise strategy and the fact that exposure falls to zero after the exercise.

A peculiar problem that we encounter in CVA computation is the presence of the so-called wrong-way risk (WWR), that is the non-negligible dependency between the value of a derivatives portfolio and counterparty default probability. In particular we face WWR if a deterioration in counterparty creditworthiness is more likely when portfolio exposure increases. Several attempts have been made to deal with WWR. From a regulatory point of view, the Basel III Committee currently requires to correct by a multiplicative factor $\alpha = 1.4$ the CVA computed under hypothesis of market-credit independence. In this way the impact of WWR is considered equivalent to a 40 % increase in standard CVA. However, the Committee leaves room for financial institutions with approved models to apply for lower multipliers (floored at 1.2). This opportunity opens the way for more sophisticated models in order to reach a more efficient risk capital allocation.

Relevant contributions on alternative approaches to manage WWR include copula-based modeling as in [6], introduction of jumps at default as in [13], the backward stochastic differential equations framework developed in [7], and the stochastic hazard rate approach in [11]. In particular [11] introduces the idea to link the counterparty hazard rate to the portfolio value by means of an arbitrary monotone function. The dependence structure is, then, described uniquely by one parameter that controls the impact of exposures on the hazard rate. Additionally, a deterministic time-dependent function is introduced to match the counterparty credit term structure observed on the market. In this framework CVA pricing in the presence of WWR involves just a small adjustment to the pricing machinery already in place in financial institutions. We only need to take into account the randomness incorporated into the counterparty default probabilities by means of the stochastic hazard rate and price CVA with standard techniques. This is probably the most relevant property of the model: as soon as we associate a WWR parameter to a given counterparty–portfolio combination, we are able to deal with WWR using the same pricing engine underlying standard CVA computation. As pointed out in [14], leveraging as much as possible on existing platforms should be one of the principles an optimal risk model should be shaped on. However, the original approach in [11] relies on a Monte Carlo-based technique to determine the auxiliary deterministic function in order to calibrate the model on the counterparty credit structure. Obtaining this auxiliary function is the trickiest part in the calibration procedure, because it involves a "delicate" path-dependent problem that is difficult to implement for realistic portfolios. In [1], it is shown how it is possible to overcome such a limitation by transforming the path-dependent problem into a recursive one with a considerable reduction in the overall computational complexity. The basic idea is to consider discrete market factor dynamics and induce a change of probability such that the new set of (transition) probabilities are computed recursively in time. We presented a straightforward implementation of our approach via tree methods. Trees are also a straightforward and well understood tool to manage the early termination in derivatives pricing. So combining tree-based dynamic programing and the recursive algorithm in [1] leads to a simple and effective procedure to price CVA with WWR when American or Bermudan features are considered. The paper is organized as follows: in Sect. 2 we review the Hull–White model for CVA in the presence of WWR and the recursive approach in [1]. In Sect. 3 we analyze the effects of early termination on CVA adjustments via numerical tests and in Sect. 4 we study the relevant case of a long position on a Bermudan swaption. Finally Sect. 5 reports some final remarks.

2 CVA Pricing and WWR

For a given derivatives portfolio we can define the unilateral CVA[2] as the risk-neutral expectation of the discounted loss that can be suffered over a given period of time

$$CVA = (1 - R) \int_{t_0}^{T} B(t_0, t)\; EE(t)\; PD(dt), \tag{1}$$

where usually t_0 is the value date (hereinafter we set $t_0 = 0$ if not stated otherwise) and T is the longest maturity date in the portfolio. Here R is the recovery rate, $PD(dt)$ is the probability density of counterparty default between t and $t + dt$ (with no default before t), and $B(t_0, t)EE(t)$ is the discounted expected exposure in t. If interest rates are stochastic, the expected exposure is defined

$$B(t_0, t)\; EE(t) \equiv \mathbb{E}[D(t_0, t)\; E(t)],$$

with $\mathbb{E}[\cdot]$ the expectation operator given the information at value date t_0, $D(t_0, t)$ the stochastic discount, and $E(t)$ the (stochastic) exposure at time t. The latter is inherently defined by the collateral agreement that the parties have in place: for example in uncollateralized transactions, $E(t)$ is simply the max w.r.t. zero of $v(t)$, the portfolio value at time t. For practical computation, the integral in (1) is approximated

[2]The party that carries out the valuation is thus considered default-free. Even if it is a restrictive assumption, unilateral CVA is the only relevant quantity for regulatory and accounting purposes. For a detailed discussion on other forms of CVA, see e.g. [9].

by choosing a discretized set of times $\mathscr{T} = \{t_i\}_{i=0,\dots,n}$ with $t_n = T$. In particular, the Basel III standard approach for CVA valuation is

$$CVA = (1 - R) \sum_{i=1}^{n} \frac{B_i \; EE_i + B_{i-1} \; EE_{i-1}}{2} PD_i, \tag{2}$$

with B_i that stands for[3] $B(t_0, t_i)$ and

$$PD_i \equiv SP_{i-1} - SP_i,$$

where SP_i is the counterparty survival probability up to t_i. Assuming that the default is modeled by means of a generic intensity-based model, we can link survival probabilities to the so-called hazard rate function $h(t)$, (see e.g. [15]):

$$SP_i = \exp\left(-\int_{t_0}^{t_i} h(t)\, dt\right).$$

A common assumption is to consider $h(t)$ constant between two consecutive dates in the set $\mathscr{T}$. Pricing CVA with (2) holds if there is no "market-credit" dependency. However, in case of wrong-way risk (WWR) a new, more sophisticated, model is needed because exposure and counterparty default probabilities are no more independent: exposure is conditional to default and a positive "market-credit" dependence originates the WWR. Recently Hull and White [11] have proposed an approach to WWR that is financially intuitive: the conditional hazard rate is modeled as a stochastic quantity related to the portfolio value $v(t)$ through a monotonic increasing function. In the following we focus on the specific functional form

$$\tilde{h}(t) = \exp\Big(a(t) + b\, v(t)\Big), \tag{3}$$

where $b \in \Re^+$ is the WWR parameter. However, results still hold for an arbitrary order-preserving function. The function $a(t)$ is a deterministic function of time, chosen in such a way that on each date

$$SP_i = \mathbb{E}\left[\exp\left(-\int_{t_0}^{t_i} \tilde{h}(t)\, dt\right)\right] \qquad \forall i = 1, \dots, n. \tag{4}$$

Combining (3) and (4) we clearly see that function $a(t)$ depends also on the value specified for the parameter b.

The main advantage of this model is that once we know b and $a(t)$, WWR can be implemented easily by means of a simple generalization of (2):

[3]From now on we use the notation x_i to represent a discrete-time variable while $x(t)$ indicates its analogous variable in continuous-time. For avoidance of doubt, any other form of dependency $(\cdot)$ does not refer to the temporal one, unless stated otherwise.

$$CVA_W = (1-R)\sum_{i=1}^{n} \mathbb{E}\left[\frac{D_i\ E_i + D_{i-1}\ E_{i-1}}{2}\ \widetilde{PD}_i\right], \tag{5}$$

where $\widetilde{PD}_i$ is the stochastic probability to default between t_{i-1} and t_i defined in terms of $\tilde{h}_i$. We want to stress that expectation in (5) can be computed via any feasible numerical method: this fact implies that, given b and $a(t)$, taking into account WWR just requires a slight modification in the payoff of existing algorithms used for the calculation of CVA.

We now briefly recall the recursive approach presented in [1] that avoids the path dependency in the determination of $a(t)$ so that Eq. (4) is satisfied. Hereinafter we refer to the technique to get such a function as either the calibration of $a(t)$ or the "calibration problem": once the three sets of parameters (the recovery R, the default probabilities PDs, and the WWR parameter b) for dealer's clients are estimated (e.g. with statistical methods) it is the most complicated issue in the calibration of Hull–White model.

Let us assume that the market risk factors underlying the portfolio are discrete and we indicate with j_i the discrete state variable that describes the market at time t_i. In this framework market dynamics is described by a Markov chain with

$$q_i(j_{i-1}, j_i) \quad \forall i = 1, \ldots, n$$

the transition probability between j_{i-1} at time t_{i-1} and j_i at time t_i. Typical examples where such a discrete approach is natural are lattice models. In particular, in [1], we applied tree methods to the pricing of CVA for linear derivatives portfolios.

Embedding the Hull–White model (3) in our setting, the stochastic survival probability between t_{i-1} and t_i becomes

$$\tilde{P}_i(j) \equiv \exp\Big(-(t_i - t_{i-1})\ \tilde{h}_i(j)\Big) \equiv P_i\ \eta_i(j) \qquad \forall i = 1, \ldots, n, \tag{6}$$

where

$$P_i \equiv \frac{SP_i}{SP_{i-1}}$$

is the forward survival probability between t_{i-1} and t_i valued in t_0. For notational convenience, we also set $\tilde{P}_0(j_0) = \eta_0(j_0) = 1$. The η process introduced in (6) can be seen as the driver of the stochasticity in survival probabilities and it plays a key role in circumventing path-dependency in the calibration of $a(t)$, as shown in the following proposition.

Proposition

In the model with discrete market risk factors, the calibration problem (4) becomes

$$\sum_{j_i} p_i(j_i)\ \eta_i(j_i) = 1 \qquad \forall i = 1, \ldots, n, \tag{7}$$

where $p_i(j_i)$ are probabilities and they can be obtained via the recursive equation

$$p_i(j_i) = \sum_{j_{i-1}} q_i(j_{i-1}, j_i)\ \eta_{i-1}(j_{i-1})\ p_{i-1}(j_{i-1}) \qquad \forall i = 1, \ldots, n, \tag{8}$$

with the initial condition $p_0(j_0 = 0) = 1$.

Proof See [1].

Thus the calibration problem (4) can be solved at each discrete date t_i via (7) by simply exploiting the fact that the process η, non-path-dependent, is a martingale under the probability measure p. Equation (8), in addition, specifies an algorithm to build this new probability measure recursively. In this framework $\widetilde{PD}_i$ can be readily obtained from (6). Let us mention that, although this is just one of the viable approaches to solve (4), it turns out to be, as shown in the next section, a natural way to handle the additional complexity induced by early exercises within the Hull–White approach to WWR modeling.

3 The Impact of Early Exercise

As already anticipated in Sect. 1, CVA when early exercise is allowed gives rise to additional features. In this section we want to highlight the differences in CVA figures when both European and American options are considered, implementing the tree-based procedure described in the previous section. It is well known that backward induction and dynamic programing applied on (recombining) trees are, probably, the simplest and most intuitive tool to price derivatives with an early exercise as American options. For these options, indeed, Monte Carlo techniques turn out to be computationally intensive in case of CVA: the exercise date, after which the exposure falls to zero, depends on the path of the underlying asset and on the exercise strategy. In such a case we are asked to describe two random times: the optimal exercise time and the counterparty default time.

3.1 The Pricing Problem

Since our goal is to study the effects of early exercise clauses on CVA, we focus on the case of a dealer that enters into a long position[4] on American-style derivatives with a defaultable counterparty. That is, the dealer is the holder of the option and she has the opportunity to choose the optimal exercise strategy in order to maximize the option value. In particular, following [3], we would need to differentiate between two possible assumptions depending on the effects of counterparty defaultability on

[4] A short option position does not produce any potential CVA exposure.

the exercise strategy. The option holder would or would not take into account the possibility of counterparty default when she chooses whether to exercise or not. In the former case, the continuation value (the value of holding the option until the next exercise date) should be adjusted for the possibility of default. However, following the actual practice in CVA computation, we assume that counterparty defaultability plays no role in defining the exercise strategy of the dealer. This means that the pricing problem (before any CVA consideration) is the classical one for American options in a default-free world.

Let us assume to have a tree for the evolution of market risk factors[5] up to time T. Hereinafter, without loss of generality, we can set a constant time step Δt and denote the time partition on the tree by means of an index i in $\mathscr{T} = \{t_i\}_{i=0,\dots,n}$ with $t_i = i\,\Delta t$. We further introduce an arbitrary set of m exercise dates $\mathscr{E} = \{e_k\}_{k=1,\dots,m}$ with $\mathscr{E} \subseteq \mathscr{T}$ at which the holder can exercise her rights receiving a payoff ϕ_k that could depend on the specific exercise date e_k. In this setting we can deal indistinctly with European ($m = 1$), Bermudan ($m \in \mathbb{N}$), and American options ($m \to \infty$). The standard dynamic programing approach then allows us to compute the derivative value at each node of the tree:

$$v_i(j_i) = \begin{cases} \phi_m(j_i) & \text{for } i \text{ s.t. } t_i = e_m = T, \\ \max(c_i(j_i),\ \phi_k(j_i)) & \text{for } i \text{ s.t. } t_i \in \mathscr{E}\backslash\{e_m\}, \\ c_i(j_i) & \text{otherwise.} \end{cases} \tag{9}$$

with c_i the continuation value of the derivative defined as

$$c_i(j_i) = B(i, i+1; j_i) \sum_{j_{i+1}} q_i(j_i, j_{i+1})\, v_{i+1}(j_{i+1}), \tag{10}$$

where the sum must be considered over all possible t_{i+1}-nodes connected to j_i at time t_i and $B(i, i+1; j_i)$ is the future discount factor that applies from t_i and t_{i+1} possibly depending on the state variable j_i on the tree.

We describe in detail the simple 1-dimensional tree; however, extensions to the 2-factor case (as, for example, the G2++ model in [4] or the recent dual curve approach in [12]) are straightforward. Once the derivative value is computed for all nodes and the WWR parameter b is specified,[6] we can calibrate the auxiliary function $a(t)$ in (3) by means of the recursive approach in [1]. The advantages of such an approach are, in this case, twofold: we avoid path-dependency in the calibration of $a(t)$, as in any other possible application, and we deal with early exercises via (9) and (10) in a very intuitive way.

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[5]If we describe the dynamics of the price of a corporate stock, we assume—for the sake of simplicity—that such entity is not subject to default risk.

[6]We refer the interested reader to the original paper [11] for a heuristic approach to determine the parameter and to [14] for comprehensive numerical tests with market data.

3.2 The Plain Vanilla Case

We now want to assess the impact of early termination on CVA in order to understand the potential differences that could arise between European and American options from a counterparty credit risk management perspective.

In the first test we study the plain vanilla option case: we assume that the dealer buys a call option from a defaultable counterparty. Counterparty default probabilities are described in terms of a CDS flat curve at 125 basis points as in [11]. More precisely, with a flat CDS curve we can approximate quite well the survival probability between t_0 and t_i as

$$SP_i = \exp\left(-\frac{s_i\, t_i}{1-R}\right),$$

where s_i is the credit spread relative to maturity t_i and R the recovery rate, equal to 40 %. We further assume that trades are fully uncollateralized.[7] The underlying asset is lognormally distributed and represented by means of a Cox-Ross-Rubinstein binomial tree. We can thus apply the dynamic programing approach described above to price options on the tree and calibrate the function $a(t)$ recursively via (7). This procedure turns out to be quite fast: the Matlab coded algorithm takes less than 0.1 second to run on a 3.06 GHz desktop PC with 4 GB RAM when $n = m = 500$. Figure 1 shows CVA profile[8] for both European and American call options as function of WWR parameter b and for different levels of cost of carry. From standard non-arbitrage arguments, we indeed know that the optimality of early exercise for plain vanilla call options is related to the cost of carry (defined as the net cost of holding positions in the underlying asset).[9]

As shown in Fig. 1, CVA profiles are significantly different for European and American options when early exercise can represent the optimal strategy (black and dark gray lines). In particular the impact of WWR is significantly less pronounced for American options compared to the corresponding European ones. On the other hand, when early exercise is no more optimal, the two options are equivalent: light gray lines in Fig. 1 are undistinguishable from each other. In addition, the upward shift in CVA exposures is due to the fact that an increase in cost of carry (e.g. a reduction in the dividend yield) is reflected entirely in an augmented drift of the underlying asset dynamics that makes, *ceteris paribus*, the call option more valuable.

The effect of early exercise on exposure profiles is depicted in Fig. 2 where a possible underlying asset path is displayed along with the optimal exercise boundary

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[7] Here we are interested in analysing the full exposure profile as function of early exercise opportunities. On the other hand, more realistic collateralization schemes can be taken into account in a straightforward manner within the described framework.

[8] Once b and $a(t)$ are determined we can use whatever numerical technique to compute (5). Here we simply implement a simulation-based scheme that uses the tree as discretization grid. The number of generated paths is 10^5.

[9] The classical example is an option written on a dividend paying stock. This frame includes also a call option on a commodity whose forward curve is in backwardation or on a currency pair for which the interest rate of the base currency is higher than the one of the reference currency.

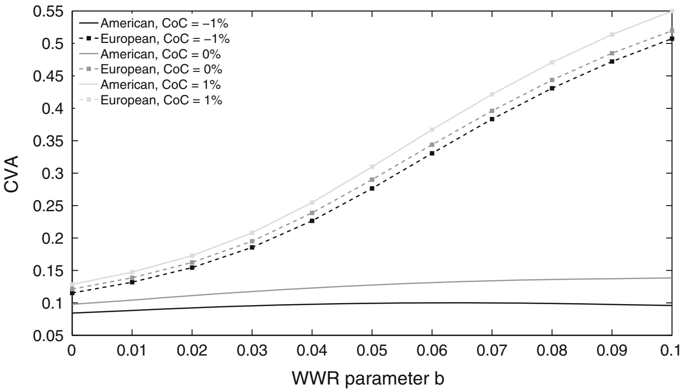


Fig. 1 CVA profiles for European and American options as function of WWR parameter b for several levels of cost of carry (CoC). Parameters are $S_0 = 100$, $K = 100$, $\sigma = 25\,\%$, $r = 1\,\%$, $T = 1$, $n = m = 500$. Counterparty CDS *curve flat* at 125 basis points

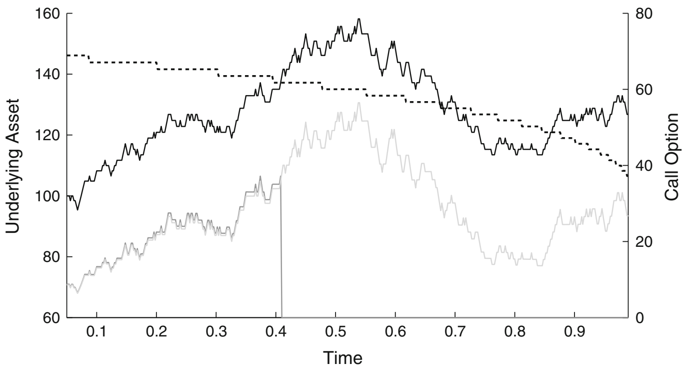


Fig. 2 The effect of early exercise on exposures. Parameters are $S_0 = 100$, $K = 100$, $\sigma = 25\,\%$, $r = 1\,\%$, CoC $= -2\,\%$, $T = 1$, $n = m = 500$. *Left hand scale* Asset path (*black solid line*) and optimal exercise boundary (*dashed line*). *Right hand scale* European option (*light gray line*) and American option (*dark gray line*)

(reconstructed on the binomial tree) and the corresponding value of European and American options. Until the asset value remains within the continuation region (the area below the dashed line), the two options have a similar value with the only difference given by the early exercise premium embedded in the American style derivative. However, if the asset value reaches or crosses the exercise boundary, the exposure due to the American option falls to zero while the European option remains alive until maturity. From the definition of CVA (1), we can see that early exercise, if optimal, reduces the exposure of the holder to the counterparty default by shortening the life of the option. The effect is even more pronounced when we introduce the WWR: early redemption, indeed, would occur as soon as the portfolio value is large enough with the consequence to eliminate the exposure just when counterparty

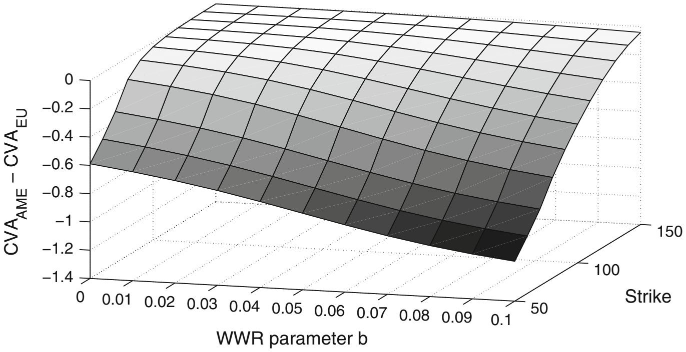


Fig. 3 Difference in CVA between European and American options as function of WWR parameter b and moneyness. Parameters are $S_0 = 100$, $\sigma = 25\,\%$, $r = 1\,\%$, CoC $= -2\,\%$, $T = 1$, $n = m = 500$. Counterparty CDS *curve flat* at 125 basis points

default probabilities become more relevant. It is possible, then, to identify in the early termination clause an important mechanism that limits CVA charges, particularly when market-credit dependency is non-negligible as shown in [8] in the case without WWR. Any change that makes early exercise more likely tends to enhance such a mechanism. We see this effect in Fig. 3 where we display the difference in CVA between European and American options as function of WWR parameter and option moneyness. With a given underlying asset dynamics, potential early exercise date is closer for more in the money options: the right of the holder is more likely to be exercised sooner. This shortens the life of the option and reduces both CVA charge (with respect to European options) and WWR sensitivity (with respect to the corresponding European option and the American options with lower moneyness). In this section we have shown that WWR can play a very different role for European and American options. In our opinion, however, WWR should be analyzed on a case-by-case basis in order to determine its magnitude and the adequate capital charge: a 40 % increase in standard CVA could overestimate the losses for an American option that can be optimally exercised in a short period while could be reductive in cases where early termination is less likely.

4 The Bermudan Swaption Case

Probably the most relevant case of long position on options with early exercise opportunities in the portfolios of financial institutions is represented by Bermudan swaptions. Such exotic derivatives are, indeed, used by corporate entities to enhance the financial structure related to the issue of callable bonds. Often, by selling a Bermudan receiver swaption to a dealer, the callable bond issuer can reduce its net borrowing cost. Usually the swaption is structured such that exercise dates match

Table 1 Diagonal implied volatility of European ATM swaptions used to calibrate the 1-factor Hull–White model

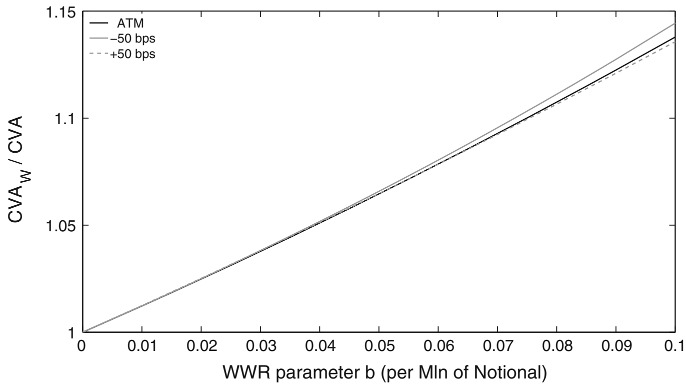
Swaption	1y9y	2y8y	3y7y	4y6y	5y5y	7y3y
Volatility %	40.4	37.6	35.1	32.8	30.8	27.7

Calibrated parameters are $\hat{a} = 0.0146$ and $\hat{\sigma} = 0.0089$

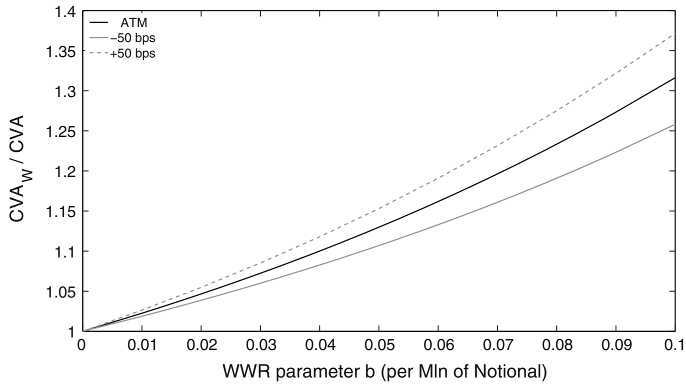
the callability schedule of the bond.[10] Let $\hat{T}$ be the bond maturity date. The dealer has the right, at any exercise date $e_k \in \mathscr{E} \backslash \{e_m\}$, to enter into an interest rate swap with maturity $\hat{T}$, where she receives the fixed rate K (equal to the fixed coupon rate of the bond) and pays the floating rate to the bond issuer with first payment made on date e_{k+1}. In our test we use the Euro interbank market data as of September 13, 2012 as given in [2]. We assume that the dealer buys a 10-year Bermudan receiver swaption where the underlying swap has, for simplicity, both fixed and floating legs with semiannual payments. The swaption can be exercised semiannually and its notional amount is Eur 100 million. We describe interest rates dynamics with a 1-factor Extended Vasicek model on a trinomial tree as in [10]. Model parameters are calibrated to market prices of European ATM swaptions with overall contract maturity equal to 10 years as shown in Table 1. As done in the previous section, we value the Bermudan swaption on the tree via dynamic programing and calibrate the WWR model function $a(t)$. Once again the combined approach on the tree allows to perform both tasks in a negligible amount of time. Figure 4 reports the WWR impact[11] for uncollateralized transactions struck at different levels of moneyness: at the money (swaption strike set equal to the market 10 years spot swap rate) and ± 50 basis points. The upper graph reports the case with no initial lockout period while in the lower one we assume that the option cannot be exercised in the first 2 years. When the option can be exercised with no restrictions, we observe a moderate inverse relationship between moneyness and WWR impact due to the protection mechanism: the opportunity to early exercise when the exposure is large limits the effect of increased counterparty default probabilities. On the other hand, the introduction of a lockout period intensifies the WWR impact. Intuitively, by expanding the lockout period we move toward the limiting case of a European option. In this case the moneyness–WWR effect is reversed: the more in the money the option is, the more relevant the WWR effect becomes. During the lockout period the in-the-money option has a considerably higher exposure to counterparty default that cannot be mitigated via early termination.

[10] Often the bond can be called at any coupon payment date after an initial lockout period.

[11] We define it to be the ratio CVA_W / CVA as given, respectively, by (5) and (2).



No lockout



2-year lockout

Fig. 4 Impact of WWR on Bermudan receiver swaptions as function of WWR parameter *b* for several levels of moneyness. Market data as of September 13, 2012. Counterparty CDS *curve flat* at 125 basis points

5 Concluding Remarks

Nowadays WWR is a crucial concern in OTC derivatives transactions. This is particularly true for uncollateralized trades that a financial institution could have in place with medium-sized corporate clients. The presence of early termination clauses in vulnerable derivatives portfolios makes the CVA computation even more tricky. We have shown a simple and effective approach to deal with calibration and pricing of CVA within the Hull–White framework [11] for American or Bermudan options. We extended the procedure in [1] to the dynamic programing algorithm required to take into account the free boundary problem inherent in the pricing of such derivatives. Numerical tests carried out underline the importance of adequate procedures to differentiate CVA profiles for European and American options. The possibility of early exercise, indeed, plays a remarkable role in mitigating the WWR: an undifferentiated CVA pricing for contingent claims with different exercise styles would then lead to severe misspecification of regulatory capital charges.

An interesting topic for further research would consider the impact of counterparty defaultability in defining the dealer's optimal exercise strategy. Even if intuitive, this poses nontrivial problems mainly due to the interrelation among derivative pricing, WWR, and calibration of function $a(t)$. It is our opinion, however, that the described framework could be extended in this direction.

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Simultaneous Hedging of Regulatory and Accounting CVA

Christoph Berns

Abstract As a consequence of the recent financial crisis, Basel III introduced a new capital charge, the CVA risk charge to cover the risk of future CVA fluctuations (CVA volatility). Although Basel III allows for hedging the CVA risk charge, mismatches between the regulatory (Basel III) and accounting (IFRS) rules lead to the fact that hedging the CVA risk charge is challenging. The reason is that the hedge instruments reducing the CVA risk charge cause additional Profit and Loss (P&L) volatility. In the present article, we propose a solution which optimizes the CVA risk charge and the P&L volatility from hedging.

Keywords CVA risk charge · Accounting CVA · Hedging · Optimization

1 Introduction

Counterparty credit risk is the risk that a counterparty in a derivatives transaction will default prior to expiration of the trade and will therefore not be able to fulfill its contractual obligations. Before the recent financial crisis many market participants believed that some counterparties will never fail ("too big to fail") and therefore counterparty risk was generally considered as not significant. This view changed due to the bankruptcy of Lehman Brothers during the financial crisis and market participants realized that even major banks can fail. For that reason, counterparty risk is nowadays considered to be significant for investment banks. The International Financial Reporting Standards (IFRS) demand that the fair value of a derivative incorporates the credit quality of the counterparty. This is achieved by a valuation adjustment which is commonly referred to as credit valuation adjustment (CVA), see e.g. [3–5]. The CVA is part of the IFRS P&L, i.e. losses (gains) caused by changes of the counterparties credit quality reduce (increase) the balance sheet equity.

Basel III requires a capital charge for future changes of the credit quality of derivatives, i.e. CVA volatility. Banks can either use a standardized approach to

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compute this capital charge or an internal model [2]. The latter charge is commonly referred to as CVA risk charge. Many banks have implemented a CVA desk in order to manage actively their CVA risk. CVA desks buy CDS protection on the capital markets to hedge the counterparty credit risk of uncollateralized derivatives which have been bought by the ordinary trading desks. Recognizing that banks actively manage CVA positions, Basel III allows for hedging the CVA risk charge using credit hedges such as single name CDSs and CDS indexes. However, the recognition of hedges is different depending on whether the standardized approach or an internal model is used [2].

Summarizing, we can look at counterparty credit risk from two different perspectives: the regulatory (Basel III) and the accounting (IFRS) one. Depending whether we consider counterparty risk from a regulatory or accounting perspective, different valuation methods are applied for this risk. In general, the regulatory treatment of counterparty risk is more conservative than the accounting one, cf. [6]. The difference between the regulatory and the accounting treatment of counterparty risk causes the following problem in hedging the CVA risk charge: eligible hedge instruments such as CDSs would lead to a reduction of the CVA risk charge. On the other hand, under IFRS, a CDS is recognized as a derivative and thus accounted at fair value through profit and loss and therefore introducing further P&L volatility.

The current accounting and regulatory rules expose banks to the situation that they cannot achieve regulatory capital relief and low P&L volatility simultaneously. Deutsche Bank, for instance, has largely hedged the CVA risk charge in the first half of 2013. The hedging strategy that reduced the CVA risk charge has caused large losses due to additional P&L volatility, cf. [7]. This example illustrates the mismatch between the regulatory and accounting treatment of CVA.[1] The mismatch demands for a trade-off between these two regimes, cf. [8]. For this reason, we propose in this article an approach which leads to an optimal allocation between CVA risk charge reduction and P&L volatility. Our considerations are restricted to the standardized CVA risk charge.

We start with an explanation of the standardized CVA risk charge, i.e. the regulatory treatment of CVA. Afterwards, we show that the standardized CVA charge can be interpreted as a (scaled) volatility/variance of a portfolio of normally distributed positions. This interpretation reveals the modeling assumptions of the regulator and will be crucial for the later considerations. In a next step, we explain the counterparty risk modeling from an accounting perspective and we compute the impact of the hedge instruments (used to reduce the CVA risk charge) to the overall P&L volatility, assuming that the risk factor returns are normally distributed. Without the mismatch between the regulatory and the accounting regime, the hedge instruments would move anti-correlated to the corresponding accounting CVAs and the resulting common volatility would be small. Due to the mismatch, the CVA and the hedge instrument changes will not offset completely. For this reason we introduce a

[1]Due to the exclusion of DVA from the Basel III regulatory calculation, the mismatch potentially intensifies.

synthetic[2] total volatility σ_{syn} consisting basically of the sum of the additional accounting P&L volatility σ_{hed} caused by fair value changes of the hedge instruments (hedge P&L volatility) and the regulatory CVA volatility $\sigma_{CVA,reg}$ (i.e. basically the CVA risk charge)[3]:

$$\sigma_{syn}^2 = \sigma_{hed}^2 + \sigma_{CVA,reg}^2. \tag{1}$$

Hence, (1) defines a steering variable describing the common effects of CVA risk charge hedging and resulting P&L volatility. One should mention that formula (1) may suggest statistical independence of the two quantities. However, there exists a dependence in the following sense: both the regulatory CVA volatility and the hedge P&L volatility depend on the hedge amount. The more we hedge, the smaller the $\sigma_{CVA,reg}$. On the other hand, the more we hedge, the larger the σ_{hed}. The definition of the synthetic volatility as a sum of σ_{hed}^2 and $\sigma_{CVA,reg}^2$ can be motivated by the following consideration: the term $\sigma_{CVA,reg}^2$ is related to the regulatory capital demand for CVA risk. The other term, σ_{hed}^2, can be interpreted as capital demand for market risk of the hedge instruments. Although the hedge instruments are excluded from the regulatory capital demand computation for market risk, they potentially reduce the balance sheet equity and therefore may reduce the available regulatory capital. The sum in (1) is now motivated by the additivity of the total capital demand.

In the following we will consider σ_{syn} as function of the hedge amount and search for its minimum. The hedge amount minimizing σ_{syn} leads to the optimal allocation between CVA risk charge relief and P&L volatility. We will derive analytical solutions. The discussion of several special cases will provide an intuitive understanding of the optimal allocation. For technical reasons we exclude index hedges in the derivation of the optimal hedge strategy. However, it is easy to generalize the results to the case where index hedges are allowed.

2 Counterparty Risk from a Regulatory Perspective: The Standardized CVA Risk Charge

In this section we introduce the standardized CVA risk charge. A detailed explanation of all involved parameters is given in the Basel III document [2]. The formula for the standardized CVA risk charge is prescribed by the regulator and is used to determine the amount of regulatory capital which banks must hold in order to absorb possible losses caused by future deteriorations of the counterparties credit quality. We will see that the standardized CVA risk charge can be interpreted as volatility (i.e. standard deviation) of a normally distributed random variable. More precisely, we will show that the CVA risk charge can be interpreted as the 99 % quantile of a portfolio of

[2] We use the word synthetic since σ_{syn} mixes a volatility measured in regulatory terms and a volatility measured in accounting terms.

[3] This connection will be explained later.

positions subject to normally distributed CVA changes (i.e. CVA P&L) only. This gives some insights into the regulators modeling assumptions for future CVA. It is worth to mention that the regulators modeling assumptions may hold or not hold. A detailed look at the regulators modeling assumptions can be found in [6].

In order to be prepared for later computations, we introduce in this section some notations and recall some facts about normally distributed random variables.

The standardized CVA risk charge K is given by [2]:

$$K = \beta\sqrt{h}\Phi^{-1}(q) \tag{2}$$

with

- $h = 1$, the 1-year time horizon,
- Φ the cumulative distribution function of the standard normal distribution
- $q = 99\,\%$ the confidence level and
- β defined by[4]

$$\beta^2 = \left(\sum_{i=1}^{n} 0.5 \cdot \omega_i \left(M_i EAD_i - M_i^{hed} B_i\right) - \omega_{ind} M_{ind} B_{ind}\right)^2 + \sum_{i=1}^{n} 0.75 \cdot \omega_i^2 \left(M_i EAD_i - M_i^{hed} B_i\right)^2 \tag{3}$$

with

- ω_i a weight depending on the rating of the counterparty i, n is the number of counterparties
- M_i, M_i^{hed}, and M_{ind} the effective maturities for the ith netting set (corresponding to counterparty i), the hedged instrument for counterparty i and the index hedge
- EAD_i the discounted regulatory exposure w.r.t. counterparty i
- B_i, B_{ind} the discounted hedge notional amounts invested in the hedge instrument (CDS) for counterparty i and the index hedge.

Formula (2) is determined by the regulator. In order to get a better understanding of this formula, we will derive a stochastic interpretation of it. Before that, we need to recall a fact about normal distributions: if the random vector $\vec{X}$ has a multivariate normal distribution, i.e. $\vec{X} \sim \mathcal{N}(0, \Sigma)$ with mean 0 and covariance matrix Σ, then, for a deterministic vector $\vec{a}$, the scalar product

$$\langle \vec{a}, \vec{X} \rangle := \sum_i a_i X_i \tag{4}$$

[4]For simplicity we consider only one index hedge. The results in this article can easily be generalized to more than one index hedge.

has a univariate normal distribution with mean 0 and variance

$$\sigma^2 = \langle \vec{a}, \Sigma \vec{a} \rangle. \tag{5}$$

Now we are able to derive the stochastic interpretation of the CVA risk charge, more precise the interpretation as volatility.

2.1 Standardized CVA Risk Charge as Volatility

In this section we will show that the regulators' modeling assumptions behind the standardized CVA risk charge are given by normally distributed CVA returns which are aggregated by using a one-factor Gaussian copula model.[5] We consider n counterparties. By R_i, we denote the (one year) CVA P&L (i.e. those P&L effects caused by CVA changes) w.r.t. counterparty i.

Lemma 1 *If one assumes $R_i \sim \mathcal{N}(0, \sigma_i^2)$ and further, if one assumes that the random vector*[6]

$$\vec{R} = (R_1, \ldots, R_n)^t$$

is distributed according to a one-factor Gaussian copula model, i.e. $\vec{R} \sim \mathcal{N}(0, \Gamma)$ with $\Gamma_{ii} = \sigma_i^2$ and $\Gamma_{ij} = \rho\sigma_i\sigma_j$ with ρ independent of i and j for $i \neq j$, then the 99 % quantile of the distribution of $\vec{R}$ is equal to the CVA risk charge (2).

Proof Using (4) and (5), we find that the aggregated CVA return (common CVA P&L) $R_{CVA,reg} := \sum_{i=1}^{n} R_i = \langle \vec{1}, \vec{R} \rangle$[7] has the distribution $\mathcal{N}(0, \sigma_{CVA,reg}^2)$ with

$$\sigma_{CVA,reg}^2 = \langle \vec{1}, \Gamma \vec{1} \rangle = \sum_{i,j=1}^{n} \Gamma_{i,j} = \left(\sqrt{\rho} \sum_{i=1}^{n} \sigma_i \right)^2 + (1-\rho) \sum_{i=1}^{n} \sigma_i^2 \tag{6}$$

If we compare the above expression with (3), we see that this expression is equal to β^2 (with $B_{ind} = 0$, i.e. no index hedges) if we set $\rho = 0.25$ and $\sigma_i = \omega_i(M_i EAD_i - M_i^{hed} B_i)$. The quantile interpretation of the CVA risk charge (i.e. Formula (2)) follows from standard properties of the normal distribution.

The above lemma shows that the standardized CVA risk charge is basically the volatility of the sum $\sum_i R_i$ of n normally distributed random variables. The normally distributed random variables are equicorrelated: $\rho(R_i, R_j) = 0.25$. Each CVA return R_i has the volatility

$$\sigma_i = \omega_i(M_i EAD_i - M_i^{hed} B_i). \tag{7}$$

[5]This is a very strong assumption that might not be true in reality.

[6]By $\cdot^t$ we denote the transpose of a vector/matrix.

[7]By $\vec{1}$ we denote the vector $(1, \ldots, 1)^t$.

Hence, buying credit protection on counterparty i reduces the corresponding CVA volatility. If we assume $M_i = M_i^{hed}$, the optimal hedge w.r.t. counterparty i is given by a CDS with notional amount B_i equals

$$B_i = EAD_i. \tag{8}$$

3 Counterparty Risk from an Accounting Perspective

As explained in the introduction, counterparty risk from an accounting perspective is quantified by a fair value adjustment called credit valuation adjustment (CVA). The CVA reduces the present value (PV) of a derivatives portfolio in order to incorporate counterparty risk:

$$PV = PV_{riskfree} - CVA,$$

whereby $PV_{riskfree}$ denotes the market value of the portfolio without counterparty risk and CVA is the adjustment to reflect counterparty risk. For the modeling of CVA, banks have some degrees of freedom. Typically, the accounting CVA is computed by means of the following formula (see e.g. [4]):

$$CVA = \int_0^T D(t)EE(t)dP(t) \tag{9}$$

with T the effective maturity of the derivatives portfolio, $D(t)$ the risk-free discount curve, $EE(t) = E[\max\{0, PV(t)\}]$ the (risk-neutral) expected positive exposure at (future time point) t, and $dP(t)$ is the (risk-neutral) default probability of the counterparty in the infinitesimal interval $[t, t + dt]$. For the implementation of (9), a discretization of the integral is necessary. Many banks assume a constant EE profile (i.e. $EE(t) = EE^*$ for all t). In that case, (9) simplifies to

$$CVA = EE^* \int_0^T D(t)dP(t). \tag{10}$$

Further, the (risk-neutral) default probabilities are typically modeled by a hazard rate model, i.e. one assumes that the default time is exponentially distributed with parameter λ. Using this assumption, we can write:

$$CVA = \lambda EE^* \int_0^T D(t)e^{-\lambda t}dt. \tag{11}$$

The approximation (11) will be helpful in the next section, where we describe the hedging of CVA from an accounting perspective

3.1 CVA Hedging from an Accounting Perspective

In previous sections we have seen that the regulatory CVA hedging (i.e. CVA risk charge hedging) can be achieved by buying credit protection. Effectively, (7) says that the regulatory exposure is reduced by the notional amount of the bought credit protection. At this place, we describe CVA hedging from an accounting perspective.

Let us consider a derivatives portfolio with a single counterparty. In order to hedge the corresponding counterparty risk, one can buy, for example, a single name CDS such that the CVA w.r.t. the counterparty together with the CDS is Delta neutral (i.e. up to first order, CVA movements are neutralized by the CDS movements). The condition for Delta neutrality is

$$\Delta CVA = \Delta CDS \tag{12}$$

whereby Δ describes the derivative of the CVA and CDS respectively (w.r.t. the credit spread of the counterparty). To be more precise, the default leg of the CDS should compensate the CVA movements. Using a standard valuation model for a CDS (see e.g. [4]) and computing the derivatives in (12), it is easy to see that (12) is equivalent to

$$B = EE^*, \tag{13}$$

i.e. the optimal hedge amount is given by EE^*. Typically, EE^* is given by the average of the expected positive exposures $EE(t)$ at future time points t:

$$EE^* = \frac{1}{T}\int_0^T EE(t)dt. \tag{14}$$

If we compare (13) with (8) we see that the optimal hedge notional amount for hedging CVA risk from a regulatory perspective is the regulatory exposure EAD, while the optimal hedge notional amount for hedging accounting CVA risk is given by EE^*. In general it holds $EAD > EE^*$, due to conservative assumptions made by the regulator[8] (we refer to [6] for a detailed comparison of these two quantities). Thus, hedging CVA risk differs whether it is considered from an accounting or a regulatory perspective. This mismatch causes additional P&L volatility in the accounting framework, if the CVA risk is hedged from a regulatory perspective (i.e. if the CVA risk charge is hedged).

Finally we remark that we can write the CVA sensitivities $\Delta_{CVA} = \frac{d}{ds}CVA$ as

$$\Delta_{CVA} = EE^* \Delta_{CDS}, \tag{15}$$

whereby Δ_{CDS} is the sensitivity of (the default leg of) a CDS with notional amount $B = 1$.

[8]For example, the alpha multiplier in the IMM context overstates the EAD by a factor of 1.4. Further, the non-decreasing constraint to the exposure profile leads to an overstatement, see [6] for details.

4 Portfolio P&L

As explained above, the hedge instruments reduce the (regulatory) counterparty credit risk. But they may cause new market risk due to additional P&L volatility. However, although in accordance with Basel III eligible hedge instruments are excluded from market risk RWA calculations, the additional P&L volatility of the hedge instruments leads to fluctuations in reported equity. In order to describe the effects of hedging to the overall P&L, we introduce in the present section the corresponding framework. We divide the overall P&L in different parts: the P&L of the hedge instruments, the P&L of the remaining positions, and the CVA P&L. The framework will be helpful later on, when we want to quantify the impact of the CVA risk charge hedges to the accounting P&L.

4.1 Portfolio P&L Without CVA

Let us assume that a bank holds derivatives with n different counterparties for which single name CDS exists. The bank has to decide to which extent it hedges the counterparty risk w.r.t. these counterparties by either single name CDS or index hedges. By Σ we denote the correlation matrix (of dimension $N \times N$, $N > n$) of all risk factors r_i, $i = 1, \dots, N$ the banks (trading) portfolio is exposed to. Without loss of generality, we assume that the correlations between the CDS of the considered n counterparties are given by the first $n \times n$ components of Σ, i.e. $\Sigma_{i,j} = \rho(CDS_i, CDS_j)$, $i, j = 1, \dots, n$. Further, $\Sigma_{n+1,i}$ denotes the correlation between the index hedge and the CDS on counterparty $i \in \{1, \dots, n\}$. The whole portfolio Π of the bank contains the hedge instruments (CDS and index hedge) as well as other instruments (e.g. bonds): $\Pi = \Pi_{hed} \cup \Pi_{rest}$. The sub-portfolio Π_{hed} is driven by the credit spreads of the counterparties. Note that Π_{rest} may depend on some of these credit spreads as well. In the following, we will assume the P&L of the portfolio Π is given by:

$$P\&L = \sum_{i=1}^{n} (B_i \Delta_i + \Delta_{i,rest}) dr_i + B_{ind} \Delta_{ind} dr_{ind} + \sum_{j=n+2}^{N} \Delta_j dr_j, \tag{16}$$

whereby Δ_i denotes the sensitivity of CDS_i w.r.t. the corresponding credit spread, $\Delta_{i,rest}$ denotes the sensitivity of the remaining positions which are sensitive w.r.t. the credit spread of counterparty i as well,[9] B_i (resp. B_{ind}) denotes the notional of CDS_i (resp. the notional of the index hedge), and dr_i describes the change of the risk factor r_i (the first n risk factors are the credit spreads) in the considered time period.

[9]For example, if Π_{rest} contains a bond emitted by the counterparty i, then (ignoring the Bond-CDS Basis) $\Delta_{i,rest} = -\Delta_i$).

4.2 Impact with CVA

This section extends the above considerations to the case where we allow for a CVA component. We define the total P&L as the difference between the P&L given by (16) and the CVA P&L:

$$P\&L_{tot} = P\&L - P\&L_{CVA}, \tag{17}$$

whereby $P\&L_{CVA}$ is defined in a similar manner as in (16)[10]:

$$P\&L_{CVA} = \sum_{i=1}^{n+1} \Delta_{i,CVA} dr_i. \tag{18}$$

In (18), the risk factors r_i are the same risk factors which appear in the first $n+1$ summands of (16). This is because the CVAs are driven by the same risk factors as the corresponding hedge instruments. Recall that in a setup where counterparty risk is completely hedged, the P&L of the hedge instruments is canceled out by the P&L of the CVAs. This is the case, if the corresponding sensitivity is equal. In Sect. 3.1 we have shown how one can achieve this (using the condition of Delta neutrality) by choosing the right hedge notional amounts.

4.3 Impact of CVA Risk Charge Hedging on the Accounting P&L Volatility

The additional P&L volatility caused by the hedge instruments is basically given by the residual volatility of the hedge instruments which is not canceled by the CVAs. In order to derive an expression for this volatility, we start with the derivation of the volatility of the total portfolio P&L. The residual volatility will consist of those parts of the total volatility which are sensitive w.r.t. the hedge instruments.

In order to proceed, we have to introduce the following notations: the vector $\vec{\Delta}_{CVA} \in \mathbb{R}^{n+1}$ contains the CVA sensitivities and the return vector $\vec{dr} \in \mathbb{R}^N$ describes the changes of the N risk factors the trading book is exposed to. We further introduce the sensitivity vectors[11] $\vec{\Delta} = (\Delta_1, \ldots, \Delta_{ind}, \ldots, \Delta_N)^t \in \mathbb{R}^N$

[10] We consider only the credit spreads as risk factors. Exposure movements due to changes in market risk factors are not considered. This is unproblematic for the considerations in this article since we will end up with dynamic CVA hedging strategy (cf. Sect. 5) which incorporates the exposure changes.

[11] The first n components of $\vec{\Delta}$ are the CDS sensitivities w.r.t. credit spread changes and the $n+1$th component is the sensitivity of the index hedge.

and[12] $\vec{\Delta}_{rest} = (\Delta_{1,rest}, \dots, \Delta_{n,rest})^t \in \mathbb{R}^n$, the notional vector $\vec{B} = (B_1, \dots, B_n, B_{ind})^t \in \mathbb{R}^{n+1}$ and the diagonal matrix $Q_\Delta = diag(\Delta_1, \dots, \Delta_n, \Delta_{ind}) \in \mathbb{R}^{(n+1)\times(n+1)}$.

Lemma 2 *If we assume that the portfolio P&L is given by (17) and if we further assume $\vec{dr} \sim \mathscr{N}(0, \Sigma)$ (for some correlation matrix Σ), then the squared volatility (i.e. the variance) of (17) is given by*[13]

$$\begin{aligned}\sigma^2_{P\&L_{tot}} &= \left\langle \begin{pmatrix} Q_\Delta \vec{B} \\ \vec{0} \end{pmatrix} \Sigma \begin{pmatrix} Q_\Delta \vec{B} \\ \vec{0} \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} \vec{\Delta}_{CVA} \\ \vec{0} \end{pmatrix} \Sigma \begin{pmatrix} \vec{\Delta}_{CVA} \\ \vec{0} \end{pmatrix} \right\rangle \\ &+ \left\langle \begin{pmatrix} \vec{\Delta}_{rest} \\ \vec{\Delta}_{N-n-1} \end{pmatrix} \Sigma \begin{pmatrix} \vec{\Delta}_{rest} \\ \vec{\Delta}_{N-n-1} \end{pmatrix} \right\rangle - 2\left\langle \begin{pmatrix} Q_\Delta \vec{B} \\ \vec{0} \end{pmatrix} \Sigma \begin{pmatrix} \vec{\Delta}_{CVA} \\ \vec{0} \end{pmatrix} \right\rangle \\ &+ 2\left\langle \begin{pmatrix} Q_\Delta \vec{B} \\ \vec{0} \end{pmatrix} \Sigma \begin{pmatrix} \vec{\Delta}_{rest} \\ \vec{\Delta}_{N-n-1} \end{pmatrix} \right\rangle - 2\left\langle \begin{pmatrix} \vec{\Delta}_{rest} \\ \vec{\Delta}_{N-n-1} \end{pmatrix} \Sigma \begin{pmatrix} \vec{\Delta}_{CVA} \\ \vec{0} \end{pmatrix} \right\rangle. \end{aligned} \tag{19}$$

Proof With the above defined vectors, we can write:

$$\begin{aligned} P\&L_{tot} &= \langle Q_\Delta \vec{B} - \vec{\Delta}_{CVA}, \vec{dr}_{n+1} \rangle + \langle \vec{\Delta}_{rest}, \vec{dr}_{n+1} \rangle + \langle \vec{\Delta}_{N-n-1}, \vec{dr}_{N-n-1} \rangle \\ &= \left\langle \begin{pmatrix} Q_\Delta \vec{B} \\ \vec{0}_{N-n-1} \end{pmatrix} - \begin{pmatrix} \vec{\Delta}_{CVA} \\ \vec{0}_{N-n-1} \end{pmatrix} + \begin{pmatrix} \vec{\Delta}_{rest} \\ \vec{\Delta}_{N-n-1} \end{pmatrix}, \vec{dr} \right\rangle \\ &= \langle \vec{a} - \vec{b} + \vec{c}, \vec{dr} \rangle \end{aligned} \tag{20}$$

whereby $\vec{dr}_{n+1}$ denotes the $n+1$-dimensional vector consisting of the first $n+1$ components of $\vec{dr}$, $\vec{dr}_{N-n-1}$ consists of the remaining $N-n-1$ components of $\vec{dr}$, $\vec{\Delta}_{N-n-1}$ denotes the vector of the remaining $N-n-1$ sensitivities, and $\vec{0}_{N-n-1}$ is the $N-n-1$-dimensional vector whose components are all equal to 0.[14] Clearly, the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ coincide with the respective summands of the left hand side of the scalar product in (20). If we use $\vec{dr} \sim \mathscr{N}(0, \Sigma)$, it follows from (4) to (5):

$$\begin{aligned} \sigma^2_{P\&L_{tot}} &= \langle \vec{a} - \vec{b} + \vec{c}, \Sigma(\vec{a} - \vec{b} + \vec{c}) \rangle \\ &= \langle \vec{a}, \Sigma\vec{a} \rangle + \langle \vec{b}, \Sigma\vec{b} \rangle + \langle \vec{c}, \Sigma\vec{c} \rangle - 2\langle \vec{a}, \Sigma\vec{b} \rangle + 2\langle \vec{a}, \Sigma\vec{c} \rangle - 2\langle \vec{c}, \Sigma\vec{b} \rangle. \end{aligned} \tag{21}$$

If we plug in the expressions for $\vec{a}$, $\vec{b}$ and $\vec{c}$, we obtain (19). □

In order to be prepared for later computations, we will further simplify Expression (19). To this end, we introduce the following notations: by Σ_{n+1} we denote the $(n+1) \times (n+1)$ matrix consisting of the first $n+1$ column and row entires of Σ only, i.e. $\Sigma_{i,j}$, $i, j = 1, \dots n+1$. The matrix $\Sigma_{N,n+1}$ is the $N \times (n+1)$ matrix

[12] The vector $\vec{\Delta}_{rest}$ contains the n sensitivities w.r.t. credit spread changes of those trading book positions which are different from the CDSs used for hedging but are sensitive w.r.t. to the credit spreads of the hedge instruments as well.

[13] The vector $\vec{\Delta}_{N-n-1}$ is defined in the proof.

[14] In the following, we will omit the index $N-n-1$ and simply write $\vec{0}$.

obtained from Σ by deleting the last $N-n-1$ columns and $\Sigma^t_{N,n+1}$ denotes its transpose matrix. With this notation and using that $\vec{0}$ cancels many components in (19), we can write:

$$\begin{aligned}\sigma^2_{P\&L_{tot}} &= \langle Q_\Delta \vec{B},\, \Sigma_{n+1} Q_\Delta \vec{B}\rangle + \langle \vec{\Delta}_{CVA}\Sigma_{n+1},\, \vec{\Delta}_{CVA}\rangle - 2\langle Q_\Delta \vec{B},\, \Sigma_{n+1}\vec{\Delta}_{CVA}\rangle \\ &\quad + 2\left\langle \vec{B},\, Q_\Delta \Sigma^t_{N,n+1}\begin{pmatrix}\vec{\Delta}_{rest}\\ \vec{\Delta}_{N-n-1}\end{pmatrix}\right\rangle - 2\left\langle \vec{\Delta}_{CVA},\, \Sigma^t_{N,n+1}\begin{pmatrix}\vec{\Delta}_{rest}\\ \vec{\Delta}_{N-n-1}\end{pmatrix}\right\rangle \\ &\quad + \left\langle \begin{pmatrix}\vec{\Delta}_{rest}\\ \vec{\Delta}_{N-n-1}\end{pmatrix} \Sigma \begin{pmatrix}\vec{\Delta}_{rest}\\ \vec{\Delta}_{N-n-1}\end{pmatrix}\right\rangle.\end{aligned} \tag{22}$$

In (22), the first summand describes the volatility of the hedge instruments if they are considered as isolated from the remaining positions (i.e. those positions which are different from the hedge instruments). Analogously, the other quadratic terms (i.e. the second and the last summand in (22)) represent the volatility of the CVA and the remaining positions respectively. The cross terms (third, fourth, and fifth summand) describe the interactions between the volatility of the hedge instruments, the CVA and the remaining positions. For example, the third term describes the interaction between the CVA and the hedge instruments.

The P&L volatility σ^2_{hed} caused by the hedge instruments is given by those terms of (22) which depend on the hedge instruments, i.e. those terms which depend on $\vec{B}$. These are the first, the third, and the fourth term of (22), i.e.

$$\sigma^2_{hed} = \langle Q_\Delta \vec{B},\, \Sigma_{n+1} Q_\Delta \vec{B}\rangle - 2\langle Q_\Delta \vec{B},\, \Sigma_{n+1}\vec{\Delta}_{CVA}\rangle + 2\left\langle \vec{B},\, Q_\Delta \Sigma^t_{N,n+1}\begin{pmatrix}\vec{\Delta}_{rest}\\ \vec{\Delta}_{N-n-1}\end{pmatrix}\right\rangle. \tag{23}$$

The other terms of (22) describe the volatility caused by the remaining positions.

In order to simplify the notation, we write σ^2_{hed} in the following way:

$$\sigma^2_{hed} = \langle A\vec{B}, \vec{B}\rangle + \langle \vec{B}, \vec{b}\rangle \tag{24}$$

with

$$A = Q_\Delta \Sigma_{n+1} Q_\Delta \tag{25}$$

and

$$\vec{b} = Q_\Delta \Sigma^t_{N,n+1}\begin{pmatrix}\vec{\Delta}_{rest}\\ \vec{\Delta}_{N-n-1}\end{pmatrix} - Q_\Delta \Sigma_{n+1}\vec{\Delta}_{CVA}. \tag{26}$$

Note that σ^2_{hed} is not simply given by a quadratic form but also incorporates a linear part. The quadratic form describes the volatility of a portfolio consisting of the hedge instruments, while the linear part describes the correlations of the hedge instruments with the remaining positions and with the CVAs.

4.3.1 Definition of the Steering Variable

We now define a steering variable aiming to define a unified framework for CVA risk charge hedging and P&L volatility. The steering variable is given by a synthetic volatility consisting of the sum of the regulatory CVA volatility and the volatility of the accounting P&L caused by the hedge instruments:

$$\sigma_{syn}^2 = \sigma_{CVA,reg}^2 + \sigma_{hed}^2. \tag{27}$$

The synthetic volatility unifies both the regulatory and the accounting framework. It can be considered as a function of the hedge notional amounts. The minimum of $\sigma_{tot,syn}^2$ describes the optimal allocation between CVA risk charge reduction and P&L volatility. Note that $\sigma_{tot,syn}^2$ contains now the matrices Γ and Σ, who describe the correlations between the same risk factors. This mismatch can be resolved, if the advanced CVA risk charge is used [2]. However, the use of different CVA sensitivities cannot be resolved. The most significant differences arise due to different exposure definitions: while the exposures EAD_i contained in the regulatory CVA sensitivities are based on the effective EPE and multiplied by the alpha multiplier (for IMM banks), this is not the case for the exposures used to compute the accounting CVA sensitivities. In general, these mismatches will lead to smaller accounting CVA sensitivities. Thus, a complete hedging of the CVA risk charge leads to an overhedged accounting CVA. See [6] for a complete description of the sources of the mismatch. Another source of potential overhedging is the following: if accounting CVA is already hedged by instruments which are not eligible hedge instruments in the sense of Basel III, additional hedge instruments are necessary for the hedging of the CVA risk charge. These hedge instruments will cause additional P&L volatility, since their offsetting counterparts (i.e. the CVAs) are not present (since they are already hedged).

5 Determination of the Optimal Hedge Strategy

This section describes concretely how the mismatch between the regulatory regime and the accounting regime can be mitigated. The result will be a dynamic CVA hedging strategy based on an optimization principle of the steering variable introduced in the previous section. We will ignore index hedges but all results can easily be generalized to the case where index hedges are included.

As opposed to the previous sections, the vector $\vec{B}$ will not contain the component B_{ind} in this section. As explained before, we want to minimize the synthetic volatility[15]

$$\sigma_{syn}^2(\vec{B}) = \sigma_{hed}^2(\vec{B}) + \sigma_{CVA}^2(\vec{B}) \tag{28}$$

[15] We ignore the index tot.

as a function of $\vec{B}$. The component B_i^* of the minimum $\vec{B}^*$ describes the optimal notional amounts of CDS_i, used to hedge the counterparty risk w.r.t. counterparty i. We now determine $\vec{B}^*$ by computing the zeros of the first derivative of σ_{syn}^2.

Theorem 1 *Under the same assumptions as in Lemma 2, the minimum $\vec{B}^*$ of (28) is given by*[16]

$$\vec{B}^* = H^{-1}\vec{f} \tag{29}$$

with

$$H := 2(A + Q_{M^{hed}} \Gamma Q_{M^{hed}}) \tag{30}$$

and

$$\vec{f} := 2Q_{M^{hed}} \Gamma Q_M \overrightarrow{EAD} - \vec{b}. \tag{31}$$

Proof In order to keep the display of the computations clear, we introduce the diagonal matrices $Q_M := diag(\omega_1 M_1, \ldots, \omega_n M_n)$ and $Q_{M^{hed}} := diag(\omega_1 M_1^{hed}, \ldots, \omega_n M_n^{hed})$ and the n-dimensional vector $\overrightarrow{EAD}$ whose components are given by the counterparty exposures. Using these definitions, we can write:

$$\sigma_{CVA}^2 = \langle Q_M \overrightarrow{EAD} - Q_{M^{hed}} \vec{B}, \Gamma (Q_M \overrightarrow{EAD} - Q_{M^{hed}} \vec{B}) \rangle. \tag{32}$$

whereby Γ describes the constant correlation between the CVAs (all diagonal elements given by 1). Using (32) and (24), we can write:

$$\begin{aligned}
\frac{\partial \sigma_{syn}^2}{\partial \vec{B}} &= \frac{\partial}{\partial \vec{B}} (\langle A\vec{B}, \vec{B} \rangle + \langle \vec{b}, \vec{B} \rangle) \\
&\quad + \frac{\partial}{\partial \vec{B}} \langle Q_{M^{hed}} \vec{B}, \Gamma Q_{M^{hed}} \vec{B} \rangle \\
&\quad - 2 \frac{\partial}{\partial \vec{B}} \langle Q_{M^{hed}} \vec{B}, \Gamma Q_M \overrightarrow{EAD} \rangle \\
&= 2A\vec{B} + \vec{b} + 2Q_{M^{hed}} \Gamma Q_{M^{hed}} \vec{B} - 2Q_{M^{hed}} \Gamma Q_M \overrightarrow{EAD} \\
&= H\vec{B} - \vec{f},
\end{aligned} \tag{33}$$

where we have used the notations (30) and (31). This shows (29). Further, we note that the matrix H is derived from correlation matrices and therefore positive semi-definite. As a result, H is indeed invertible. Moreover, it holds

$$\frac{\partial^2 \sigma_{syn}^2}{\partial^2 \vec{B}} = H.$$

Hence, the second derivative of σ_{syn}^2 is positive semi-definite and B^* is indeed a minimum.

[16] All terms are introduced in the proof.

Remark The implementation of the optimal hedge strategy works as follows: one has to compute on a regular basis (e.g. daily, weekly, etc.) the optimal solution (29). To do this one needs the CVA sensitivities,[17] the trading book sensitivities, and the correlation matrix of the risk factors.[18] Afterwards, the CVA desk needs to buy credit protection described by the optimal solution. This reduces the capital demand for counterparty risk and (by construction) minimizes the accounting P&L of the bought credit protection.

The approach presented in this article is based on many simplifying assumptions and restricted to the standardized CVA risk charge. Obviously, one could relax these assumptions and apply a comparable optimization principle. In such a case, it would possibly be hard to derive an analytical solution. Instead, one would obtain a numerical solution.

5.1 Special Cases

For illustration purposes, we consider the case $n = 1$, i.e. the special case of a single netting set. In that case both H and $\vec{f}$ are scalars:

$$H = 2\Delta^2 \Sigma_{1,1} + 2\omega^2 (M^{hed})^2$$

and

$$f = 2\omega^2 M M^{hed} EAD + 2\Delta \Delta_{CVA} \Sigma_{1,1} - \left(\Delta \Sigma_{1,1} \Delta_{rest} + \Delta \sum_{j=2}^{N} \Sigma_{1,j} \Delta_j \right), \tag{34}$$

whereby Δ describes the sensitivity of the hedge instrument of the considered counterparty, Δ_{rest} the sensitivity of the remaining positions (i.e. all positions without the CDS used for hedging purposes), Δ_{CVA} the sensitivity of accounting CVA and Δ_j are the sensitivities to the risk factors of the remaining positions. Thus, the optimal solution is

$$B^* = \frac{2\omega^2 M M^{hed} EAD + 2\sigma^2 \Delta \Delta_{CVA} - \left(\Delta \sigma^2 \Delta_{rest} + \Delta \sum_{j=2}^{N} \Sigma_{1,j} \Delta_j \right)}{2\Delta^2 \sigma^2 + 2\omega^2 (M^{hed})^2} \tag{35}$$

where we have used that $\Sigma_{1,1}$ is equal to the volatility σ^2 of the hedge instrument. First, in order to get a better understanding of B^*, let us assume that the risk factor (credit spread) of the hedge instrument is independent of the remaining positions, i.e.

[17] Banks which actively manage their CVA risk usually compute these sensitivities.

[18] Larger banks usually have these data available, e.g. for market risk management purposes.

$\Delta_{rest} = 0$ and $\Sigma_{1,j} = 0$, for $j = 2, \ldots, N$. In that case (35) (we assume additionally $M = M^{hed}$) becomes

$$B^* = \frac{2\omega^2 M^2 EAD + 2\Delta\Delta_{CVA}\sigma^2}{2\omega^2 M^2 + 2\Delta^2\sigma^2}. \tag{36}$$

We see already that B^* is (at least from a certain volatility level) a decreasing function in σ^2, as we would expect it. Obviously, if we ignore the fact that the hedge instrument introduces further volatility (i.e. we assume $\sigma^2 = 0$), it holds

$$B^* = EAD.$$

It is easy to see that this is the optimal hedge amount if we minimize the CVA risk charge alone. As explained above, the most significant differences between the IFRS CVA and the regulatory CVA are the different exposure computation methodologies. In (36), these differences are reflected in EAD and Δ_{CVA}: while EAD is based on the regulatory methodology, Δ_{CVA} is based on accounting CVA methodology.[19] For illustration purposes, let us assume that Δ_{CVA} is based on the same exposure methodology as the regulatory CVA sensitivities (and that the modeling assumptions Sect. 3 holds). This means, that cf. (15)

$$\Delta_{CVA} = EAD\Delta, \tag{37}$$

i.e. we use the regulatory exposure EAD in (15) instead of the economical exposure EE^*. If we plug in (37) in (36), we obtain:

$$B^* = \frac{(2\omega^2 M^2 + 2\Delta^2\sigma^2)EAD}{2\omega^2 M^2 + 2\Delta^2\sigma^2} = EAD. \tag{38}$$

Thus, if we ignore the mismatch between the accounting and the regulatory CVA, the optimal hedge solution is given by the optimal hedge solution of the CVA risk charge only. If we include the mismatch, we can approximate the accounting CVA sensitivity by (cf. (15))

$$\Delta_{CVA} = EE^*\Delta. \tag{39}$$

As explained in Sect. 4.3.1, EE^* is smaller than EAD. Using (36) and (39) yields:

$$B^* = \frac{2\omega^2 M^2 EAD + 2\Delta\sigma^2 EE^*}{2\omega^2 M^2 + 2\Delta^2\sigma^2} < \frac{2\omega^2 M^2 EAD + 2\Delta\sigma^2 EAD}{2\omega^2 M^2 + 2\Delta^2\sigma^2} = EAD. \tag{40}$$

Hence, the mismatch leads to a smaller optimal hedge amount than the current regulatory exposure.

[19]Note that Δ_{CVA} depends on the exposure as well (while Δ is based on a unit exposure, cf. (16)). But this exposure is computed based on accounting methodology. This is the main source of differences between the accounting and regulatory regimes.

We remark that it cannot be excluded that B^* becomes negative. This is the case if the risk factors of the remaining positions are strongly correlated to the risk factor of the hedge instrument. In such a situation it seems to be reasonable to set $B^* = 0$.

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Capital Optimization Through an Innovative CVA Hedge

Michael Hünseler and Dirk Schubert

Abstract One of the lessons of the financial crisis as of late was the inherent credit risk attached to the value of derivatives. Since not all derivatives can be cleared by central counterparties, a significant amount of OTC derivatives will be subject to increased regulatory capital charges. These charges cover both current and future unexpected losses; the capital costs for derivatives transactions can become substantial if not prohibitive. At the same time, capital optimization through CDS hedging of counterparty risks will result in a hedge position beyond the economic risk ("overhedging") required to meet Basel II/III rules. In addition, IFRS accounting rules again differ from Basel, creating a mismatch when hedging CVA. Even worse, CVA hedging using CDS may introduce significant profit and loss volatility while satisfying the conditions for capital relief. An innovative approach to hedging CVA aims to solve these issues.

Keywords CVA · Hedging · CDS · Contingent financial guarantee · Risk charges · OTC derivatives

1 Preface

In the following the nexus between credit risk (counterparty risk), liquidity, and market risk is analyzed and a solution with respect to CVA hedging of OTC derivative contracts is proposed.

The starting point is the consideration of collateral and its respective recognition in different but "basic" financial instruments like repos and (partially un-) collateralized

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OTC derivative contracts as well as the comparison to corresponding uncollateralized financial instruments like money market loans or uncollateralized OTC derivative contracts. The role of collateral is analyzed with respect to its legal basis, its treatment in Financial Accounting (IFRS, refer to [4]) and regulatory reporting according to Basel II/III (cf. [1, 2]).

The analysis leads to a definition of the concept of liquidity and its relation to the use of collateral in financial markets. As will be shown, the concept of liquidity, inherent in the legal framework related to collateral of basic financial instruments, can be considered as a transformation of secured into unsecured financing and vice versa. Moreover, with respect to the associated valuation and risk the liquidity transformation exhibits similarities to the concept of wrong-way risk. The transformation of unsecured into secured financing can be used to derive new types of financial instruments, e.g. in the application to CVA hedging issues of OTC derivative contracts. In this case the hedging instrument also solves the issue of disentangling funding value adjustments (FVA) and counterparty value adjustments (CVA), which is intensively discussed by practitioners in context with the pricing of OTC derivatives.

2 The Role of Collateral in OTC Contracts and Its Legal Basis

In the following the main legal basis with respect to the role of collateral is outlined.

2.1 The Role of Legal Versus Economic Ownership

There are two main properties which are of relevance in connection with the role of collateral, the transfer of legal ownership (i.e. the possibility of "re-hypothecation") in contrast to the economic ownership and the value of the collateral.

By entering into a repurchase agreement the legal title to the securities is transferred to the counterparty but economically the securities stay with the selling counterparty since the buying counterparty has the obligation to compensate the selling counterparty for income (manufactured payments) associated with the securities and to redeliver the securities. In case of an Event of Default, both obligations terminate. The treatment in an Event of Default provides that the residual claim is settled in cash and determined taking into account the cash side as well as the value of the collateral. In this case the obligation to redeliver securities transferred as collateral expires and the buying counterparty remains the legal owner. Thus the price risk of the collateral (uncertainty of value) is entirely borne by the legal owner.

In case of (only) economic ownership, e.g. a pledge, this is not necessarily the case, since the treatment in an Event of Default differs as e.g. this kind of "collateral" is part of the bankrupt/legal estate and therefore underlying the insolvency procedure.

Despite these legal differences, the regulatory rules according to Basel II/III and the accounting rules under IFRS also require different treatment of collateral. In general IFRS follows the economic ownership concept irrespective of the legal basis of the collateral while Basel II/III rather follows the legal ownership concept.

2.2 Affected Market Participants

Not all market participants are affected by the same accounting and regulatory rules. Banks have to follow IFRS and Basel II/III rules, while e.g. investment funds are not affected by Basel II/III rules but are governed by investment fund legislation, e.g. UCITS directive. These different legal frameworks for market participants impact the usage of collateral in OTC contracts, e.g. the assets of an investment fund under UCITS represent special assets and the use of repos and cash collateral is limited. In addition, these investment funds have no access to sources of liquidity other than the capital paid which limits the use of cash and the provision of cash collateral in context of derivatives exposure. For example, cash collateral received from OTC derivative contracts has to be kept in segregated accounts and cannot be used for any kind of (reverse) repo transaction. Alternatively, the use of a custodian for optimizing the provision of cash collateral can be considered.

2.3 Financial Instruments Involving Collateral and Standard Legal Frameworks (Master Agreements)

Analyzing the legal basis of collateral facilitates the definition of liquidity and liquidity transformation.

2.3.1 Derivatives Under ISDA Master Agreement

The type and use of collateral are governed in the CSA (credit support annex), which represents an integral part of the ISDA Master Agreement framework[1] and cannot be considered separately. The ISDA Master Agreement forms the legal framework and is applicable for the individual derivative contracts supplemented by the CSA. For example, default netting in the Event of Default (default of a counterparty) is governed by the ISDA Master Agreement including the netting of the collateral which in turn is defined in the CSA. The CSA defines the type(s) of collateral and the terms of margining/posting, while the transfer of the legal ownership is governed in the ISDA Master Agreement. In general ISDA Master Agreements contracted under English Law provide the legal transfer of ownership of the collateral while ISDA Master

[1] ISDA®, International Swaps and Derivatives Association, Inc., 2002 Master Agreement.

Agreements contracted under New York Law do not. In the latter re-hypothecation, i.e. the re-use of the received collateral for counterparties is prohibited.

In case of ISDA Master Agreements under English Law the derivative contracts are terminated in case of an Event of Default and the collateral is taken into account in order to determine the residual claim. The determination of the residual claim is performed independently from the estate of the insolvent party.

2.3.2 Repos Under GMRA

A repo or repurchase agreement under GMRA[2] can economically be seen as a collateralized loan and is typically motivated by the request for cash. In case of repurchase agreements, the legal title to the securities provided as collateral is transferred to the counterparty (buyer) in exchange of the desired cash (purchase price). The credit risk and liquidity of the underlying securities determine the haircut in the valuation of the collateral. Adverse changes in the inherent credit risk of the securities are offset by an increase in haircut and induce in terms of margining additional posting of collateral to the counterparty. At maturity the securities are legally transferred back to original owner (seller) in exchange for the agreed cash amount (repurchase price). In case of a counterparty's default the securities are not returned and the recovery risk of the securities is borne by their legal owner (the buyer).

2.3.3 Securities Lending Under GSLMA

In contrast to a repo, a securities lending under GSLMA[3] is motivated by the need for securities but is (commonly) also a secured financing transaction since the securities as well as the collateral are legally transferred to the respective counterparty. In the secured case the collateral can be cash or other securities.

2.4 Credit and Counterparty Risk Related to Collateral

Consider the case that Bank 1 and Bank 2 enter into a repo transaction, where Bank 2 receives cash from Bank 1 in return for securities. There are two features of importance: Bank 1 needs cash funding, which requires an assumption with respect to the sources of funding, e.g. central bank, deposits. The corresponding assumption represents a component in determining the profitability of the repo. An additional feature is the inherent wrong-way risk within the repo transaction. In this case the

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[2]Sifma, Securities Industry and Financial Markets Association and ICMA, International Capital Market Association, 2011 version Global Master Repurchase Agreement.

[3]ISLA, International Securities Lending Association, Global Master Securities Lending Agreement, Version: January 2010.

wrong-way risk for Bank 1 is defined as an adverse correlation (positive in the example above) between counterparty credit risk toward Bank 2 and market value of the collateral (securities). Assuming a long position in the underlying securities (collateral) for Bank 1, the wrong-way risk constitutes a decrease in value of the securities (collateral) and a simultaneous decrease in credit worthiness of Bank 2. In this case the risk for Bank 1 is the failure of Bank 2 in balancing the collateral posting. Since in a repo transaction the legal ownership is transferred to Bank 1, the net risk position comprises the price risk (in the Event of Default of Bank 2) associated with the collateral (securities) including the haircut and the cash claim (cash loan). A similar rationale holds in case of a short position in securities (collateral) since an event of default affects the ability to post as well as to return posted collateral. Similar considerations hold in case of a (partially) collateralized OTC derivative transaction, e.g. an interest rate swap.

3 Terms of Liquidity and Definition of Liquidity Transformation

Dealing with the concept of liquidity reveals that the term is not defined consistently or not uniformly in financial regulations. A natural way is to adopt legal definitions.

3.1 Terms of Liquidity

There is a variety of definitions for the term liquidity, e.g. meeting payment obligations (liquidity of an entity), liquid marketable securities (ability to buy and sell financial instruments), etc. The analysis above reveals the interdependence of “liquidity” and counterparty credit risk, respectively credit risk. As such liquidity of an entity can be considered as the relatively measured ability for a bank to raise cash from a credit line or in return of collateral which in turn is dependent on the liquidity of financial instruments. The collateral itself is only accepted if the price of the collateral can be reliably determined, e.g. it is traded with sufficient frequency on an active market.

3.2 Comparison of Secured and Unsecured Financing

The best way to illustrate the concept formation of liquidity respectively liquidity transformation is the comparison of unsecured and secured financing in case of a default event. Continuing the example above, the following comparison considers Bank 1 as cash provider.

1. Financial action

   Secured: Exchange of cash versus collateral
   Unsecured: Paying out cash of a loan granted

2. Prerequisite and term of liquidity

   Secured: "Liquid" collateral (price of collateral can be reliably determined)
   Unsecured: Credit line loan illiquid - not marketable

3. Net (relative) risk position in case of default

   Secured: Market value of collateral: Default Probability (issuer of the security received as collateral) $\times$ recovery rate of collateral $\times$ amount of collateral (proximate representation via haircut)
   Unsecured: Recovery rate of cash loan $\times$ exposure at default (EAD)

4. Relation to estate of insolvent party

   Secured: Only residual claim part of the estate of the insolvent party but amount of residual claim is determined independently of the estate of the insolvent party
   Unsecured: Entirely part of the estate of the insolvent party

5. Risk

   Secured: Credit risk of collateral issuer, correlation between counterparty risk and price of collateral (wrong-way risk in an adverse case)
   Unsecured: Credit risk with respect to the borrower

Note that in the comparison above the net (relative) risk position in both cases, for secured and unsecured financing, involves a recovery rate but the associated risk relates to different counterparties. In case of secured financing the default risk is coupled with the recovery risk (price risk) of the collateral and the risk position can be settled promptly in case of a default while in case of the unsecured financing the settlement of the recovery depends on the insolvency process.

This comparison in particular shows that the credit risk toward the counterparty in the unsecured financing transaction being rather illiquid is opposed to the market value risk of the received collateral which is assumed to be liquid in the secured case plus the correlation of this risk and the credit risk of the issuer of the securities taken as collateral. In the adverse case this risk correlation is also known as "wrong way risk".

3.3 Liquidity Transformation

Accordingly considering liquidity as an absolute quantity is not useful but as a relative quantity: a relation between secured financing and unsecured financing, which we term liquidity transformation. This transformation is not independent from credit respective counterparty risk, since each type of financing is associated with a different type of credit risk. The liquidity transformation is dependent on the type of entity and cannot be considered separately from its legal status. A bank has different access and a higher degree of freedom to assign liquidity irrespective of the purpose than, e.g. an investment fund.

4 New Approach to CVA Hedging

The new CVA hedging approach outlined below represents a response to current challenges in banking regulation and reveals the importance of liquidity transformation. The legal-based background described above can be used to explain current challenges of banking industry if in addition to prevailing market conditions the regulatory and financial accounting environments are taken into account. Recent environmental changes have immediate impact on banking business activities concerning counterparty risk and can be summarized as follows:

Regulatory and Accounting Aspects

- CCR (counterparty credit risk) is under scrutiny of regulators and financial accounting standard setters.
- Increased regulatory requirements on bilateral collateralization and clearing.
- Increased (regulatory) capital requirements for banks.
- Increased P/L volatility due to IFRS fair value accounting rules (e.g. recognition of CVA).

Business Impact

- Increased (regulatory) capital affects resp. limits banking business.
- Intensified application of credit risk mitigation by netting, collateralization and hedging.
- Increased demand for secured (collateralized) transactions
- Increased demand for (liquid / high quality) collateral.
- Increased demand for optimization of collateral.

4.1 Issue

During the financial crises regulators and financial accounting setters notified the relevance of counterparty credit risk in OTC derivative contracts. In response to this relevance several regulatory (legislative) initiatives have been undertaken like central clearing, increased regulatory capital, etc. These impacted the business of banking industry in several ways: intensified use of credit risk mitigation techniques and increased demand for secured transactions (demand for collateral, cf. also [3]).

Despite the environmental changes credit risk mitigation is and remains essential to continue banking business. Considering equity as a scarce source, banks are forced to tighten their credit exposure in order to offset the increase in capital charges due to increased costs for CCR and other factors. The tightening of credit exposure limits banking business and increases the demand for credit risk mitigation techniques (including hedging).

The mentioned regulatory changes induce tremendous costs for the banking industry. Therefore, managing credit risk by commonly used CDS hedging strategies becomes expensive in presence of the banking regulation, so credit risk management will be rearranged, e.g. more offsetting positions, avoiding exposures (reducing limits) or transferred ("outsourced") outside the regulated banking sector, so e.g. investment funds are in a favorable position to manage a bank's risks. This also holds for counterparty credit risk following the idea to transfer counterparty credit risk to market participants outside the banking sector that are in the situation to manage this risk economically at lower cost than banks.

Additionally banking industry is faced with various different regulations. With respect to counterparty credit risk a bank is confronted with conflicting objectives resulting from regulatory requirements, i.e. Basel II/III, and financial accounting rules. Therefore, under current regulatory and accounting requirements banks cannot manage counterparty credit risk (CCR) of derivatives uniformly in respect of capital requirements and P/L volatility. This results from the fact that the hedging of counterparty credit risk exposure (in terms of Basel II/III requirements) requires the hedging of current and future changes of exposure, while IFRS only considers current exposure. So a bank is required to hedge more than the current exposure ("overhedging") in terms of Basel II/III. But since hedging is mainly carried out by derivatives as CDS, these CDS cause P/L volatility under IFRS, since derivatives are recognized at fair value through P/L.

As described above secured and unsecured financing is common practice in finance industry and can be observed in counterparty credit risk of OTC derivative contracts. As illustrated below in an uncollateralized OTC derivative trade between Bank A and counterparty B, the parties enter into an unsecured financing relationship. If the market value of the derivative trades of Bank A against counterparty B increases then Bank A is exposed to counterparty credit risk (CVA risk). Bank A implicitly provides counterparty B an illiquid credit line in the sense, that the positive exposure amount ("market value") is recognized as an asset which becomes a legal claim in the Event of Default. This exposure is not a tradable asset but needs to be funded thus it could be interpreted as an illiquid asset. In comparison to standard banking credit business, this credit line is unlimited and varies with the market value of the underlying derivative trades, which implies also unlimited funding. The current focus of discussions and research concentrates on measuring counterparty credit risk by exposure and default probability modeling (CVA risk) and the assignment of the appropriate discount rate for the OTC derivative trades reflecting the FVA. The discussed approaches share the following assumptions:

1. No market segmentation between collateralized and uncollateralized OTC derivative trades.
2. The application of the absence of arbitrage principle, which in particular assumes the unlimited use of liquidity by market participants.
3. Liquidity risk and credit risk cannot be decoupled.
4. The coincidence of counterparty credit risk and credit risk, which can be both hedged by the same type of hedging instruments (credit default swaps (CDS), contingent credit default swaps (CCDS)).
5. The absence of transaction costs, which are represented by regulatory costs (e.g. CVA risk charges according to Basel II/III) and reported earnings volatility under IFRS stemming from fair value accounting of counterparty fair value adjustments and derivative valuation.

These ideal assumptions are not necessarily met in reality, therefore alternative approaches have to be explored.

4.2 Solution

Since banks with significant activities in derivatives markets can be affected quite heavily by the aforementioned issues, a workable solution should solve the build-in conflict of regulatory and accounting requirements. As a result, the solution contributes to an improved competitiveness of the bank in the context of derivative risk management, derivatives' pricing, and support the bank in conducting derivatives business which will ultimately benefit the economy as a whole. Consequently, a potential solution is about developing a financial instrument ("credit risk mitigating instrument") which reduces the Basel II/III CCR capital requirements and CVA risk charge without resulting in additional P/L volatility under IFRS. Such a financial instrument represents a solution to the issues described above since it creates:

- A market for counterparty credit risk exposure
  The positive exposure of an (un-) collateralized derivative portfolio can be considered as an illiquid asset in contrast, e.g. to a liquid issuance of a bank.
- A new asset type—make the derivative claim a tradable asset
  The idea is to make this exposure tradable in exchange for collateral by means of an instrument like Collateral Support Annex (CSA) which directly refers to the possibly varying positive exposure of a derivative portfolio.
- An active market involving banks and investment funds
  In order to increase liquidity and to avoid only a shift of capital charges from one institution to another due to hedging activities for the taken credit risk a transfer to a market participant outside the banking sector is considered.

The outline of a solution follows the liquidity transformation. The unsecured financing for OTC derivatives would be represented by uncollateralized OTC derivatives while secured financing requires corresponding posting of collateral. Pursuing the

Fig. 1 Secured OTC derivative transaction

aim of decoupling liquidity and counterparty risk, at least three parties are necessary to involve as demonstrated in the analysis on repos above. Therefore, the aim could not be achieved by cash collateralized bilateral OTC derivatives commonly used in the interbank market, since there is still a one-to-one correspondence between liquidity requirements (e.g. cash collateral postings) and counterparty risk. Additionally a bilateral CSA assumes that both counterparties have unlimited access to liquidity, which represents a difficulty if counterparty B is a corporate according to its limited access to collateral/cash. Therefore a secured financing transaction for CVA hedging has to be structured differently.

The secured financing transaction outlined in Fig. 1 involves a third party "Default Risk Taker" C, who is posting collateral to Bank A on behalf of counterparty B, i.e. whenever the value of the derivative trade is positive for Bank A. This transaction represents a tri-party CSA and works similar to a margining. The transaction between "Default Risk Taker" C and Bank A is an asymmetric contract, since if the value of the derivative trade is negative for Bank A, no collateral is provided to or by Bank A. In case of a default of counterparty B the posted collateral is not returned to "Default Risk Taker C". The structure described above represents the appropriate complement for a bilateral uncollateralized OTC derivative transaction.

The structure reveals the concept of liquidity transformation including a decoupling of liquidity and counterparty risk, since by using the contract the unsecured financing transaction is transformed into a secured financing transaction. Referring to the comparison of unsecured and secured financing described above (cf. Sect. 2.3), the proposed structure goes one step further by linking both market segments and transforming liquidity within one single transaction. By definition of the liquidity transformation, the transaction exchanges different types of credit risk.

4.3 Application

The table in Fig. 2 shows the contemplation of the new CVA hedge structure (cash collateral with contingent financial guarantee, "CCCFG"; for more detail refer to [5]) to existing credit risk mitigation techniques applied in the banking industry. Its main features are summarized as follows:

- The proposed structure represents a credit risk mitigating instrument, which reduces the Basel II/III CCR capital requirements CVA risk charge, since the cash collateral provided by a third party is permitted under Basel II/III requirements and reduces the exposure according to Basel II/III.

Aspects	Netting	Bilateral Collateralization	Credit Default Swap (CDS) based hedges	Contingent CDS
Economics	Reduction of the risk position by netting of exposures	Counterparty risk is reduced by posted (cash) collateral	Hedges of the counterparty default risk	Hedges of the counterparty default and exposure risk
Operational	▪ Legally enforceable ▪ Default netting (ISDA standard)	▪ Changes in OTC derivative contracts (CSA) ▪ Requires liquidity (an issue for corporates)	▪ Delta Hedging required ▪ Liquid CDS	▪ No Delta Hedging required ▪ Less liquid than CDS
Financial Accounting (IFRS)	IAS 32 requires simultaneous payment- and default netting	Posted collateral reduces fair value volatility	Fair value accounting through P&L	Fair value accounting through P&L
Regulatory (Basel II/III)	Basel differs from IFRS due to default netting	Reduces derivative exposure (credit risk mitigation)	Credit risk mitigation if requirements are met	Credit risk mitigation if requirements are met

Fig. 2 Current and new approaches for credit risk mitigation in banking industry

- Accordingly there is immediate regulatory capital relief, which results in an immediate saving respectively reduction of the cost of equity.
- The proposed structure simultaneously qualifies as a contingent financial guarantee such that there is no additional P/L volatility under IFRS. In particular the financial guarantee accounting under IFRS applies to the proposed structure by considering the case of default. In case of a default of OTC derivatives contracted under ISDA the final claim is determined. The financial guarantee under IFRS comes into effect only at default—not before—and "guarantees" the value of the final claim, which is recognized at amortized cost and physically transferred to counterparty C in return for cash to Bank A. The final claim takes into account the posted collateral until the Event of Default. For a more detailed description refer to [5].
- As becomes apparent from the table above the new CVA hedge structure is a separate financial instrument. This cash collateral with contingent financial guarantee ("CCCFG") differs from a "traditional" CDS/CCDS, since the collateral postings are directly related to the counterpartys exposure. In case of a CCDS the cash collateral refers to the CCDS contract itself reflecting its value and there is no direct legal link to the exposure subject to hedging by the CCDS. Additionally CCDS represents derivatives in terms of IFRS and not necessarily qualify as credit risk mitigation instrument under Basel II/III. If a CCDS qualifies as credit risk mitigation instrument it applies to the Basel II/III PD, while the CCCFG directly affects the exposure.
- Operationally the new CVA hedging instrument is more effective and less costly than CDS delta hedging approaches, since a constant adjustment of a hedging position using CDS induces transaction costs and depends also on the gamma of the risk position. Accordingly the hedge position is never "perfect".

- The approach is flexible with respect to counterparty risk profiles, since it applies to linear and nonlinear exposure profiles.
- The legal framework of the approach is based on ISDA, which ensures the operational effectiveness in terms of legal certainty and the recognition in front office IT systems in order to process the transaction.
- It has to be noticed that investment funds have to observe certain rules and regulations which come with the specific fund format and domicile. For example, funds fulfilling the highest standards are limited to invest in eligible assets which are characterized by sufficient liquidity in order to ensure that the fund is in a position to meet potential redemptions. Bilateral transactions that are illiquid by definition require a buy-and-hold investment strategy which may not be suitable for all investment funds.

4.4 Example

In the following for the sake of simplicity only a qualitative example is provided, since by comparing the induced costs the CVA hedge already indicates its profitability.

- Bank A holds a portfolio of uncollateralized derivatives (e.g. interest rate swaps (IRS)) with Counterparty B (e.g. a corporate) a netting set is considered.[4]
- Bank A enters into a CVA hedge transaction with Investment Fund C who is taking over credit (counterparty credit risk of B) and market risk and provides liquidity with reference to the uncollateralized derivative transaction(s) between Bank A and Counterparty B in terms of the cash collateral postings to Bank A. The transaction between Investment Fund C and Bank A is a unilateral (asymmetric) collateral contract in favour of Bank A (and on behalf of Counterparty B). The transaction chart follows Fig. 1.
- In the following table the impact for Bank A with and without CVA hedge is summarized:

With respect to the risk illustrated in the first line in the table above, the CVA hedge transaction mitigates entirely the risk of Bank A by transferring the risk to investment fund C. This results from the posted cash collateral of Investment Fund C to Bank A on behalf of counterparty C. Comparing the induced costs (second line in the table above) reveals that the (uncollateralized) derivative business is exposed to regulatory and cost of equity charges as well as funding costs. In case of the CVA hedge transaction all these costs are inapplicable, since the posted cash collateral by Investment Fund C to Bank A on behalf of counterparty B leads to entire regulatory capital and cost of capital relief and serves as funding to the derivative exposure between Bank A and counterparty B. On the other hand Bank A pays a fee to Investment Fund C for taking over the counterparty credit risk of B and also interest

[4]In order to keep legal and operational complexity in an event of default low one netting set is considered.

Bank A	without CVA Hedge	with CVA Hedge
Risk	▪ **Counterparty Credit Risk of B,** ▪ **Market Risk and** ▪ **Liquidity/Funding Provision**	**0**
Induced Costs	▪ **Costs related to Counterparty Credit Risk B** ▪ **Cost of Equity on Regulatory Capital** (Basel II/III charges)) ▪ **Costs associated with the Leverage Ratio (Basel III)** ▪ **Costs of Funding costs** (depending on Bank A's rating and funding model assumption)	▪ **CVA hedge spread („fee payment")** ▪ **Interest on received cash collateral**
Cash Flows	▪ **No Default:** derivative payments ▪ **Default:** residual claim times recovery rate	▪ **No Default:** derivative payments, fee payment, receive cash collateral, pay interest on cash collateral, repayment of collateral at maturity (since exposure reduces to zero). ▪ **Default**: physical delivery of residual claim vs. keeping posted cash collateral and receiving difference payment up to the amount of residual claim

Fig. 3 Comparison derivatives exposure with and without CVA Hedge transaction from bank A's perspective

on the posted cash collateral. Describing the associated cash flow profiles the two situations, default and non-default of the counterparty, are distinguished (third line in the table above). While in case without CVA hedge structure the cash profiles are straightforward, with CVA hedge transaction in addition fee and interest payments on the collateral have to be considered in the non-default situation. In the event of default of counterparty B, the residual claim of the transaction is physically delivered to Investment Fund C in return for cash equal to the notional of the residual claim. This procedure follows standard ISDA rules (Fig. 3).

5 Conclusion

The new CVA hedging instrument is used in order to transfer counterparty credit risk to entities which are able to manage the risk on an economic basis at lower cost. Investment funds can act as "credit risk taker" and manage counterparty credit exposure at a lower cost than banks, since investment funds are not subject to regulatory capital requirements according to Basel II/III. It has to be noted though that an implementation of the solution described above requires an intense capability and knowledge of dealing with derivatives at the risk taking investment funds. On the other hand, since investment funds are not subject to the same regulations as those for banks described above they may become a natural partner for banks in this context.

The proposed structure bridges the difference between capital rules and financial accounting standards in order to optimize capital requirements and charges for CVA. This is achieved by its liquidity transformation property—the liquidity and credit risk transformation of the counterparty's exposure—and by meeting the Basel II/III and

IFRS requirements: simultaneous CCR capital and CVA risk charge relief as well as reduced P/L volatility in IFRS resulting from CVA accounting. While the objective outlined herein is predominantly to provide a suitable solution for CVA issues in context of derivatives transactions, it may also create interesting opportunities for investors of the risk taking investment funds.

This solution also contributes to valuation and the discussion on FVA and CVA, since it requires the pricing of the collateral between counterparties "at arm's length". This price determines the discount rate by applying the absence of arbitrage principle. As a consequence FVA is disentangled from CVA by using the proposed structure as a mean.

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FVA and Electricity Bill Valuation Adjustment—Much of a Difference?

Damiano Brigo, Christian P. Fries, John Hull, Matthias Scherer, Daniel Sommer and Ralf Werner

Abstract Pricing counterparty credit risk, although being in the focus for almost a decade by now, is far from being resolved. It is highly controversial if any valuation adjustment besides the basic CVA should be taken into account, and if so, for what purpose. Even today, the handling of CVA, DVA, FVA, ... differs between the regulatory, the accounting, and the economic point of view. Eventually, if an agreement is reached that CVA has to be taken into account, it remains unclear if CVA can be modelled linearly, or if nonlinear models need to be resorted to. Finally, industry practice and implementation differ in several aspects. Hence, a unified theory and treatment of FVA and alike is not yet tangible. The conference *Challenges in Derivatives Markets*, held at Technische Universität München in March/April 2015, featured a panel discussion with panelists representing different points of view: John

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Hull, who argues that FVA might not exist at all; in contrast to Christian Fries, who sees the need of all relevant costs to be covered within valuation but not within adjustments. Damiano Brigo emphasises the nonlinearity of (most) valuation adjustments and is concerned about overlapping adjustments and double-counting. Finally, Daniel Sommer puts the exit price in the focus. The following (mildly edited) record of the panel discussion repeats the main arguments of the discussants—ultimately culminating in the awareness that if everybody charges an electricity bill valuation adjustment, it has to become part of any quoted price.

Keywords Counterparty credit risk · Credit valuation adjustment · Debit valuation adjustment · Wrong way risk

1 Welcome

Matthias: Welcome back from the coffee break. After the many interesting talks we already enjoyed today, we will now continue the conference with a panel discussion on current issues in counterparty credit risk. And we are very proud to present you such prestigious speakers on this topic—our anchorman Ralf Werner will introduce them to you in a minute (Fig. 1).

We hope that this discussion will provide you with insights on the current discussion about CVA, DVA, FVA, etc. that go beyond what you can read in scientific papers. In my personal view, these valuation adjustments are a special topic in financial mathematics, because they are not simply expressed by formulas some mathematicians invent and you implement in a spreadsheet. In contrast, these adjustments are chal-

Fig. 1 View on the panel. From *left* to *right*: Matthias, Ralf, Daniel, Christian, Damiano, and John

lenges a whole bank has to work on as a team, because they can involve different departments, different asset classes, different trading desks, the IT-infrastructure, lots of data, etc. Hence, it is not something that is “done” after a scientific paper has been published. Moreover, there is no consensus—neither in academia nor in practice—on what adjustments should be used and how they must be computed. In this regard, I am very happy to see representatives from the financial industry as well as from academia gathering for this discussion.

I will now pass the microphone to Ralf Werner who will be our anchorman. Ralf is professor for “Wirtschaftsmathematik” at Augsburg University. Prior to this he was professor at the University of Applied Sciences in Munich, and prior to this he worked for several financial institutions—most of which have defaulted.

Ralf: Yes, indeed. Three in total.

Matthias: In any case, he gained quite some experience—practical and theoretical—with credit defaults that he is now sharing with you. Thank you very much Ralf!

Ralf: Thank you, Matthias, and a warm welcome to everybody from my side. I’m very honoured to chair this discussion. I don’t think I will need to do much because we already had an excellent warm-up over lunchtime, and my experience is that these four experts in the panel won’t need much input from my side to keep the discussions controversial, yet fruitful.

For the unlikely event that the discussion might get stuck, we have prepared a few additional questions. Further, any question or comment from the audience will be addressed immediately, i.e. we will interrupt whenever possible and whenever meaningful.

The idea is that each discussant has about ten minutes to address one or more topics he deems important. I’ll try to dig a bit deeper and if you like you join in asking and eventually after 15 minutes we hand over to the next discussant. This means that in one hour we should be able to pretty much cover everything concerning DVA, FVA, CVA, multi-curve, whatsoever, within the scope of the conference.

Let me now introduce the participants in reverse alphabetical order. I would like to start with Daniel Sommer to my left. Daniel is not only representing one of the main sponsors of this conference, but he’s further representing almost 20 years of experience in financial consulting. Daniel is a member of the financial risk management group at KPMG, and for more than ten years he’s responsible partner for risk methodology. Daniel holds a PhD on interest-rate models from the University of Bonn, he has published several papers, he is working for all major banks in Germany, so in short he comes with a broad experience of what’s going on in the market. I think this is an excellent opportunity for us to challenge his knowledge and his experience.

On the other end of the panel we have John Hull. I both asked John as well as Damiano during the lunch break, and we agreed that re-introducing both of them after we had such great and detailed introductions this morning prior to their talks is saying the same thing twice over. John will hopefully talk a bit about FVA, and I assume all of you have read his 2012 paper, see [7]. If not, my introduction may last another 60 s, so please at least run through the abstract of this great paper. It’s an excellent work, starting heavy discussions in the community, I’d like to say—fruitful discussions, raising lots of interesting questions on FVA: *Is it really there? Should it be zero or not?* For me, somehow, the discussion is not yet over, so I am looking forward to what John has to say.

Besides John, between Daniel and Damiano, we have Christian Fries, our local panel member from the LMU. Christian was appointed professor for Financial Mathematics a few years ago. I should emphasise that besides his academic duties he is still mainly working at DZ BANK where he is responsible for model development, heading this department. Of course, I think you all know Christian from his open-source library and from his book, resp. on Monte Carlo methods in finance [5], and I'm sure we will gain a lot of insight from this mixed-role in practice and academia.

And, finally, we have Damiano Brigo with us, whom I would like to start right away without any further notice, so please, Damiano.

2 Damiano Brigo

Damiano: Okay, thank you. I made some of the points during the presentation, but I think it's worth summing up a little bit what's been happening from my point of view. I worked on what is now called CVA since, I think, 2002 or 2003 at the bank. At the time it was called counterparty risk pricing, not CVA, and nobody was really very interested because the spreads were small for most of the trades and so on, so the work was recycled a few years later, especially in 2007. But as we did that it was clear that this was only a small part of a much broader picture where we had to update the valuation paradigms used in investment banking and not only there (Fig. 2).

The big point that seems to come out, at least methodologically, from that big picture is nonlinearity, which shows up in a number of aspects that can or may be neglected in many cases but not always. So one of the aspects is the close-out, what

Fig. 2 Damiano Brigo giving his presentation on "Nonlinear valuation under credit gap risk, collateral margins, funding costs, and multiple curves"

happens at default. What do you put in your simulation? Should you use a risk-free close-out, where at the first default you just stop and present-value the remaining cash flows without including any further credit, collateral, and funding liquidity effects? Or should you rather use a replacement closeout, where those effects are all included in the valuation at default?

This is a big question. If you go for the replacement, then the problem as we have seen becomes recursive, if you like, or nonlinear from a different point of view. And that's not because we mathematicians are trying to push BSDEs or semi-linear PDEs on you. It's simply because of the accounting assumptions. It's a basic fact, an accounting rule that says that you have to value your deal at default using a replacement value. This is a simple accounting rule, but it translates into a quite nightmarish nonlinear constraint in the valuation. Then when borrowing and lending rates are asymmetric in financing your hedge, if they are justified to be, then you have another source of nonlinearity because to price these costs of carry you need to know the future value of the hedge accounts and of the trade itself. And this induces another component of nonlinearity (see [4] and [3]).

If it's there or not depends on the funding model you adopt for your treasury. If the trading desk is always net borrowing and possible liquidity bases are symmetric, you don't have that, and you can more or less have a symmetric problem, but if it's not net borrowing then you do have an asymmetry in the funding rate: one is the credit risk of your bank, one is the credit risk of the external funder, plus liquidity bases. So, we all know that borrowing and lending don't happen at the same rates usually (well, we experience it personally, at least).

So, the nonlinearity is there. The big question is *Should we embrace it or keep it at arm's length?*, because it makes things too complicated in practice. The answer is the second one, and basically if there is any real nonlinearity in the picture, the required methods like BSDE's or semilinear PDE's are very hard to implement on large portfolios in an efficient way that ensures that you can value the book many times during trading activity very quickly—especially because nonlinearity means the price or the value is not obtained by adding up the values of the assets in the portfolio, so you need to price the portfolios at all the possible aggregation levels that you need, and if each component of such a run is slow, you can imagine what kind of operational nightmare you get into. So I don't think it's realistic or feasible at the moment that we embrace nonlinearity. We need to linearise, which means, in the two cases I mentioned, we assume that borrowing and lending rates are the same, which is true for some funding policies, and you also assume that you don't use a replacement closeout at default in the CVA calculation of the valuation adjustment for credit.

Then the other problem I would mention is keeping all the risks in separate boxes with a label on each box: *CVA: this is credit risk, FVA: this is funding cost, LVA: this is collateral cost* and so on. This is a little misleading because these risks interact in the way that I just described. Each cash flow involves the whole future value which depends on all the risks together. The classification in boxes is useful managerially because you want to assign responsibility in an organisation; you cannot have everyone responsible for everything unless you have a very illuminated kind of workplace, but if you don't, you want to assign responsibility for credit risk to the CVA desk, and maybe the funding costs to a different team in the CVA or XVA desk and so on. But if these aspects are so connected as I said, it's very hard to separate the risks in different boxes. Wrong-way risk is another aspect of the fact that the dependence makes the idea that you can have risk taken care of separately by the CVA desk for credit risk and by the traditional trading desk for the trade main market risk not very realistic. To some extent, you can do it, but it's not precise.

So, these are labels that we apply in order to be able to work operationally in a realistic setting, but they don't have the amount of rigour or precision that we would sometimes think they have in practice. So, should we, again, monitor and watch out for manifestations of nonlinearity like overlapping adjustments? We saw that in some set-ups DVA almost completely overlaps with the funding adjustment. And, so, should we be aware of these and avoid the double-counting, or should we forget it and just compute the different adjustments, add them up, and forget about all these overlaps and analyses?

I think it's important to have at least an initial understanding of these issues before throwing ourselves into very difficult calculations. There are many other things I could say. The nonlinearity makes the deal pricing very difficult—in funding costs especially. When you don't know the funding policy of the other institution, or maybe you don't agree with the funding policy of the other institution, but you're still asked to pay their funding prices, you might object and go to another bank, or you might in turn say, *I also have some funding costs, and I want to charge you.* And there is no transparency in the funding model of the treasury process. How can bilateral valuation be achieved in a transparent way? This is another problem.

So a number of authors conclude by saying the funding-adjusted value is a value; it's not a price. You can use it for profitability analysis internally, but you shouldn't charge it outright to a client because it's hard to justify this charge fully, as we have seen. On the other hand and this is the final point I want to raise, which is kind of a meta-topic, I would like to talk about the self-fulfilling aspect in financial methodology, that if two or three top banks start doing something, everybody else follows because this becomes the new standard. *Top bank A is doing this, top bank B is doing this, so we have to do this as well.* And then even if something is not justified based on financial principles, or it is not reasonable methodologically or even mathematically it doesn't matter because if you don't do it you place yourself out of the market.

This is very frustrating for a scientist, for someone who thinks there are underlying sound principles behind what's going on, but in the end you are forced to set the problem aside, because that's what the market is doing, and if you don't follow, you are automatically out.

I would like to conclude with that kind of provocative point, and I'm sure my colleagues will have more interesting points to make on it. Thank you.

Ralf: Thank you, Damiano. Is there anyone in the panel who wants to take up one of these points? Or in the audience?

Christian: I'd like to ask you, Damiano: you said *close-out* value. This is a very important discussion. So, from my point of view, is this an issue for the lawyers, or is this an issue for financial mathematics? What would you say?

Damiano: I think it's an issue for both in a sense, in that the lawyers should tell us if it makes sense to have this close-out there or not based on legal considerations. In the end, I don't think we can decide this with mathematics alone. With mathematics we can say, *If you adopt this close-out, the valuation problem is like this, and if you adopt this other, the valuation problem is like that*, but the decision must be taken based on accounting, financial, and legal principles, not based on mathematics.

I would say that the regulations should converge. We've had ISDA pushing a little towards the replacement close-out, but very mildly. ISDA wrote in 2009 that in determining a close-out amount, the determining party may consider any relevant information, including quotations (either firm or indicative) for replacement transactions supplied by one or more third parties (!) that may take into account the creditworthiness of the determining party at the time the quotation is provided (notice the use of *may*). In the end I think it's a decision for the regulators and the policy-makers. We discussed this earlier, but let me be more explicit. Are you thinking, with respect to your operational model, let's say, when the deal has defaulted do you think to actually replace it with a new one or simply to liquidate everything and close the position? This is the real question. If you think to replace it with another physical deal, and you intend to re-start the trade with another contracting party, then you should assume a replacement close-out. If you're thinking of liquidating the position, then it stops here, with a cash settlement, and you may use a risk-free closeout. However, from the point of view of continuity, mathematics seems to suggest that you should include the replacement because you value the trade, mark it to market every day, including credit and funding costs, and all of a sudden at the default event, you remove this. You create a discontinuity in valuation this way, which shows up as some funny effect, which I don't want to go into right now.

I think mathematics gives you some hint, but it's really a regulatory / accounting / legal discussion that we should have, and then use the maths to include the outcome properly into the valuation. That's my view.

Ralf: Let me exaggerate a bit, but will this lead into a situation where your line of reasoning is also applied to mortgages or government debt? Would Greece say, *I'll only pay 60 because I'm valued at 50 anyway, so this is the right replacement value?* Will this lead us into such kind of discussions?

Damiano: That is very hard to model because when you have such a large market effect, then the close-out itself could change the economy basically, so I don't think it's very realistic in that sense.

In fact, we found in the published paper [1] that there is no superior close-out. If you use the replacement close-out, you have some advantages in terms of continuity and consistency, but you'll have some problems when the correlation goes up towards the systemic risk scenario. In that case the risk-free close-out becomes more sensible economically. There is no clear-cut case, and you cannot make a regulation that depends on correlation or the level of perceived systemic risk switching from one close-out to the other. Can you imagine what happens when you are in the middle. I don't even want to go there (see [2]).

So I think we have to be very careful about the maths, and we have to clearly understand which level of aggregation, of size, we're talking about, and in the case of a country, I think that would be quite dangerous.

Daniel: I agree.

Damiano: At the global derivatives conference a couple of years ago, I was talking to some of the banking quants and I said, *Which close-out are you using?*, and they would say *We're using the risk-free close-out because that's the only thing we can implement on a large portfolio.*

Ralf: I agree. I've heard this is hidden in the recovery rate, anyway.

Christian: So maybe I'd like to comment or offer a question on this self-fulfilling prophecy because I do not understand it. I do understand that if there is some idiot in the market who's trading options at the wrong price, then I can use his incorrect pricing to have an implied volatility. Hence, I can imply his dumbness into my model and that's fine. But now you say that everybody is doing it, so we should do it. And I believe this does not apply to FVA. For me, FVA is a real cost and, for example, the market will now decree not to account for FVA, I still picture that I have lost, for example, if I issue a bond at LIBOR plus spread, and just put the money to the ECB for a zero interest rate, I have a loss, right? So, then I would say I would rather go out of the market instead of making the loss.

Damiano: Okay, so let me ask you another question. Suppose electricity bills become prohibitive and electricity skyrockets, will you start charging your client an electricity bill valuation adjustment because that's a real cost you're having? Or will this be embedded in the prices like in the old days.

Christian: It is.

Damiano: When you go and buy some bread from the baker, the baker doesn't charge you a running water and electricity bill valuation adjustment because he needs some water to run his bakery, you know ...

Christian: Yeah, but if you go to the bakery, he charges you such that he is covering all his costs.

Damiano: That's right.

Christian: It's just not transparent.

Damiano: That's right.

Christian: But the cost is inside the price.

Damiano: But then if you add these valuation adjustments one by one, one after the next, every year a new one, with the nonlinearity effects we see that they possibly overlap, you are overcharging sometimes, and this is not good, and that's what I feel is happening.

KVA. Think about it. KVA is a valuation adjustment on capital requirements, but the future CVA potential losses trigger capital requirements—so you have your valuation adjustment on a valuation adjustment. This is getting out of hand.

Christian: This point I understand, but that is regulatory …

Damiano: But going back to the *self-fulfilling prophecy*, the other thing I wanted to say: think about base correlation. For CDOs base correlation is a model. You use a Gaussian copula, flatten 7,750 correlations into one, apply different flat correlations to each different tranche on the same pool. To explain a panel of 15 CDOs you have 15 different and inconsistent models and then … I kid you not, once at an international conference I met one quant from a top bank who was lecturing about base correlation along the lines of *here's an example of calibration, this is a great model, you should use it, CDOs are great, invest in this.* And when I asked, after his talk, I have some questions for you about this model, he'd say, *Oh, I'm the marketing quant. I don't do models really*. And I said, *Take me to your leader!*, meaning the real quants then, and he said, *Oh, you cannot talk to them; they don't talk to the public. My function is to convince people, investors and the market that this is a great model, this is a great product, and everybody must come into this market.*

However what you are saying is partly true. If the market is kind of complete in a way, then by hedging your strategy according to the correct hedge you can prove that your price is right against an opponent, but if the market is largely incomplete, this is very hard to do. And this is what we look at when we look at funding costs. We don't know the hedging or the funding policy of another entity. It's not transparent. You don't know what they're doing, how they're financing, their short-term/long-term funding policy, their internal fund transfer pricing, their bases. You don't know many things.

Christian: This is exactly the point. The market is not complete here, and I cannot pass this risk to someone else. This is my example with the volatility: if someone is on the wrong volatility I can pass this risk to him, but with my funding it's still my risk and it's my cost to cover it. I believe it has to be in there. If you make it transparent, it's something different, maybe.

Damiano: Okay, but then you have to really watch out for the overlap as you add new risk. For example, in some formulations if you take into account the trading DVA and also the full funding benefit, you have the same thing twice. You have to be very careful there. So this practice of adding a new adjustment on top of the old ones every year is very dangerous because you may miss some of the overlaps. The banks are paying attention to it; it's not that bad. If it develops in the fact that in ten years we'll have 15 new valuation adjustment, this will be out of control.

Audience member: I have a question because I really like this bakery example, so let's say you have one bakery who sells bread for 1.80 and who doesn't have very high electricity costs, and you have another bakery which sells it for 2.00 because they have a lot higher electricity costs. So what is the market price, then? Is it 1.80?

Damiano: The price, if you look at a clean price versus an adjusted price, the price would be the clean price without costs. But then, of course, the price is adjusted into an operational price that takes into account the bill, but the bill is not quoted explicitly, it's embedded in the price, so that if you think this baker is too expensive, you'll go to the other one. Maybe the other one is out of town, so they have lower costs because of that.

But in the other industries, we always knew that the price of a good that you end up buying depends on many circumstances that are not in a theoretical price in a way. Somehow ironically, part of the finance industry arrived at this realisation quite late. But that's another matter. I took too much time and I don't want to monopolise this panel.

John: Don't forget that we have bid-offer spreads in this industry. Those bid-offer spreads are designed to cover overhead costs, so adding in costs for electricity and other things is not really the way to do it.

Ralf: Thank you, John. Let me hand over to Christian. Christian, maybe you want to tell us your opinion on what's going on in financial institutions at the moment. Maybe with some more focus on the practitioner's point of view.

3 Christian Fries

Christian: You've asked me to make a few statements and I take the role of the practitioner.

I have the same opinion as Damiano, but I'd like to make the point that I don't like the adjustments. And why? Maybe because the word "adjustment" already implies that you did something wrong. If I have to adjust something, it tells me that the original value is wrong. For example, in my car there is this small device that tells me how long it takes to get from Frankfurt to Munich, and what would I like to see there? With my car it takes five hours. I could also fly. It would take one hour if you take the plane, but you have to add four hours' adjustment. So I would prefer just to see the five because the five is correct. The one hour is no information for me.

Then, let me give you another example. Consider a swap which exchanges LIBOR against a fixed rate, and this swap is traded at a bank, usually at a swap desk, sometimes it's called flow trading. And then we have another swap that exchanges LIBOR capped and floored against the fixed rate; this swap is called a structured swap, and it's traded at a different desk. This desk is sometimes called nonlinear trading desk because these people are doing the nonlinear stuff, but except sometimes for information purposes, we do not express the price of the swap as the price of the linear product plus the nonlinear trading premium. So there is no such thing as an option-valuation adjustment, so we do not have an OVA or something like that.

Daniel: Going back a few years, people tried to calculate option-adjusted bond spreads.

Christian: Yes, I know, and I am sometimes reminded of it. And so there is one desk in the bank that is taking the responsibility for all this complex stuff. This desk is also making transactions through the swap desk because the desk needs to hedge its interest rate risk, so he's hedging out all linear stuff to the other guys, and he keeps all the nonlinear risks. Let me make a remark about FVA; I will come back to CVA. For me FVA has a strong analogy to cross-currency, to multi-currency models—at least if you have the same rate for borrowing and lending. Each issuer has its own currency. So what is his currency? His currency is the bond he's issuing. Everything has to be denominated in his own interest rate, his funding rate. There are even instruments on the market which profit from this arbitrage between two banks which have different funding. These are the total return swaps where one bank with poor funding goes to another bank with good funding and they exchange funding and they both profit from this deal. I mean, the market for total return swaps is currently dead because funding is for free, but these things existed. I have a little paper with my colleague Mark Lichtner on this (see [6]).

This currency analogy: we had this in multi-currencies for years. We know how to value instruments in different currencies, and we have the same phenomena in currencies. For example, the cross-currency swap exchanges a floating rate in one currency for a floating rate in another currency. From the theory, this should be zero: both are floaters which are at par, but cross-currency swaps trade at a premium. There is a cross-currency basis spread. The reason is that there is a preference in the market, that one likes to finance oneself in U.S. Dollars and not in Euros (or vice versa), so, for example, a Euro bank would prefer to go to Euro financing instead of U.S. Dollar financing. I believe that FVA is something very natural. Also in mathematical theory it has been there in this currency analogy since, and it should be recognised inside the valuation because we wouldn't value Euro derivatives using the U.S. Dollar curve, would we?

One more word to CVA. If I'm provocative, I would say, like Damiano already pointed out, counterparty risk isn't something new. We had a defaultable LIBOR market model years ago, and counterparty risk was used years ago maybe only for credit derivatives, but it's not so new, and what is actually new here is that we suddenly have to look at netting. So the big change for me in this valuation adjustment topic is that we are talking about portfolio effects. What Damiano said this morning: the sum of each individual product valuation doesn't give you the value of the portfolio. So you have portfolio effects, you have to value everything in a single huge valuation framework, but if you define all the products of a bank as a portfolio, as one single product—I believe that the theory to be able to do this is actually to some extent known—the big problem is how do you implement numerically what you do on the computational side. For me this is the main motivation for these valuation adjustments. It is because we have computational problems, and we like to decompose the valuations into valuations for which we can sum up the products.

Going back to FVA, I do not understand why many people still use the risk-free interest rate as the basis for this valuation, for your reference valuation—because, first of all, I don't believe there's such a thing as a risk-free interest rate; it's just a misnomer. And wouldn't it be better to keep the adjustments as small as possible such that the price which you calculate is already as close as possible to the true price? So, for example, my navigation system in the car tells me, from Frankfurt to Munich you need four hours and thirty minutes. Okay, when I drive you need five hours and thirty minutes, but it gives me a good proxy. The proxy is using the average information available.

So coming back to Damiano's talk, maybe we should simplify things. I like to have things simplified, and my question is how can you simplify things such that you can implement them in a bank. For example, we can simplify and say that treasury uses an average funding rate which is in the middle of the bid-offer, and we use that rate to calculate the funding costs so that we have symmetry there and so on.

Finally, I would like to have just one desk where nonlinear effects are managed. We could have this set-up, so the question is how can we have this set-up in a bank. We could have this set-up if we have internal transactions in the bank, and these transactions are fully collateralised. So we have these linear traders who trade collateralised transactions with this nonlinear trading desk, and the nonlinear trading desk has the residual.

My conclusion is that I would like to have one formula or one model which gives me the true price, and then we can set up internal transactions, but what is the good way to set up these internal transactions such that we can implement this in a bank? This is my concern.

Audience member: Talking about implementation in the bank: What can you implement? Where is banking nowadays? CVA, we have all the data for CVA, I assume. No clue on wrong-way risk on these correlations you need and you already think about FVA and adjustments on adjustments but still didn't manage to find a decent proxy for wrong-way risk? The question is, are we looking and are we solving the right problems? What is your impression?

Christian: The data is actually the critical thing here. We can include more and more effects in a nonlinear trading valuation framework by improving the model—for example like the approaches we have seen here including wrong-way risk, copulas, whatever, but the problem is that we actually do not have the data to calibrate the model.

For example, going back to John's talk this morning, I have a little comment here: you'll see the effect of this multi-curve switch from LIBOR to OIS, but in this calculation there is an assumption. The assumption is that the swap, which is LIBOR-collateralised, so we use LIBOR discounting, trades at the same swap rate as the swap rate that is OIS-collateralised, so we use OIS discounting, so if you have the same rates for the swap, you get different forward rates. That's what we saw this morning.

The problem is you do not observe the swap rate for a LIBOR-collateralised swap. So it could even be that the swap rates are different and the forwards are the same. If we value, for example, an uncollateralised product, we do not even know what the correct forward rate is because we would need the uncollateralised swap to calibrate this forward rate. Data already start at the very beginning. The problem is data.

Ralf: Do you agree, Damiano?

Damiano: I talked to one of the CVA traders at a top tier 1 bank. They told me they have what they call zero-order risks in mind more than cross-gamma hedging. What they don't have for many counterparties is a healthy default-probability curve because there's no liquidity in the relevant CDS, so maybe they have a product with the airport of Duckburg, and this airport hasn't issued a liquid bond and there is no CDS. Where do you obtain the default probability? From the rating? But that's a physical measure, not a risk-neutral measure. And then the wrong-way correlations: you should use market-implied correlations because you are pricing, but then, where do you get them? It's almost impossible to get them for many assets, and also, finally, I would say that with CVA, you're right—we talk about KVA, but CVA is still very much a problem—and there is what I call payout risk, so depending on which close-out you use, and whether you include the first default check or not (some banks don't, because by avoiding it you avoid credit correlation, which is a bad beast in many cases), so depending on the type of CVA formula you implement—you have five, six different definitions of CVA—and that is payout risk. With old-style exotics, you had a very clear description of the payout, then you implemented the dynamics; you would get a price and hedge, and that would change with the model, and that would be model risk. Now with CVA we have payout risk. We don't even know which payout we are trading exactly, unless we have a very precise description of the CVA calculation.

But it's not like when you ask another bank, *What CVA charge are you applying to me?*, they tell you *It's a first-to-default inclusive, risk-free closeout …* They don't tell you that. … *And I'm using this kind of CDS curve.* Sometimes they don't tell you that, and you don't know.

Ralf: Daniel, do you have the same experience?

Daniel: Absolutely. I think even as many banks are talking about FVA these days, I think CVA is still an unresolved topic, and our observation is that even in a small market like the German market, there are a lot of different approaches taken by the banks to calculate CVA. The problem is becoming more difficult by the minute as the observable CDS prices, or tradable and liquid CDS prices get fewer and fewer. So this is an issue that gets more complicated by the minute.

And then another observation: we had a talk about wrong-way risk this morning, and we learned about the difficulties that this involves, and not surprisingly it's our observation that many banks are far from including wrong-way risk in their CVA calculations, so there's a long way to go before even CVA is settled.

Ralf: Okay, thank you very much, Daniel.

John: Maybe I should just respond to the point that Christian made about my presentation this morning. My swap rates were all fully collateralised swap rates, which would today reflect OIS discounting. I think Wolfgang [Rungaldier] called them the clean rates. As soon as you look at the uncollateralised market, any rates you see are contaminated by CVA and DVA.

You say, *Use LIBOR discounting.* I would say the correct thing to do even with uncollateralised transactions is still to use OIS discounting and calculate your CVA and DVA using spreads relative to the OIS curve. Forget about the LIBOR curve. The LIBOR curve is no longer appropriate for valuing derivatives. It could by chance be that LIBOR is the correct borrowing rate for the counterparty you're dealing with, but in most cases the borrowing rate of an uncollateralised end user is different from LIBOR, so LIBOR is not a relevant rate. I don't care whether we call the OIS rate the risk-free rate or not, but it is the best close-to-risk-free benchmark that we have.

Ralf: Thank you, John. It's now your turn, so please continue with your statement.

4 John Hull

John: Hard to know where to start because I have written quite a bit on FVA in the last few years. I've actually consciously decided to stop doing it because I realise I could spend the whole of the rest of my academic career writing about this, and I'd never convince most people.

Actually, my interest in FVA has got an interesting history. In the middle of 2012, I got a call from the editor of Risk magazine saying, *We're bringing out the 25th anniversary edition of Risk magazine. We'd like you to write an article for it.* I agreed to write the article. (No academic ever says no to writing an article.) I asked *What would you like me to write about?* He said, *We don't mind what you write about, so long as it's interesting to our readership. But, by the way, we need the article in three weeks.*

I went down the corridor to discuss this with my colleague Alan White. We had a number of interesting ideas for the article. After two and a half weeks we settled on FVA. The trouble was that we then had only three days to write the article. In retrospect, I wish we'd had longer. So what did that article say? That article said, you should not make an FVA adjustment. I'll explain why in a minute. The reaction to the article was interesting. Usually when you write these articles, nothing much happens. You get maybe a little bit of a response from a few other academics. But in this case we were absolutely inundated with emails from people about this article. Two-thirds of emails were saying *You're crazy. You don't know what you're talking about. Clearly there should be an FVA adjustment. We've been doing for a while now* … and so on.

The other one-third were a little bit more positive, and some of them even went so far as to say, *We're glad someone's finally said this because we were a little uncomfortable with this FVA adjustment.* And, of course, Risk magazine realised that this was an exciting topic for them, so they started organising conferences on FVA.

Two people from Royal Bank of Scotland wrote a rejoinder to our article, which appeared in the next issue of Risk. And we were invited to write a rejoinder to the rejoinder, and so it went on. It was a really crazy time.

What I very quickly found out was that: Alan and I had a different perspective from most of the people we were corresponding with on this, and the reason was that we've been trained in finance. We've moved from finance into derivatives, and most of the people we were talking to had moved from physics or mathematics into derivatives. One important idea in corporate finance is that when you're valuing an investment, the discount rate should be determined by the riskiness of the investment. How you finance the investments is not important. Whether you finance it with debt or equity, it's the riskiness of the investment that matters. In other words, you should separate out the funding from the valuation of the investment (Fig. 3).

That was where we were coming from. In the case of derivatives a complication is that we can use risk-neutral valuation, so we've got a nice way of doing the valuation, but that does not alter the basic argument. Expected cash flows that are directly related

Fig. 3 John Hull giving his presentation on "OIS discounting, interest rate derivatives, and the modeling of stochastic interest rate spreads"

to the investment should be taken into account. In the case of derivative transactions these expected cash flows include CVA and DVA.

So that's where we were coming from. We've modified our opinion a little bit recently. I think I'm more or less in the same camp as Damiano here, judging by his presentation. Let's suppose that you fund at OIS plus 200 basis points. If the whole of the 200 basis points is compensation for default risk, then you are actually getting a benefit from that 200 basis points, in that that 200 basis points is reflecting the losses to the lender (and benefits to you) of a possible default on your borrowings. That is what we call DVA 2, and what Damiano called DVA(F), and other people have called it FDA. This is not what we usually think of as DVA. What we usually think of as DVA is the fact that as a bank you might default on your derivatives, and that could be a gain to you. Here we are applying the same idea to the bank's debt.

DVA 2 cancels out FVA, and that was the main argument we made in that Risk magazine article. But if you say that the bank's borrowing rate is OIS plus 200 basis points where 120 basis points is for default risk, and 80 basis points is for other things—maybe liquidity—we can argue that 80 basis points is a dead-weight cost. It's part of the cost of doing business, you're not getting any benefit from that 80 basis points. You are getting benefit from the 120 basis points: a DVA-type benefit because you can default on your funding.

So I think I am in the same camp as Damiano. I think he called it LVA. This component of your funding cost which is not related to default risk, is arguably a genuine FVA. The problem is, of course, that it's very, very difficult to separate out the bit of your funding cost that's due to default risk and the bit of your funding cost that's due to other things.

And then another complication is, of course, that accountants assume—for example when calculating CVA—the whole of your credit spread reflects default risk.

I have lots and lots of discussions with people on this. You realise very quickly that you're never going to convince somebody who's in a different mindset from yourself on this. One important question, though, is what are we trying to do here? With these sorts of adjustments, are we trying to calculate a price we should charge a customer? (Obviously in this day and age, we would be talking about the price we should charge an end user because transactions with other banks are going to be fully collateralised.) Or are we concerned with internal accounting? Or is it financial accounting that is our objective? I've always taken the view that what we're really talking about here is what we record in our books as the value of this derivative. But if you take the view that what we're trying to do is to work out what we should charge an end user, a customer, then actually I have no problems doing whatever you like, even trying to convince a customer that the customer should pay an ECA, an electricity cost adjustment. We all know that what you're trying to do is get the best price you can and hopefully cover your costs.

What I found was when I was talking to people about FVA is you start talking about how derivatives should be accounted for and very quickly you slip into talking about how much the customer should be charged, which is a totally different issue. Obviously, there's all sorts of costs you've got to recover in terms of what you charge the customer.

Where are accountants coming from? As you all know, accountants want you to value derivatives at exit prices. The accounting bodies are quite clear, that the exit prices have nothing to do with your own costs. Exit prices should be related to what's going on in the marketplace. Therefore, your own funding costs can't possibly come into an exit price. If other dealers are using FVA in their pricing, their funding costs may be relevant, but your own funding costs are not relevant. An interesting question is how should we determine exit prices in a world where all dealers are incorporating FVA into their pricing. Should we build into our exit price an average of the funding costs of all dealers or the funding cost of the dealer that gives the most competitive price? You can argue about this, but it is difficult to argue that it is your own funding costs that should be used in accounting.

What we have found is there's a lot of confusion between DVA and FVA, and as I said there's really two distinct parts to DVA. There's the DVA associated with the fact that you may default on your derivatives. That's what we call DVA 1. It's the usual DVA. Your DVA 1 is your counterparty's CVA and vice versa. And then there's what we call DVA 2, which is the fact that you might default on your funding.

Banks have always been uncomfortable with DVA. Even though accounting bodies have approved DVA they dislike the idea of taking their own default risk into account. This has led some banks to replace DVA by FVA. In this context, FVA is sometimes divided into a funding benefit adjustment and a funding cost adjustment with the funding benefit adjustment being regarded as a substitute for DVA.

When you look at what's actually going on right now, banks are all over the place in terms of how they make funding value adjustments. I agree with Damiano that once JP Morgan announced that it is taking account of FVA, then everybody felt they had to do it as well. The correctness of FVA becomes a self-fulfilling prophecy. A bank's auditors are going to say, *Everybody else is doing this? Why aren't you doing it?* Whether or not you believe the models used by everyone else are correct, you have got to use those models to determine accounting values.

You can have research models for trading, but for accounting you've just got to do what everybody else does. When a critical mass of people move over to doing something, whether it's right or wrong, you've got to do it.

I notice from a recent article in Risk that the Basel committee is getting interested in funding value adjustments. And U.S. regulators are getting interested in funding value adjustments as well. In addition, I can tell you that a few months ago, Alan White and I were invited to FASB to talk to them about funding value adjustments. They have concerns about the use of FVA in accounting. They like derivatives accounting valuations to be based on market prices not on internal costs.

I think we are in a fairly fluid situation here. When JP Morgan has said, *We're doing it this way, and we're taking a one-and-half billion dollar hit* it is tempting to believe that everyone else will follow suit and that is the end of the story. I don't think it is the end of the story because we have not yet heard from accountants and regulators. Also, I think it is fair to say that the views of banks and the quants that work for them are evolving.

There's some good news. (Maybe it's not good news if you're a quant working for a bank.) The good news is that we're clearly moving to a world where all derivatives are fully collateralised. We're now in a situation where if you deal with another financial institution or another systemically important entity, you've got to be fully collateralised. Dealing with an end user, you don't have to be fully collateralised. But there's a lot of arguments (we talked about some of them at this conference) suggesting that end users will get a better deal if they are fully collateralised.

FVA is not going to be such a big issue going forward. Indeed, I think it's going to fade away as full collateralisation becomes the norm. But no doubt arguments about some other XVAs will continue.

Ralf: Thank you, John. I take away that for PhD students it is wise not to pursue too much research on FVA, then, it might not be worth the effort …

Audience member: Sorry, just if you'll allow me a little comment. Since the issue of the self-fulfilling prophecy was picked up also by John Hull, just a little comment from a mathematical point of view. If you do mathematics for the application, you need a model. Possibly a true model. So what is a true model? Now, if you do applications for the natural or physical sciences, possibly there is a true model. It is very complicated, and what you do, you choose a model that is a good compromise between representativity and tractability, right, so you can deal with this model and it's still relatively good.

Now we come to social / economic sciences. What is the true model? If, at some point, the majority sort of implicitly uses a sort of model, isn't that all of a sudden the true model that other people should follow, or am I wrong here?

Damiano: Like base correlation, for example?

John: Yes, I don't see it quite that way, though. I think opinions will fluctuate through time. Nearly all large global banks do make funding value adjustments now. There are two or three holdouts, but most of them do.

I think FVA is going to be more of a fad than a truth. I think that in five years' time we could be in the opposite position to today: everybody just decides they don't want to make these funding value adjustments. That's just my own personal opinion.

One thing I meant to say is that there are interactions between CVA and DVA. If one side defaults, you don't care about the other side defaulting later, and there are a number of other close-out issues. I agree with what Damiano says. Those create a lot of complications. And they are relevant because those are complications in assessing expected cash flows arising from the derivatives portfolio that you have with a counterparty. They're nothing to do with funding. They're to do with expected future cash flows, which are the relevant things to calculate a valuation. It does make the valuation more complicated, but to overlay that with funding adjustments I don't think is correct except insofar as some part of the funding value adjustment is the dead-weight cost I was talking about.

Ralf: Thank you, John. You mentioned valuation, so maybe this is the keyword to hand over to Daniel.

5 Daniel Sommer

Daniel: First, John, as you immediately addressed the accounting profession, I'm not an accountant but I work for a firm that does audit and accounting as some part of its business. Are our accountants just people who tell the banks to do what everybody else does? The story is slightly more complicated than that because what accountants are interested in, and I pick up this story about self-fulfilling prophecies, what they are interested in eventually is fair value. And, indeed, for financial instruments that's defined as the exit price. But then the big question is: How do you find out what the exit price actually is?

Because it's not like for all the instruments that we're talking about in this seminar here, it's not something that you can read on Bloomberg or any other data provider. It's nothing that people will tell you in the street immediately. It's rather a complicated exercise to find out what fair value actually is. What would be the exit price at which you could actually exit your position? It's at that point where that whole reasoning comes up with the notion of how other people are thinking about valuing a certain position. How are my counterparties, my potential counterparties in the market, thinking about it? And that gives a bit more sense to the statement *Do what everybody else does*. Because if everybody else is taking certain aspects of a financial instrument into consideration when valuing this asset, it's very likely that your exit price that you are offered will also take that into consideration. It's for that reason that accountants are interested in what everybody else is doing, and frankly speaking, yes, at KPMG, that was indeed the discussion we had with many banks over the last three/four years where we met the banks in London on various panels to discuss FVA with them. Those were quite open discussions. From one year to the other, we sort of made a roll call and asked who's going to do what next year and when do you think you will be moving to FVA, etc., just to get a feeling for where the market was going in order to have a better understanding of what the market thought fair value would be. In that sense, I think that gives a bit more meaning to accountants telling the banks to do what everybody else is doing.

Now, coming to the current situation, indeed, I think there is no major bank globally left who has not declared they were doing something on funding valuation adjustments, with a lot of banks having come up with that in their 2014 year-end accounts. So I think the pressure on those banks who have not yet done that is actually rising. That's something which I think is a matter of fact.

I'm happy to comment or give my personal opinion about FVA, and perhaps talk about it by going back to some anecdotal evidence which I came across during the financial crisis. Before that, let me just mention a few more things.

Indeed the regulators become interested in FVA, and I think that there are at least two big issues that will have real effects on the banks that will enter the regulatory discussion or should enter the regulatory discussion. One thing is, indeed, the overlap between FVA and DVA, where many banks are happy to scrap DVA to a certain extent and replace it with FVA because that will have an immediate effect on their available regulatory capital. Because as they do the calculations these days, they offset FVA benefits and FVA costs. Thus to reduce DVA, where they need to deduct DVA from core Tier 1 capital, has a real effect on the bank's balance sheets and profitability calculations regarding regulatory capital.

The other thing people mentioned and it is true: hedging FVA just as is the case with CVA is a complicated issue and involves also hedging the related market risk. And so the question that we have been debating for CVA for a long time already is whether you are allowed to include the market risk hedges in your internal model for market risk or not. We've seen some movements in this direction recently by the regulators, but I think that those are two questions that at least should be quite prominent in the regulatory debate coming up.

That's one thing. The other thing is related to accounting. People quite leisurely mentioned that, well, yes, we need to go from a single deal valuation to portfolio valuation. And indeed for CVA that's absolutely inevitable. If you do that, nevertheless, for an accountant that raises a few uncomfortable questions because it raises the question: What is actually the unit of account? Apparently it's not a single deal. It may be the netting set as far as CVA is concerned, but when you look at funding, the netting set may even be too small, so it may be some sort of funding set, so all the deals that you have in one currency or so. When you look at effects on the balance sheet, do you need to value your whole bank before you can actually value your derivatives correctly? That's a bit of an uncomfortable direction we're going into.

Those are a few comments on things that people have said up to now, but on FVA itself, let me give you a little anecdote that occurred to me during the financial crisis. During the financial crisis, the CFO and CEO of one of our top-ten German banks asked me: *Look, all the banks have to reduce the values of their ABS and CDO books. Actually, don't you think that if a book is match-funded, it should be worth more than if a book is not match-funded?* And this goes back to the real fundamental question of liquidity risk and whether liquidity should play a role in pricing. And everybody who's read Modigliani and Miller, would say, *By no means.* That would be the standard answer. Nevertheless, when you come to think about the situation that the banks were in during the financial crisis, actually having a match-funded book gave you at least the option to wait. And there's real value in that option, as the banks who were able to wait were able to realise this because much of the write-downs that happened during the financial crisis actually came back as defaults were not as heavy as would have been thought at the time and indicated through the quotes at the peak of the crisis. It wasn't even traded prices at the time; it was basically quotes that banks were valuing their books on.

One might think—a very personal view at this point—one might think that if banks go for match-funding their books, it's like buying a very, very deep out-of-the-money option that they can then exercise when things get really bad. So that's one comment I would like to make.

The other point is somewhat more disconcerting. What does being liquid mean in a world that has had the experience of the financial crisis? Is it sufficient to say that a bank is liquid if it can generate enough funds through the collateralised inter-bank money market? Or does a bank have to have access to sufficient central bank money to prove that it is liquid? At least the experience of the financial crisis showed the vulnerability of the inter-bank market and the importance of central bank money to keep the system afloat. In that case at least part of the liquidity costs of banks would be due to ensuring it has enough central bank money or assets that can swiftly be turned into the latter. But if that was so then this would change our whole valuation paradigm, which after all is based on the general equilibrium theory and the theory of value by Gérard Debreu and others. In this theory there is no need for a central bank to keep the system working. Therefore, acknowledging the existence of funding costs through the introduction of FVA may have far reaching consequences on the derivatives pricing theory compared to just the calculation of some odd valuation adjustment and quarreling about which funding curve to use to determine an exit price.

Ralf: Thank you, Daniel. John, do you want to comment on this? Is Miller and Modigliani still valid in such an environment?

John: Well, I think it is, but what Modigliani and Miller say is that if you cut the pie up, the sum of the pieces is worth the same as the whole. Now, the question is, who are the potential stakeholders you've got to look at when you cut the pie up.

I agree with pretty much everything that Daniel said. It makes a lot of sense.

Ralf: Christian?

Christian: I have a question maybe from the practitioner's side, also being a little bit of a quant with respect to the exit price, which keeps me puzzling. Just to make that clear, for me there are two prices at least. The exit price, I can realise it only once: by going out of business. There's only one opportunity to realise the exit price. There is, of course, the price which I use in calculating my risk sensitivities, my hedge, which I use in solving my optimal control problem, in my risk management problem.

So, for example, if the exit price would include a tax, there would be some kind of going-out-of-business tax, the exit price would clearly include this tax, but of course as long as I'd like to stay in business I would never charge that tax, and I would not include it in my hedging because it would never occur to me.

What is strange for me is that I believe that the good price for doing the optimal control problem, so how do you hedge and so on, is actually the price which is going to concern and not the exit price, but the balance sheet is using the exit price, and it appears to me as if management is always looking at the balance sheet. Isn't there some kind of contradictions? What is the price that should be used to find the optimal path for the company? To make the investment decisions and so on?

Daniel: First of all, it's very clear that what the accounting standards mean by exit price is by no means the price at which the bank would go out of business. It's a going concern still. Of course it's an artificial concept in the sense that you will never … even if you were to sell just a portfolio of your trading book, you would probably not realise what accountants think of as the fair value because they explicitly rule out including portfolio effects on this fair value.

What this exit price actually means is, two people meet in the market and they agree on a certain price at which to exchange a position without changing the market equilibrium, it has to be small relative to the market.

Christian: For example, for my own bonds, the exit price is my bond value, which obviously includes my funding, and for uncollateralised derivatives it is the derivative valued with some average market funding, and if I take your example of fully matched funding, this is puzzling me because the bonds are on funding and the uncollateralised derivatives are not on funding.

Ralf: I think this goes in the same direction as my question to Damiano about the close-out value—what value to use. I think we probably will not solve this puzzle today. Looking at the time, I would like to thank all of you for your attention. Thank you very much to all panelists, and I suppose there's plenty of time for further discussions during the dinner tonight. Thank you!

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