Chapter 3
Summary and Looking Ahead

This topical survey, with its sketch of the history and evolution of the field of early algebra, has highlighted research related to the nature of early algebra, its learning, and its teaching. We have noted, in particular, that early algebraic thinking does not develop on its own without appropriate instructional support. And so, as we look ahead to the future, we recommend that further research be carried out in the following areas:

- The nature of classroom culture and the role of the teacher in fostering early algebraic reasoning.
- The forms of curricular activity that support early algebraic thinking.
- The nature of professional development that supports teachers' capacity to foster early algebraic thinking in the classroom.
- Theorizing about the study of number, operations, and properties in the context of early algebra.
- The use of neuroimaging techniques to inform the learning and teaching of early algebra.
- The development and use of digital tools to facilitate the teaching and learning of early algebra.
- The impact of early algebraic thinking on students' later study of algebra.

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DOI 10.1007/978-3-319-32258-2_3

References

Ainley, J. (1999). Doing algebra type stuff: Emergent algebra in the primary school. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematic Education* (Vol. 2, pp. 9–16). Haifa, Israel: PME.

Ainley, J. (2001). Research forum on early algebra. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 129–159). Utrecht, The Netherlands: PME.

Ainley, J., Nardi, E., & Pratt, D. (1998). Constructing meaning for formal notation in Active Graphing. In I. Schwank (Ed.), *Proceedings of the First Conference of the European Society for Research in Mathematics Education* (Vol. I, pp. 189–200). Osnabrück, Germany: CERME.

Bass, H. B., & Ball, D. L. (2003). Foreword. In T. P. Carpenter, M. L. Franke, & L. Levi, *Thinking mathematically: Integrating arithmetic and algebra in elementary school* (pp. v–vii). Portsmouth, NH: Heinemann.

Blanton, M. L. (2008). *Algebra in elementary classrooms: Transforming thinking, transforming practice*. Portsmouth, NH: Heinemann.

Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2015a). A learning trajectory in six-year-olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education,46*, 511–558.

Blanton, M. L., & Kaput, J. J. (2004). Elementary grade students' capacity for functional thinking. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 135–142). Bergen, Norway: PME.

Blanton, M. L., & Kaput, J. J. (2008). Building district capacity for teacher development in algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 361–388). New York: Routledge.

Blanton, M., Levi, L., Crites, T., & Dougherty, B. (2011). Developing essential understanding of algebraic thinking for teaching mathematics in grades 3–5. In B. J. Dougherty & R. M. Zbiek (Eds.), *Essential understandings series*. National Council of Teachers of Mathematics: Reston, VA.

Blanton, M., Stephens, A., Knuth, E., Gardiner, A. M., Isler, I., & Kim, J.-S. (2015b). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education,46*, 39–87.

Bolea, P., Bosch, M., & Gascón, J. (1998). The role of algebraization in the study of a mathematical organization. In I. Schwank (Ed.), *Proceedings of the First Conference of the European Society for Research in Mathematics Education* (Vol. II, pp. 135–145). Osnabrück, Germany: CERME.

Booth, L. R. (1984). *Algebra: Children's strategies and errors*. Windsor, UK: NFER-Nelson.

Bourke, S., & Stacey, K. (1988). Assessing problem solving in mathematics: Some variables related to student performance. *Australian Educational Researcher,15*, 77–83.

C. Kieran et al., *Early Algebra*, ICME-13 Topical Surveys,
DOI 10.1007/978-3-319-32258-2

Britt, M. S., & Irwin, K. C. (2011). Algebraic thinking with and without algebraic representation: A pathway for learning. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 137–159). New York: Springer.

Brizuela, B., & Schliemann, A. (2004). Ten-year-old students solving linear equations. *For the Learning of Mathematics,24*(2), 33–40.

Cai, J. (2004). Developing algebraic thinking in the earlier grades: A case study of the Chinese elementary school curriculum. *The Mathematics Educator,8*(1), 107–130.

Cai, J., & Knuth, E. (Eds.). (2011). *Early algebraization*. New York: Springer.

Cai, J., Ng, S. F., & Moyer, J. (2011). Developing students' algebraic thinking in earlier grades: Lessons from China and Singapore. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 25–42). New York: Springer.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann; Reston, VA: National Council of Teachers of Mathematics.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Carraher, D. W., & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In F. K. Lester Jr (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 669–705). Charlotte, NC: Information Age.

Carraher, D. W., & Schliemann, A. D. (2015). Powerful ideas in elementary school mathematics. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd ed., pp. 191–218). New York: Taylor & Francis.

Carraher, D., Schliemann, A. D., & Brizuela, B. M. (2001). Can young students operate on unknowns? In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 130–140). Utrecht, The Netherlands: PME.

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education,37*, 87–115.

Cho, S., Ryali, S., Geary, D. C., & Menon, V. (2011). How does a child solve 7 + 8? Decoding brain activity patterns associated with counting and retrieval strategies. *Developmental Science,14*(5), 989–1001.

Collis, K. F. (1975). *A study of concrete and formal operations in school mathematics: A Piagetian viewpoint*. Melbourne, Australia: Australian Council for Educational Research.

Cooper, T. J., & Warren, E. (2011). Years 2 to 6 students' ability to generalise: Models, representations and theory for teaching and learning. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 187–214). New York: Springer.

Curriculum Planning & Development Division. (2006). *2006 Mathematics syllabus: Primary*. Singapore: Ministry of Education.

Cusi, A., & Malara, N. A. (2013). A theoretical construct to analyze the teacher's role during introductory activities to algebraic modelling. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the 8th Congress of the European Society for Research in Mathematics Education* (pp. 3015–3024). Antalya, Turkey: CERME.

Cusi, A., Malara, N., & Navarra, G. (2011). Theoretical issues and educational strategies for encouraging teachers to promote a linguistic and metacognitive approach to early algebra. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 483–510). New York: Springer.

Davis, R. B. (1964). *Discovery in mathematics: A text for teachers*. Palo Alto, CA: Addison-Wesley.

Davis, R. B. (1975). Cognitive processes involved in solving simple algebraic equations. *Journal of Mathematical Behavior,1*(3), 7–35.

Davis, R. B. (1985). ICME-5 report: Algebraic thinking in the early grades. *Journal of Mathematical Behavior,4*, 195–208.

Davis, R. B. (1995). Why are they changing school algebra and who's doing it? *Journal of Mathematical Behavior,14*, 1–3.

Davydov, V. V., Gorbov, S. F., Mikulina, G. G., Savaleva, O. V. (1999). *Mathematics class 1* (edited by J. Schmittau). Binghamton: State University of New York.

Dougherty, B. J. (2003). Voyaging from theory to practice in learning: Measure Up. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17–23). Honolulu: PME.

Dougherty, B. (2008). Measure Up: A quantitative view of early algebra. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 389–412). New York: Routledge.

Dougherty, B. J., & Slovin, H. (2004). Generalized diagrams as a tool for young children's problem solving. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 295–302). Bergen, Norway: PME.

Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers, grades 6–10*. Portsmouth, NH: Heinemann.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction,25*(4), 439–478.

English, L. D., & Warren, E. A. (1998). Introducing the variable through pattern exploration. *The Mathematics Teacher,91*(2), 166–170.

Franke, M., Carpenter, T. P., & Battey, D. (2008). In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 333–360). New York: Routledge.

Fujii, T. (2003). Probing students' understanding of variables through cognitive conflict problems: Is the concept of variable so difficult for students to understand? In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 49–65). Honolulu, HI: PME.

Fujii, T., & Stephens, M. (2001). Fostering an understanding of algebraic generalisation through numerical expressions: The role of quasi-variables. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra* (pp. 258–264). Melbourne, Australia: The University of Melbourne.

Goldenberg, E. P., & Shteingold, N. (2008). Early algebra: The Math Workshop perspective. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 449–478). New York: Routledge.

Grabner, R., Ansari, D., Koschutnig, K., Reishofer, G., Ebner, F., & Neuper, C. (2009). To retrieve or to calculate? Left angular gyrus mediates the retrieval of arithmetic facts during problem solving. *Neuropsychologia,47*, 604–608.

Grouws, D. A. (Ed.). (1992). *Handbook of research on mathematics teaching and learning*. New York: Macmillan.

Hewitt, D. (2014). A symbolic dance: The interplay between movement, notation, and mathematics on a journey toward solving equations. *Mathematical Thinking and Learning,16*, 1–31.

Ischebeck, A., Zamarian, L., Egger, K., Schocke, M., & Delazer, M. (2007). Imaging early practice effects in arithmetic. *NeuroImage,36*, 993–1003.

Kail, R., & Park, Y. S. (1994). Processing time, articulation time, and memory span. *Journal of Experimental Child Psychology,57*(2), 281–291.

Kaput, J. J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum. In *The nature and role of algebra in the K-14 curriculum* (Proceedings of a National Symposium, 1997, organized by the National Council of Teachers of Mathematics, the Mathematical Sciences Education Board, and the National Research Council, pp. 25–26). Washington, DC: National Academy Press.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). New York: Routledge.

Kaput, J.J., & Blanton, M. (2001). Algebrafying the elementary mathematics experience. Part 1: Transforming task structures. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra* (pp. 344–351). Melbourne, Australia: The University of Melbourne.

Kaput, J. J., Blanton, M. L., & Moreno, L. (2008a). Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 19–55). New York: Routledge.

Kaput, J. J., Carraher, D. W., & Blanton, M. L. (Eds.). (2008b). *Algebra in the early grades*. New York: Routledge.

Karmiloff-Smith, A. (2010). Neuroimaging of the developing brain: taking 'developing' seriously. *Human Brain Mapping,31*, 934–941.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Elementary School Journal,102*, 59–80.

Khng, K. H., & Lee, K. (2009). Inhibiting interference from prior knowledge: Arithmetic intrusions in algebra word problem solving. *Learning and Individual Differences,19*, 262–268.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics,12*, 317–326.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 33–53). Reston, Virginia: NCTM.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde, & A. Pérez (Eds.), *Eighth International Congress on Mathematical Education: Selected lectures* (pp. 271–290). Seville: S.A.E.M. Thales.

Kieran, C. (1997). Mathematical concepts at the secondary school level: The learning of algebra and functions. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 133–158). East Sussex, UK: Psychology Press.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator,8* (1), 139–151.

Kieran, C. (2011). Overall commentary on early algebraization: Perspectives for research and teaching. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 579–593). New York: Springer.

Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics* (pp. 11–16). London: John Murray.

Kwon, H., Reiss, A. L., & Menon, V. (2002). Neural basis of protracted developmental changes in visual-spatial working memory. *Proceedings of the National Academy of Sciences of the United States of America,99*(20), 13336–13341.

Lannin, J. K., Barker, D. D., & Townsend, B. E. (2006). Recursive and explicit rules: How can we build student algebraic understanding? *Journal of Mathematical Behavior,25*(4), 299–317.

Lee, K., Lim, Z. Y., Yeong, S. H. M., Ng, S. F., Venkatraman, V., & Chee, M. W. L. (2007). Strategic differences in algebraic problem solving: Neuroanatomical correlates. *Brain Research,1155*, 163–171.

Lee, K., Yeong, S. H. M., Ng, S. F., Venkatraman, V., Graham, S., & Chee, M. W. L. (2010). Computing solutions to algebraic problems using a symbolic versus a schematic strategy. *ZDM —The International Journal on Mathematics Education,42*, 591–605.

Linchevski, L. (1995). Algebra with numbers and arithmetic with letters: A definition of pre-algebra. *Journal of Mathematical Behavior,14*, 113–120.

Lins, R., & Kaput, J. J. (2004). The early development of algebraic reasoning: The current state of the field. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The future of the teaching and learning of algebra: the 12th ICMI Study* (pp. 47–70). Dordrecht, The Netherlands: Kluwer Academic.

MacGregor, M., & Stacey, K. (1999). Learning the algebraic method of solving problems. *Journal of Mathematical Behavior,18*(2), 149–167.

Malara, N. A., & Navarra, G. (2003). *ArAl Project: Arithmetic pathways towards favouring pre-algebraic thinking*. Bologna, Italy: Pitagora.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (pp. 65–86). Dordrecht: Kluwer.

Mason, J. (with Graham, A. & Johnston-Wilder, S.). (2005). *Developing thinking in algebra*. London: Sage.

Matthews, P., Rittle-Johnson, B., McEldoon, K., & Taylor, R. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education,43*, 316–350.

Mavrikis, M., Noss, R., Hoyles, C., & Geraniou, E. (2013). Sowing the seeds of algebraic generalisation: Designing epistemic affordances for an intelligent microworld. In R. Noss & A. DiSessa (Eds.), *Journal of Computer Assisted Learning*, *29*(1), 68–85.

Menon, V. (2010). Developmental cognitive neuroscience of arithmetic: implications for learning and education. *ZDM—The International Journal on Mathematics Education,42*, 515–525.

Moss, J., & London McNab, S. (2011). An approach to geometric and numeric patterning that fosters second grade students' reasoning and generalizing about functions and co-variation. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 277–301). New York: Springer.

Moyer, J., Huinker, D., & Cai, J. (2004). Developing algebraic thinking in the earlier grades: A case study of the U.S. *Investigations* curriculum. *The Mathematics Educator,8*(1), 6–38.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Ng, S. F. (2004). Developing algebraic thinking in early grades: Case study of the Singapore primary mathematics curriculum. *The Mathematics Educator,8*(1), 39–59.

Ng, S. F. (2012). *A theoretical framework for understanding the different attention resource demands of letter-symbolic versus model method*. Paper presented at Topic Study Group 9 of the 12th International Congress on Mathematical Education, Seoul, Korea. Available at: http://www.icme12.org/upload/UpFile2/TSG/0748.pdf

Ng, S. F., & Lee, K. (2009). The model method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education,40*, 282–313.

Noble, T., Nemirovsky, R., Wright, T., & Tierney, C. (2001). Experiencing change: The mathematics of change in multiple environments. *Journal for Research in Mathematics Education,32*, 85–108.

Orton, A. (Ed.). (1999). *Pattern in the teaching and learning of mathematics*. London: Cassell.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics,42*, 237–268.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalizations. *Mathematical Thinking and Learning,5*, 37–70.

Radford, L. (2006). Algebraic thinking and the generalization of patterns: A semiotic perspective. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 2–21). Mérida, MX: PME-NA.

Radford, L. (2010). Layers of generality and types of generalization in pattern activities. *PNA,4*(2), 37–62.

Radford, L. (2011a). Embodiment, perception and symbols in the development of early algebraic thinking. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 17–24). Ankara, Turkey: PME.

Radford, L. (2011b). Grade 2 students' non-symbolic algebraic thinking. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 303–322). New York: Springer.

Radford, L. (2014). The progressive development of early embodied algebraic thinking. *Mathematics Education Research Journal,26*, 257–277.

Radford, L., & Roth, W.-M. (2011). Intercorporeality and ethical commitment: An activity perspective on classroom interaction. *Educational Studies in Mathematics,77*, 227–245.

Rivera, F. (2013). *Teaching and learning patterns in school mathematics: Psychological and pedagogical considerations*. New York: Springer.

Rivera, S. M., Reiss, A. L., Eckert, M. A., & Menon, V. (2005). Developmental changes in mental arithmetic: evidence for increased functional specialization in the left inferior parietal cortex. *Cerebral Cortex,15*, 1779–1790.

Rojano, T., & Sutherland, R. (2001). Arithmetic world—algebra world. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra* (pp. 515–522). Melbourne, Australia: The University of Melbourne.

Roschelle, J., Shechtman, N., Tatar, D., Hegedus, S., Hopkins, B., Empson, S., & Gallagher, L. P. (2010). Integration of technology, curriculum, and professional development for advancing middle school mathematics: Three large-scale studies. *American Educational Research Journal,47*(4), 833–878.

Russell, S. J. (2015). *Mathematical argument and productive lingering*. Teachers Development Group Leadership Seminar, Portland, OR, March 2015. Available from: http://www.teachersdg.org/2015%20Seminar%20Docs/OK_SJRussell_Lingering_Argument.pdf

Russell, S. J., Schifter, D., & Bastable, V. (2011). *Connecting arithmetic to algebra*. Portsmouth, NH: Heinemann.

Russell, S. J., Schifter, D., Bastable, V., & Franke, M. (submitted). *Bringing early algebra to the elementary classroom: Results of a professional development program for teachers.*

Russell, S. J., Schifter, D., Bastable, V., Higgins, T., & Kasman, R. (in press). *Mathematical argument in the elementary classroom: A yearlong focus on the arithmetic operations.* Portsmouth, NH: Heinemann.

Schifter, D. (1999). Reasoning about operations: Early algebraic thinking in grades K-6. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12* (pp. 62–81). Reston, VA: National Council of Teachers of Mathematics.

Schifter, D., Bastable, V., Russell, S. J., Riddle, M., & Seyferth, L. (2008a). Algebra in the K-5 classroom: Learning opportunities for students and teachers. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics. 2008 Yearbook of the National Council of Teachers of Mathematics* (pp. 263–277). Reston, VA: NCTM.

Schifter, D., Monk, S., Russell, S. J., & Bastable, V. (2008b). What does understanding the laws of arithmetic mean in the elementary grades? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 413–448). New York: Routledge.

Schliemann, A., Carraher, D. W., Brizuela, B., Earnest, D., Goodrow, A., Lara-Roth, S., & Peled, I. (2003). Algebra in elementary school. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 127–134). Honolulu, HI: PME.

Schmittau, J., & Morris, A. (2004). The development of algebra in the elementary mathematics curriculum of V. V. Davydov. *The Mathematics Educator, 8*(1), 60-87.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics,22*, 1–36.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics,26*(2/3), 191–228.

Slavit, D. (1999). The role of operation sense in transitions from arithmetic to algebraic thought. *Educational Studies in Mathematics,37*, 251–274.

Stacey, K. (1989). Finding and using patterns in linear generalizing problems. *Educational Studies in Mathematics,20*, 147–164.

Stacey, K., Chick, H., & Kendal, M. (Eds.). (2004). *The future of the teaching and learning of algebra: the 12th ICMI Study*. Dordrecht, The Netherlands: Kluwer Academic.

Stacey, K., & MacGregor, M. (1997). Ideas about symbolism that students bring to algebra. *The Mathematics Teacher,90*(2), 110–113.

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester Jr (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319–369). Greenwich, CT: Information Age Publishing.

Steinweg, A. S. (2001). Children's understanding of number patterns. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 203–206). Utrecht, The Netherlands: PME.

Sutherland, R. (1993). Connecting theory and practice: Results from the teaching of Logo. *Educational Studies in Mathematics,24*, 95–113.

TERC. (1998). *Investigations in number, data, and space*. Menlo Park, CA: Dale Seymour Publications.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. Coxford (Ed.), *Ideas of algebra: K-12* (pp. 8–19). Reston, VA: NCTM.

van Ameron, B. A. (2002). *Reinvention of early algebra—Developmental research on the transition from arithmetic to algebra* (doctoral thesis). Utrecht University, Freudenthal Institute. Available from: http://dspace.library.uu.nl/bitstream/handle/1874/874/full.pdf?sequence=18

Wagner, S., & Kieran, C. (Eds.). (1989). *Research issues in the learning and teaching of algebra. Vol. 4. Research agenda for mathematics education*. Reston, VA: National Council of Teachers of Mathematics.

Warren, E. (2002). Unknowns, arithmetic to algebra: two exemplars. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 361–368). Norwich, UK: PME.

Warren, E. (2003). Young children's understanding of equals: A longitudinal study. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 379–386). Honolulu, HI: PME.

Warren, E., & Cooper, T. J. (2008). Patterns that support early algebraic thinking in the elementary school. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics. 70th Yearbook of the National Council of Teachers of Mathematics* (pp. 113–126). Reston, VA: NCTM.

Warren, E., & Cooper, T. (2009). Developing mathematics understanding and abstraction: The case of equivalence in the elementary years. *Mathematics Education Research Journal,21*(2), 76–95.

Watanabe, T. (2011). A critical foundation for school algebra in Japanese elementary school mathematics. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 109–124). New York: Springer.

Webb, N., Franke, M., Ing, M., Wong, H., Fernandez, C., Shin, N., & Turrou, A. (2014). Engaging with other's mathematical ideas: Interrelationships among student participation, teachers' instructional practice, and learning. *International Journal for Educational Research,63*, 79–93.

Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: the tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics,49*, 379–402.

Further Reading

Blanton, M., Brizuela, B. M., & Stephens, A. C. (2016). *Elementary children's algebraic thinking.* Plenary paper prepared for presentation at Topic Study Group 10 of 13th International Congress on Mathematical Education (ICME13).

Carraher, D. W., & Schliemann, A. D. (2016). *Functional relations in early algebraic thinking.* Plenary paper prepared for presentation at Topic Study Group 10 of 13th International Congress on Mathematical Education (ICME13).

Malara, N. A., & Navarra, G. (2016). *Epistemological issues in early algebra: Offering teachers new words and paradigms to promote pupils' algebraic thinking.* Plenary paper prepared for presentation at Topic Study Group 10 of 13th International Congress on Mathematical Education (ICME13).

Mason, J. (2016). *How early is too early for thinking algebraically?* Plenary paper prepared for presentation at Topic Study Group 10 of 13th International Congress on Mathematical Education (ICME13).