Part IV
Awardees

Teaching Mathematics in Tomorrow's Society: A Case for an Oncoming Counter Paradigm

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Abstract The historical analysis of mathematics teaching at secondary level shows the succession in time of different school paradigms. The present paper describes and tries to analyse a new didactic paradigm, still at an early age, the paradigm "of questioning the world", which relies heavily on four interrelated concepts, that of inquiry and of being "Herbartian", "procognitive", and "exoteric". It is the author's ambition to show, however succinctly, how the present crisis in mathematics education could hopefully be solved along these lines, which preclude recurring to strategies seeking only to patch up the old, still dominant paradigm "of visiting works".

Keywords Anthropological theory of the didactic · Inquiry · Mathematics · Paradigm of questioning the world · Research and study path

The Anthropological Theory of the Didactic

I formally began working on mathematics education when I joined the Institute for research on mathematics teaching (IREM) in Marseilles (France) more than forty years ago—in February of 1972 to be precise. I write these lines qua 2009 recipient of the Hans Freudenthal Medal, an honour of which I am immensely proud. It is thus my wish to respond to it by indulging in a quick outline of the main conclusions at which I have arrived, letting interested readers judge for themselves the cogency of such views.

First of all, I must say that this presentation will draw upon the theoretical framework which my name has come to be associated with, I mean ATD, i.e. the *anthropological theory of the didactic*. Just as there are economic or political facts, there are *didactic* facts, which I will refer to as a whole as *the didactic*. The didactic is

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a vital dimension of human societies. In a slightly simplified way, one can say that it is made up of the motley host of social situations in which some person does something—or even manifests an intention to do so—so that some person may "study"—and "learn"—something. The something to be studied (and learnt) is known as the *didactic stake* in the situation. As you can see, this formulation formally refers to *two* persons. I will use the letter *y* to denote the first person, and the letter *x* to denote the second, so that we can say that *y* does, or intends to do, something to help *x* study (and learn) something. Of course, at times, *y* and *x* can be one and the same person. In such a (fundamental) case of self-directed learning, *x* helps him/herself study the didactic stake. The "something" that *y* does or intends to do is metaphorically called a *didactic gesture* and is part of the didactic as a whole.

Basically, didactics is the science studying the conditions that govern such "didactic situations", i.e. social situations which hinge on some "didactic triplet" comprising some *x*, some *y*, and some didactic stake *O*. The didactics of mathematics is concerned with those cases in which the didactic stake *O* is regarded as pertaining to mathematics. More generally speaking, *O* is what is called, in ATD, a "work", i.e. anything, material or immaterial, created by deliberate human action, with a view to achieving definite functions. To obtain more generality, let me substitute a set *X* of persons for the person *x*, arriving thus at the "didactic triplet" (*X*, *y*, *O*), which can model a typical high-school class—*X* being the group of students, and *y* the teacher to whom it befalls to teach the work *O*. Naturally, we can also consider triplets of the form (*X*, *Y*, *O*), where *Y* is a team of didactic "helpers" that may include a full-fledged teacher alongside "assistants" of different kinds. Let me add here that, in ATD, a condition is said to be a *constraint* for a person or an institution if it cannot be modified by this person or institution, at least in the short run. Now the basic question in didactics is somewhat the following: given a set of constraints *K* imposed upon a didactic triplet (*x*, *y*, *O*), what conditions can *x* and *y* create or modify—i.e. what *didactic gestures* can they make—in order for *x* to achieve some determined relation to *O*? This will be the starting point for what follows.

The Paradigm of Visiting Works and Its Shortcomings

The prospective view on the didactic dimension in our societies that I wish to make explicit—and, I hope, clear—can be encapsulated in a crucial historical fact: the old didactic paradigm still flourishing in so many scholastic institutions is bound to give way to a new paradigm still taking its first steps. To cut a longer story short, I define a didactic paradigm as a set of rules prescribing, however implicitly, what is to be studied—what the didactic stakes *O* can be—and what the forms of studying them are.

The "old" paradigm I've just mentioned has been preceded by a number of distinct, sometimes long-forgotten paradigms. The most archaic of these didactic paradigms disappeared, in many countries, during the nineteenth century. In the field of mathematics as well as in many other fields of knowledge, it was organised around the study of *doctrines* or *systems*—of mathematics, of philosophy, etc.—approached from outside and considered as outstanding achievements in the history of human creation. Within this paradigm, one used to study Euclid's *Elements* in the way most of us may still study (or aspire to study) Plato's or Hegel's systems of philosophy. This initial paradigm—which I call the paradigm of "hailing and studying authorities and masterpieces"—has gradually given way to the school paradigm that nowadays all of us, willingly or not, are supposed to revel in, which evolved in the course of centuries from the older paradigm of studying "grand systems". The "great men" supposed to have authored those systems were waved aside and the systems crushed into smaller pieces of knowledge of which the authorised labels—Pythagoras, Thales, Euclid, Gauss, etc., as far as mathematics is concerned—still record their origins.

In the framework of the anthropological theory of the didactic, this paradigm is known as the paradigm of "visiting works" or—according to a metaphor used in ATD—"of visiting monuments", for each of those pieces of knowledge—e.g., Heron's formula for the area of a triangle—is approached as a monument that stands on its own, that students are expected to admire and enjoy, even when they know next to nothing about its *raisons d'être*, now or in the past.

In spite of the long-standing devotion of so many teachers and educators to this unending intellectual pilgrimage, notwithstanding the often admirable docility of so many students in accepting the teacher as a guide, this once pervasive paradigm is currently on the wane. This has come to be so, it can be argued, because the paradigm of visiting monuments tends both to make little sense of the works thus visited—"Why does this one happen to be here?", "What is its utility?" remain generally unanswered questions. The interested reader may want to check how this applies to a number of mathematical entities. For example, what purpose does the notion of *reflex* angle serve? The same question can be raised about angles in general, and also about parallel lines, intersecting lines, rays, line segments, and so on. Of course, the same goes for the reduction of fractions or polynomial expansion, with the notion of decimal number, and what have you. In what situations can this mathematical entity prove useful, if not utterly unavoidable, and how? Because these questions are usually hushed up—visiting a monument is no place to raise "What for?" or "So what?" questions—, students are reduced to almost mere spectators, even when educators passionately urge them to "enjoy" the pure spectacle of mathematical works.

A number of factors explain at least partially the long dominance of the paradigm of visiting works as monuments as well as its present decline—and, I suggest, its impending demise. Historically, the first cause seems to be the congruity of this paradigm with the social structure of formerly undemocratic countries or, since more recent times, weakly or incompletely democratic. Such societies are founded on an all-pervasive pattern inseparably linking those in command positions, on the one hand, and those in obedience positions, on the other hand. Almost all institutions (be they families, schools, or nations) hinge on some replica of this fundamental, dualistic pattern. I shall not go into debate, here, about this age-old social structuring. I only want to emphasise the specific risks that the functioning of this ubiquitous power structure easily generates, in the form of abuses of authority, power, or rank—call them as you like. The existence of a dualistic configuration with one in authority and one in obedience may for sure be vindicated, on a "technical" basis, as needed to keep institutions going. But such a technically justified twofold structure is normally limited in time and, above all, *in scope*. Authority is, or should be, restricted to a specified number of specific situations, and should therefore refrain from encroaching on every aspect of life—unless it changes into tyranny. But respecting this rule is not everyone's forte. The classical paradigm of visiting "monuments of knowledge," however small, suffers today, at many levels, from the constant abuses of pedagogic power that its historical kinship with the dualistic pattern of power mechanically generates.

The consequences of this historical situation are many. First and foremost, I shall mention a consequence already alluded to: the resistless evolution of the school mathematics curriculum towards a form of epistemological "monumentalism" in which knowledge comes in chunks and bits sanctified by tradition and whose supposed "beauty" has been enhanced by the patina of age; that students have to visit, bow to, enjoy, have fun with and even "love". All this of course is but a daydream, as far as the mass of students—not the happy few, who need very little attention—is concerned.

The main effect of this long-term situation is the growing tendency among students to develop a relation to "official", scholastic knowledge in agreement with what I shall term the "Recycle bin/Empty recycle bin" principle: all the knowledge taught may legitimately be forgotten or, more exactly, *ignored*, as soon as exams have been passed. Of course this is presumably as old as the school-and-exam system. But it has shaped a relation to knowledge as driven by institutional, short-term, and labile motives, which stands away from the functional approach to knowledge based on its real-world utility—to understand a situation, be it mathematical or not, make a decision, or postpone it to allow for further study of the problem addressed.

A correlate, if not properly a consequence, is to be found in a yet more challenging fact: what little knowledge remains after the school years is rarely regarded as something that could bear on situations one might face outside school—and this seems particularly true in the case of mathematical knowledge. School-generated knowledge tends therefore to be unusable, in that its "remnants" are unable to perform their specific function. But there is more to it than that. Visiting a monument basically boils down to listening to a report or account made by the teacher-guide about the monument visited—what we call in the French of ATD an *exposé*, a word from whose meaning the negative connotation it has acquired in English must be expelled in this context. By its very nature, any account, a report, or an exposé skips "details", i.e. aspects that, more or less arbitrarily, choice-makers have ignored or altogether discarded. To give just one example, in the French curriculum—as is the case, I presume, in many other mathematics curricula across the world—, tradition has it that the algebraic solving of cubic equations is overlooked, while quadratic equations are emphatically considered. In his/her scholastic visit of the mathematical universe, the student thus reaches an endpoint beyond which lie mathematical _territories that, more often than not, will remain indefinitely terra incognita to him/her. What will be of this student if, in later life, they need to know what a cubic equation is and how it can possibly be solved? School education along the lines of the current paradigm has no clear answer to that question, it seems.

The relation to knowledge and ignorance thus associated with the visiting of mathematical works has become increasingly unsuited to people's needs and wants, up to the point that there currently exists a widespread belief that mathematical knowledge is something one can almost altogether dispense with—whereas, in a not so remote past, mathematics could be regarded as the key to a vast number of individual as well as collective problems. In this respect, the chief flaw in the paradigm of visiting monuments, which relates to the undemocratic ethos in which this paradigm originated, has to do with the choice of "monuments" to visit at school. As we know, this choice is usually the combined result of a long-lasting tradition, on the one hand, and of irregularly spaced, hectic reforms, on the other. In no way, it seems, the decisions made go beyond what the people in charge of this choice-making think opportune, fit, or even "good" for the edification of the mounting generations. In no way, it seems, is the choice of the monuments to be visited made on an experimental basis or at least on a large and supposedly relevant experiential basis. In what follow, I will try to adduce evidence that such a "feat" can be achieved provided we opt for the emerging didactic paradigm I call the "paradigm of questioning the world".

Questioning the World: Towards a New Didactic Paradigm

Up to a point, we might soon discard the current didactic world in favour of a new paradigm which, when contrasted with the old one, looks like a *counterparadigm*—although, as we shall see, it isn't doomed to break off all contact with its predecessor. The main changes that I shall stress are few but radical. Let us consider again a triplet (X, Y, O). An almost inconspicuous but crucial tenet of traditional education is that the members x of X are children or adolescents: traditionally, the educational endeavour is about young people, before they attain maturity. When maturity has been reached, everyone is supposed to be educated—well or badly, that is another question. In contrast with this view of education, in the didactic paradigm of questioning the world, education is a lifelong process. The x in the triplet (x, y, O) can be a toddler as well as a mature adult or an older person. A society's didactic endeavour is regarded (and assessed) as applying to all—to citizens no less than to future citizens. Consequently, the assessment of this crucial endeavour can no longer focus on young people only: not only should we explore what 15-year olds happen to know, but we should extend this quest to people aged 30 to (at least) 70. More than anything, society's didactic effort is not simply known by what people know: it should be appraised on the basis of what they can *learn*—and *how* they can do so.

A second, central tenet of the paradigm of questioning the world is that, in order to learn something about some work O, x has to *study* O, often with the help of some y. You don't learn to solve a cubic equation by chance; you have to stop and consider the question that arises before you. In today's common culture, many people, it seems, have a propensity to shun every question to which the answer is not obvious to them. What the new didactic paradigm aims to create is a new cognitive ethos in which, when any question Q arises, x will consider it, and, as often as possible, will *study* it in order to arrive at a valuable answer A, in many cases with a little help from some y. In other words, x is supposed not to systematically balk at situations involving problems that he/she never came across or never solved. For reasons I shall not comment on, I call *Herbartian*—after the German philosopher and founder of pedagogy Johann Heinrich Herbart (1776–1841)—this receptive attitude towards yet unanswered questions and unsolved problems, which is normally the scientist's attitude in his field of research and should become the citizen's in every domain of activity.

The new didactic paradigm wants the future as well as the full-blown citizen to become Herbartian. Let me give three easy, miscellaneous examples of possibly impending "open" questions. First example: many people engaged in social science research but who have had little contact with statistics during their school or college years may come across Pearson chi-squared test, bump into the elusive notion of degrees of freedom, and become obsessed with the question "What does the expression 'degrees of freedom' mean exactly?" Second example: physics students may be upset about having to use the curious symbol "proportional to" ($\propto$), "an eight lying on its side with a piece removed" (Miller 2011), without having the slightest idea about how the manipulation of this symbol can be justified in mathematical terms, particularly as concerns the intriguing conclusion that, if a variable z is proportional to variables x and y, then z will also be proportional to their product xy. Third example: anyone interested in the question of biodiversity may stumble upon a mathematical equation such as this:

$$H_e = 1 - \frac{1}{1 + 4N_e\mu} \qquad (1)$$

For the unrepentant non-mathematician, the first question will be: "What does that mean? What does that entail?" For all of us, I suppose, a second question will soon emerge: "Where does it come from? How can it be arrived at?" Of course, the pre-Herbartian citizen generally ignores all these questions because he/she usually recoils from anything seemingly mathematical. But the citizen in tune with the new didactic paradigm will face the questions, and, whenever possible, will come to grips with each of them. How is that possible?

In the didactic world shaped by the paradigm of visiting monuments, most people behave "retrocognitively". I use the word "retrocognition" not in its old parapsychological sense but simply to express the cognitive attitude that leads one to refer preferentially and almost exclusively to knowledge *already known* to one. Retrocognition in this sense is governed by the quasi-postulate according to which, once your school and college years are over, if you don't know in advance the answer to the question that faces you, then you'd better renounce all pretension to arrive at a sensible answer. This, of course, correlates with the propensity I mentioned earlier for staying away from unheard-of questions. By contrast, the paradigm of questioning the world calls for a very different attitude, that I dub *procognitive* (in a sense unrelated to the use of the word in denoting a drug that "reduces delirium or disorientation"), and which inclines one to behave as if knowledge was essentially still to discover and still to conquer—or to rediscover and conquer anew. In the retrocognitive bent, therefore, knowing is "knowing backwards"; whereas in the procognitive dedication, knowing is "knowing forwards".

In the scenario I present, how does one construct and validate an answer A to a question Q? Basically, inquiring into a question Q requires a twofold move. In the first place, the "inquirer" x will search the relevant literature for existing answers to question Q—a move traditionally banned at school, while to the contrary it is unavoidable in scientific research. In ATD it is common to denote an existing answer by the letter A with a small lozenge or diamond—a "thin" rhombus—in superscript, $A^{\diamond}$, in order to express that such an answer has been created and diffused by some institution which, in some sense, hallmarked it. Of course an answer $A^{\diamond}$ needs not be "true" or "valid"; but it is up to x to evaluate answers $A^{\diamond}$ to see if they are relevant—which also departs from school usage, in which answers provided by the teacher are guaranteed by the same token. In order to arrive at a proper answer—usually denoted by the letter A with a small heart in superscript regarded as the "maker's mark": $A^{\heartsuit}$—, the inquirer x has to use "tools", mathematical or not, i.e. works of different nature. It is from the combined study of the "hallmarked" answers $A^{\diamond}$ and of the works O (used as tools both to study answers $A^{\diamond}$ and to construct an answer $A^{\heartsuit}$) that the process of research for an answer $A^{\heartsuit}$ will get under way.

The inquiry led by x into Q opens up a path called a *research and study path* (or trail, or track, or course, etc.). To proceed along this path, the inquiry team X has to use knowledge—relating to answers $A^{\diamond}$ as well as to the other works O—hitherto unknown to its members, that the team will have to get familiar with to be able to continue on the trail towards answer $A^{\heartsuit}$. A necessary condition in this respect is for X and for every member x of X to behave *procognitively*, looking forward to meeting new knowledge—new works—without further ado.

Some more didactic aspects should be stressed here. Firstly, in the paradigm of questioning the world, encountering new knowledge or e-encountering old, half-forgotten knowledge along the research and study path is the way that inquirers *x learn*—they learn or relearn the answers $A^{\diamond}$, the working tools O and, finally, the answer $A^{\heartsuit}$. It should then be clear that the *contents* learnt, in this context, *have not been planned in advance*—contrary to what is usual in the paradigm of visiting monuments—and are determined essentially by two factors: by the question Q being studied, in the first place, and then by the research and study path covered, which in turn is determined by the $A^{\diamond}$ and the O encountered and studied in order to build up the answer $A^{\heartsuit}$. Secondly, it must be emphasised that studying a (mathematical or non-mathematical) work O—the same holds for the answers $A^{\diamond}$—is determined by the project of arriving at an answer $A^{\heartsuit}$. Contrary to the fiction forced upon x and y in the paradigm of visiting works, there is no such thing as a "normal" or "natural" study of a given work O. All exposés are special, none is exhaustive, and most fail to conceal their arbitrariness. The study of a work O in the context of an inquiry into some question Q will heavily depend, both quantitatively and qualitatively, on the use of O in the making of the answer $A^{\heartsuit}$. What should be clear in such a context-bound study of O is that the knowledge of O thus acquired by the investigators is *functionally coherent* because it is cohered by the inquiry into question Q, so that the *raisons d'être* of O that do explain its use in the case in point are readily apparent.

Society, School, and the New Paradigm

The paradigm of questioning the world and the inquiries that make it a reality do not exist in a vacuum. They must have a basis in society and in school. Once again let me stress here that the field of relevance of the didactic schema—called the *Herbartian schema*—outlined so far extends to the whole of society—it is not conceived as being restricted to school. Any person can represent x in a didactic triplet (x, y, O). [A didactic "helper" y may fail to exist, in which case it is common to write the triplet in the form $(x, \varnothing, O)$: the didactic triplet is then reduced in actual fact to a 2-tuple.] Of course it is easy to spot an outstanding difference. In many modern societies, going to school during the first part of one's life—while you're a youngster—is compulsory. Admittedly, there is no such thing as compulsory education for adults in general. In this respect, the scenario advocated here supposes a fundamental change, with the extension of the right to education into the right to *lifelong* education for all, provided by an adequate infrastructure that we could continue to call "school", but in a sense that goes back to ancient Greece and, more precisely, to the Greek word *skhole*, which originally designated spare time devoted to leisure (this was still its meaning in the time of Plato, for example), but which evolved to mean "studious leisure", "place for intellectual argument", and "time for liberal studies". The new role of the didactic in our societies thus implies the development of a ubiquitous institution that, in what follows, I shall term, more genuinely, *skhole*. Of course, school as we know it is a key component of *skhole*, even though, in its present form, it remains largely foreign to the new didactic paradigm. But *school* is not all of *skhole*. For example, for adults as well as for younger people, a good part of *skhole* takes place at home: home *skhole*ing will be, and already is, a master component of *skhole*. In what follows, *skhole* will be approached for its capacity to favour the development and flourishing of the paradigm of questioning the world—even though parts of it are still under the control of the old school paradigm.

I begin by considering the case of adults' *skhole*ing—of which today's "adults schooling", as we may call it, is but a meagre component. In truth, many citizens are already, though partially, equipped to inquire on their own into the many questions that may beset them, for example in their daily life. This being noted, what are the main constraints that hinder, and what are the conditions that might favour the development of adults' *skhole*ing? The first condition lies in the fact that, instead of fleeing when faced with questions, x duly confronts them. To do so, x has to formulate them explicitly, at least for him/herself. Simple as it may sound, such a move conflicts with a fundamental determinant of our cultures, the disjunction between "masters" and "underlings", if I may say so, that forbids the latter to raise questions about the world—natural or social—, or, as the saying goes, to put it "into question", while "masters" have alone the legitimacy to question the world and to change it. Sheer observation—but this conclusion can easily be submitted to experimentation—shows that most people get excited at daring to pose on their own the merest question. Historically, posing questions was the privilege of the mighty, although it has become a defining right of citizens; but it is a right not yet exercised as it should in a fully developed democracy.

Let us suppose that some citizen has decided to inquire into some question Q, becoming thus an inquirer x in a triplet (x, ?, Q). At this stage of his/her study, two problems face him/her. On the one hand, x may think of getting help from some people Y; on the other hand, he/she will have to "search the world" for answers $A^{\diamond}$ to question Q and relevant works O. The first of these two problems has no systematic solution today. The second problem has a good approximate solution. It consists in the sum total of the information provided by the Internet and especially the Web. In fact, I shall refer to the Internet *sensu latissimo*—in the broadest sense—, a sense that, against current usage, includes… all the libraries in the world, because any document is either available on the Internet or can be regarded as *not yet* available on the Internet. To take here just one example, in the case of an inquiry into the mathematics of the "proportional to" symbol ($\propto$), when starting from Jeff Miller's well-known website on the *Earliest uses of symbols of relation* (2011), one is led to Florian Cajori's classic book on the history of mathematical notations (1993, vol. 1, p. 297), which in turn refers the inquirer to three older books, authored respectively by Emerson (1768), who was the introducer of the symbol $\propto$, Chrystal (1866), and Castle (1905). Today, all of these books are available online for free. Let us also observe that the Internet allows most inquirers x to find help from occasional helpers y, for example on Internet forums and discussion threads, so that the main solution to the second problem also supplies a (partial) solution to the first problem.

Making inquiries on the Internet *sensu latissimo* meets with well-recognised difficulties. First, if x is almost certain to come across at least some relevant resources, documents allowing him/her to go further and deeper into the question studied may be scarce. Second, the inquirer x can prove unable both to find out relevant documents that do exist and to make the most of what little information he/she culled. The inquirer's intellectual equipment—or more exactly the inquirer's *praxeological* equipment, in a sense of the word *praxeology* proper to ATD—thus rests on two pillars: the capacity to locate resources, online and offline, and the knowledge necessary to take advantage of them. This leads to the question of making good use of the works O gathered. Most general questions Q entail the use of works O pertaining to different branches of knowledge, so that the study of Q is bound to be a co-disciplinary pursuit, bringing together for a common endeavour tools from different "disciplines". It should be stressed at this point that what I've called a citizen is not a person reduced to being a member of a political community. But, much to the contrary, he/she is considered according to his/her accomplishments and potential, particularly as an inquirer into questions of any breed. It results from this that a citizen does not only have to be educated in many fields but, in the procognitive perspective of the new didactic paradigm, a citizen must be ready to study and learn, even from scratch, fields of knowledge new to him/her. A citizen is not only a law-abiding person; he/she also has to become a knowledgeable person, indefinitely ready to study works hitherto unknown to him/her, just because some inquiry calls for their study.

The citizen I portray here may feel unable to live up to what is thus required of him/her. This feeling essentially results from the old didactic organisation of school and society that has imposed upon us the illusion according to which, for any knowledge need we may experience, there somewhere exists a providential person who can teach us whatever we want to know. Such a puerile belief leads to passivity and submission to events outside our reach. In the paradigm of questioning the world, attending a course or a conference on some subject of interest is certainly not disregarded. But we should take them as means to a common end—learning something on some determined work O supposed to be useful in order to bring forth an answer $A^{\heartsuit}$ to question Q. In such a situation, because of a relation to ignorance and knowledge resulting from exposure to the old school paradigm, we are prone to feel frustrated at not having all the knowledge needed—all of history, biology, mathematics, physics, chemistry, philosophy, linguistics, sociology, and so on indefinitely. The character implicitly fantasised here is what I've come to call *an esoteric* (using thus the adjective also as a noun), who is supposed to already know all the knowledge needed (the idea most people have of "a historian", "a biologist", "a mathematician", "a physicist", etc., is commonly akin to this fantasy). By contrast, *an exoteric* has to study and learn indefinitely, and will never reach the elusive status of esoteric. Indeed, all true scholars are exoteric and should remain so in order to remain scholars: esotericism, as I define it here, is a fable.

The citizen in the new paradigm is therefore called upon to become Herbartian, procognitive, and *exoteric*. How can we promote this new citizenship? Beyond being possessed by the epistemological passion necessary to go all the way from pure ignorance to adequate knowledge, a crucial condition is, for sure, the *time* allotted to study and research in an adult's life. More often than not, it seems, this time tends to zero as years pass by. In this respect, I suggest that we repeat again and again the founding trick of the ancient Greeks—that of transmuting leisure time, which some of our contemporaries seem to enjoy so abundantly, into study and research time, in the authentic tradition of *skhole*. Such a pursuit pertains to what Freud once called *Kulturarbeit*, "civilisational work"—a radical change still to come, which is a sine qua non of the emergence of the new didactic paradigm.

The problem of the time allotted to study and research has an easy solution when it comes to ordinary schooling: youngsters go to school to study, in accordance with *skhole*'s defining principle. But in what measure does school welcome the new didactic paradigm? I shall not dwell too long on this subject. I will, however, suggest that in too many cases, the so-called "inquiry-based" teaching resorts to some form or another of "fake inquiries", most often because the generating question Q of such an inquiry is but a naive trick to get students to find and study works O that the teacher will have determined in advance. Of course, this is the plain consequence of the domination of the paradigm of visiting works, which implies that curriculum contents are defined in terms of works *O*. In contradistinction, in the paradigm of questioning the world, the curriculum is defined in terms of *questions Q*. However, the works *O* studied in consequence of inquiring into these questions *Q* play a central role in the process of defining and refining the curriculum: starting from a set *Q* of "primary" questions, the curriculum contents *C* eventually studied will include the questions *Q* and answers $A^{\heartsuit}$, together with the answers $A^{\diamond}$ and the works *O*.

At this point two questions arise, though. The first question relates to the set *Q* of "primary" questions: where do these questions come from, and according to what mechanisms? In the case of a national curriculum, the set of primary questions to be studied at school constitutes the "core curriculum", and therefore the foundation of the national pact between society and school. Consequently, it is up to the nation to watchfully and democratically decide what the set *Q* will consist of and to periodically revise and update its contents on the basis of a careful monitoring of the curriculum's life-cycle. Because it is essential to the relationship between a society and its schooling system, the core curriculum—i.e. the "primary" questions—will play a decisive part in the society's *skhole*. But it should be obvious that the curriculum is not precisely defined by the primary questions alone. The inquiries entailed by these questions are in no way uniquely defined: as we know, an inquiry may follow different paths of study and research, and the questions inquired into as well as the other works encountered and, up to a point, studied, are indeed path-dependent. As a result, even if the *core* curriculum (in the sense defined above) has been made precise, the ensuing curriculum might well look fuzzily defined because of its built-in variability. How can this situation be managed for the better?

Let us consider didactic triplets (*X*, *Y*, *O*) with *O* a (finite!) family of questions. We can envisage two types of didactic triplets associated with a class of students. First, there is a *seminar*, in which *O* is a dynamic family of questions comprising the primary questions and the questions their study will generate. (Remember that the scenario delineated is supposed to apply to advanced students as well as to… toddlers, so that the words I use here must be taken in a very broad sense, which allows for their adaptation to a wide variety of concrete conditions.) This seminar will essentially be co-disciplinary, for primary questions rarely fall into a unique disciplinary domain. Second, there will be disciplinary *workshops* to study the questions and works put forward in the seminar but which pertain essentially to a given discipline—there will be for example a chemistry workshop, a mathematics workshop, a history workshop, a biology workshop, and so on. The activated workshops may vary depending on the primary questions studied in the seminar. The key fact is that, in this two-step process (seminar *plus* workshops), some works *O* and disciplines will be insistently recurrent, because they will be more often called upon in the inquiries, while others will be encountered erratically or will almost never turn up. This "degree of mobilization" of a work *O*, if averaged nationally across all the seminars held at a given school level, gives the "degree of membership" of the work *O* to the curriculum regarded, metaphorically, as a continually redefined fuzzy set—a view more adequate to the true nature of a real curriculum. As indicated above, and contrary to the age-old habit of imposing a curriculum founded essentially on opinion, the paradigm of questioning the world makes it possible to bring to light in an organic way which resources are really used in trying to question and know the world, both natural and social.

What Will Be the Place of Mathematics?

At a given point in time, an inquiry may come to a stop because some useful tool proves unavailable to the inquirers. One major reason for which an inquiry may thus grind to a halt is that the mastery of essential parts of some work *O*, ideally required to continue progress, lie well beyond the inquirers' reach. This, it should be stressed, is the common law of inquiry, be it at school or in a research team, and is definitely *not* the preserve of "low-level exoterics": it is part and parcel of the art of inquiry—such an "incident" is but one of the twists and turns in an inquirer's venture. But the path followed in a given inquiry, whatever its determinants, has crucial consequences in the didactic scenario displayed above: if a work *O* is very rarely drawn upon in seminars and workshops across the nation, then this work *O* will eventually vanish from the national curriculum. To be quite frank, this can result in the disappearing of parts of traditional school disciplines; for the place occupied by a discipline in the new curriculum will depend on its effectiveness in providing tools for inquiring into the curriculum-generated questions; it will depend no longer on any formerly or recently established hierarchy of disciplines, held to be the unquestionable legacy of the past. Traditionally flourishing disciplines should then worry about their future at school: will they continue to thrive or will they soon languish? The question is put to every discipline, and especially to mathematics.

If knowledge is valued according to what it enables us to rationally understand and achieve, the problem we are confronted with is not so much the fate of the disciplines as the value and quality of the inquiries going on in the seminars and workshops. From this point of view, the foregoing scenario can be improved substantially by allowing for the possibility to append "control questions" to any question pertaining to the curriculum. In some sense, this adds, to the bottom-up information flow emanating nationwide from the seminars and workshops, a top-down regulatory control on schools, operated by supervisory authorities. Any question *Q* can indeed be supplemented meaningfully by one or a series of "side questions" Q^* that will be touchstones for controlling the quality, thoroughness and profundity of an inquiry into question *Q*. It is in this way that it becomes possible to point out meaningfully—and not out of sheer pretentiousness—the utility of such and such work *O* to get deeper into the question studied. For example, to a question about biodiversity, one might relevantly add a question about *genetic* diversity and, in turn, a question about the meaning and interest of Eq. (1) above, a question likely to draw the inquirers' attention to the importance of… mathematics in inquiring into genetic diversity.

For mathematics as well as for a myriad of works pertaining to the most varied fields of knowledge, such a system of control questions seems indispensable to remind the *x* and the *y* that inquiring into some question may require the use of tools that will first appear, from within the cultural limits that they are precisely expected to transcend, as far removed from the matter under study. This is particularly true in the case of mathematical works. For deep-rooted historical reasons, mathematics is today both formally revered and, at the same time, energetically shunned. Numerous people flee away from mathematics as soon as they are no longer obliged to "do" mathematics. This has determined many mathematics educators to engage in a strategy of seduction, with a view to regaining the favour of "mathematical non-believers" by convincing them that, as the saying goes, "maths is fun"! Let me say tersely that this strategy has two main demerits and that, in my view, it should be as such utterly discarded. The first defect seems to be liberally ignored in today's educational world: for deep political and moral reasons, the instruction imparted at school must refrain from manipulating feelings and beliefs—we must be unimpeachable as far as the liberty of conscience of *x* (and *y*) is concerned. Consequently, mathematics educators must resist the temptation to try to induce students to "love" mathematics: their unique mission is to let them *know* mathematics, which is a bit more demanding! Love and hate are personal, intimate feelings that belong to the private sphere proper. Of course, it is highly probable that knowing mathematics better will result in some form of keenness towards mathematics. But all this entirely pertains to every single person's conscience.

The second defect of the much acclaimed seduction strategy is its very low yield, if I may say so. The problem with mathematics—as with other disciplines—is a mass problem. The root of it lies, in my view, in the process of cultural rejection that mathematics has suffered for a long time now, with the crucial consequence that, outside mathematical institutions proper, mathematics vanishes from the "lay" scene, so much so that many documents about topics not substantially foreign to mathematics can show no trace at all of mathematics, a fact which jeopardises the quality of many inquiries. Let me give here a simple example. Consider the question "Why does ice float in water?" Part of the answer is: because ice is less dense than liquid water. Now why is ice less dense than liquid water? The usual answer is that the arrangement of H_2O molecules occupies *more* space in ice than in liquid water. A closer look at this answer leads to some easy calculations (Ravera 2012). Indeed, it can be shown that, under certain conditions, the unit cell of ice has a height of 737 pm (i.e. 737×10^{-12} m), with its base a rhombus with sides of length 452 pm and an angle of 60º. The volume of the unit cell is therefore

$$V = \frac{\sqrt{3}}{2} \times 452^2 \times 737 \times 10^{-33}\mathrm{L} \tag{2}$$

The molar mass of water is approximately 18 g/mol. The mass of a unit cell of ice is known to be that of four molecules of water. Avogadro's number is taken here to be 6.02×10^{23} mol^{-1}. Hence the mass M of a unit cell:

$$M = \frac{4 \times 18}{6.02 \times 10^{23}} \text{g} \tag{3}$$

The density of ice is therefore:

$$d = \frac{M}{V} \approx 917 \text{g/L} \tag{4}$$

This (approximate) result confirms that ice is lighter than liquid water. The calculation uses elementary tools that are all (supposedly) mastered at age 15. In spite of this, this calculation is generally withheld from most relevant presentations available on the Internet. This is no exception to the rule. In a majority of cases, the mathematics of the topic being presented is decidedly absent, as if it had never existed. This is typically what mathematics educators must combat. In this respect, as far as mathematics is concerned, the "touchstone questions" that should be appended tentatively to any question proposed for study come down to this: "What are the mathematics of the matter, and how can awareness of them enhance the quality of your answer?"

Is this really a way out of the historic trap in which mathematics has been lured? I believe so. The seduction strategy, which is successful with an insignificant number of people, is but another pitfall. In my view, the only realistic solution will consist in trying to rationally persuade the citizens and, to begin with, the students that dispensing with mathematics may crucially impoverish our understanding and drastically reduce the quality of our involvement in both the natural and the social world. This, of course, will not be achieved through fine words only. It needs daily action, in schools as well as outside schools, especially in the leisure time given to learning by the citizenry to enrich their lives. In this pursuit, mathematics educators will play a crucial, though different, part.

For centuries, mathematics as a cultural institution thrived on a twofold self-presentation: it was understood as being composed, on the one hand, of "pure" mathematics, and, on the other hand, of "mixed" mathematics, with its pervasive ethos and slightly imperialistic touch. The "mixed" part, later called "applied" mathematics, has steadily declined at school during the last decades, while what remained of the former part—pure, though elementary, mathematics—tried to symbolise and maintain the old "empire". It is my belief that this time has now come to an end. Today, we have to revive the epistemological spirit of mixed mathematics, although without any cultural arrogance, but with the political and social will necessary to revitalise the idea that mathematics is for us, human beings, a solution, not a problem.

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Mathematics for All? The Case for and Against National Testing

Gilah C. Leder

Abstract National numeracy tests were introduced in Australia in 2008. Their format and scope are described and appraised in this paper. Of the various group performance trends presented in the annual national NAPLAN reports two (gender and Indigeneity) are discussed in some detail. For these, the NAPLAN findings are compared with broader international data. Recent Australian research spawned by, or benefitting from, the NAPLAN tests is also summarised. In some of this work, ways of using national test results productively and constructively are depicted.

Keywords National tests · Gender · Indigeneity

Introduction

> It should come as no surprise… that the introduction of a national regime of standardised external testing would become a lightning rod of claim and counter-claim and a battleground for competing educational philosophies. The National Assessment Program—Literacy and Numeracy (NAPLAN) is a substantial educational reform. Its introduction has been a source of debate and argument (Sidoti and Keating 2012, p. 3).

Formal assessment of achievement has a long history. Kenney and Schloemer (2001) point to the use, more than three thousand years ago, of official written examinations for selecting civil servants in China. The birth of educational assessment is, however, generally traced to the 19th century and its subsequent growth has undoubtedly been intertwined with advancements in the measurement of human talents and abilities (Lundgren 2011). Over time the development of large scale, high stake testing and explorations of its results have proliferated. "Many nations", wrote Postlethwaite and Kellaghan (2009), "have now established national assessment mechanisms with the aim of monitoring and evaluating the quality of their education systems across several time points" (p. 9). More recently, Eurydice (2011) also drew attention to the widespread practice of national testing

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throughout Europe, confined in some countries to a limited number of core curriculum subjects but in others comprising a broad testing regime. Large scale national assessment programs, with particular emphasis on numeracy and literacy[1], were introduced in Australia in 2008—after extensive consultation and much heated debate within and beyond educational and political circles.

The NAPLAN Numeracy Tests

Until 2007, Australian states and territories ran their own numeracy and literacy testing programs. Although much overlap could be found in the assessment instruments used in the different states, there were also variations—some subtle, others substantial—in these tests.

The first National Assessment Program—Literacy and Numeracy (NAPLAN) tests were administered in May 2008 and have been conducted annually since then. For the first time, students in Years 3, 5, 7, and 9, irrespective of their geographic location in Australia, sat for a common set of tests, administered nation-wide. The Numeracy tests contain both multiple choice and open-ended items. Their scope and content are informed by the *Statements of Learning for Mathematics* (Curriculum Corporation 2006). The 'what' students are taught is described by four broad numeracy strands. These are Algebra, function and pattern; Measurement, chance and data; Number; and Space, though some questions may overlap into more than one strand. Instructional strategy, the 'how' of mathematics is described by proficiency strands. "The proficiency strands—Understanding, Fluency, Problem solving and Reasoning—describe the way content is explored or developed through the 'thinking' and 'doing' of mathematics" (Australian Curriculum, Reporting and Assessment Authority ACARA 2010). In Years 3 and 5, the papers are expected to be completed without calculator use. Two distinct papers are set for Year 7 and 9 students—one is expected to be completed without the use of a calculator; for the other calculator usage is allowed.

The NAPLAN numeracy scores for Years 3, 5, 7, and 9 are reported on a common scale which is divided into achievement bands. For each of these year levels, the proportion of students with scores in the six proficiency bands considered appropriate for that level is shown. For Year 3, 5, 7, and 9 these are bands one to six; three to eight; bands four to nine; and bands five to ten respectively. Each year, results of the NAPLAN tests are published in considerable detail, distributed to each school, and made readily available to the public.

The advantages anticipated by the introduction of national tests to replace the variety of tests previously administered by the different Australian states and

[1] Sample assessment tests have been administered to selected groups of students in Years 6 and 10 in Scientific Literacy (Year 6 students only), Civics and Citizenship, and Information Communication Technology Literacy. These sample assessments were introduced respectively in 2003, 2004, 2005 and are held on rolling a three-yearly basis.

territories were similar to those commonly put forward in the wider literature (e.g., Postlethwaite and Kellaghan, 2009) as a rationale or justification for introducing national tests: assessment consistency across different constituencies, increased accountability, and a general driver for improvement.

ACARA is responsible for the development of the national assessment program and the collection, analysis, and reporting of data. The procedures followed are described clearly on the ACARA website and are consistent with those generally advocated for large scale assessment testings (Joint committee on testing practices 2004). Guidance on interpreting the vast amount of data in the National Report is provided in the document itself (ACARA, 2011a) and in multiple ancillary documents (see e.g., ACARA, 2011b; Northern Territory Government n.d). NAPLAN achievement outcomes are reported not only at the national level, but also by state and territory data; by gender; by Indigenous status; by language background status[2]; by geolocation (metropolitan, provincial, remote and very remote); and by parental educational background and parental occupation. Each of these categories which are clearly not mutually exclusive, has been shown, separately, to have an impact on students' NAPLAN score. Broad performance trends for the different groupings have been summarised as follows:

> In Australia, girls have typically performed better on tests of verbal skills…, while boys have typically performed better on tests of numerical skills… Children from remote areas, children from lower socioeconomic backgrounds and children of Indigenous background have tended to perform less well on measures of educational achievement (NAPLAN 2011b, p. 255).

It is beyond the scope of this paper to look at each of the categories mentioned above. Instead, the focus is on two groups of special interest: *girls/boys* and *Indigenous* students. What trends can be discerned in the years of NAPLAN data available at the time of writing this paper?

Trends in NAPLAN Data: Gender and Indigeneity

Data for Years 3 and 9 by gender and Indigeneity are shown in Tables 1 and 2 respectively.

From these tables it can be seen that:

Gender

- The mean NAPLAN score for males is invariably higher than that for females.
- The standard deviation for males is also consistently higher than for females, that is the range of the NAPLAN scores for males is higher than that for females.

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[2] LBOTE, language background other than English, defined as "A student is classified as LBOTE if either the student or parents/guardians speak a language other than English at home."

Table 1 Numeracy Year 3 students, NAPLAN achievement data 2008–2011

Group \year		All	M	F	Indigenous	Non-Indigenous	Indigenous year 5[a]
2008	Mean	396.9	400.6	393.1	327.6	400.5	408.0
	S.D	70.4	72.8	67.6	70.6	68.4	65.8
	≥National min[b] (%)	95.0 %	94.6 %	95.5 %	78.6 %	96.0 %	69.2 %
2009	Mean	393.9	397.5	390.2	320.5	397.7	420.5
	S.D	72.9	75.3	70.0	76.0	70.6	66.4
	≥National min (%)	94.0 %	93.5 %	94.5 %	74.0 %	95.2 %	74.2 %
2010	Mean	395.4	397.8	392.9	325.3	399.0	416.9
	S.D.	71.8	74.0	69.3	71.2	69.8	70.5
	≥National min (%)	94.3 %	93.7 %	94.9 %	76.6 %	95.3 %	71.4 %
2011	Mean	398.1	402.6	393.5	334.4	401.7	421.1
	S.D.	70.6	73.0	67.6	65.0	69.1	64.0
	≥National min (%)	95.6 %	95.2 %	96.0	83.6 %	96.4 %	75.2 %

[a] I refer to the data in the last column later in the paper. To save space the information is included in this table

[b] National minimum standards: The second lowest band on the achievement scale represents the national minimum standard expected of students at each year level

Table 2 Numeracy Year 9 students, NAPLAN achievement data 2008–2011

Group \year		All	M	F	Indigenous	Non-Indigenous	Year 7 Non-Indigenous
2008	Mean	582.2	586.5	577.6	515.1	585.7	548.6
	S.D	70.2	72.0	68.1	65.6	68.7	71.6
	≥National min (%)	93.6 %	93.7 %	93.6 %	72.5 %	94.8 %	96.4 %
2009	Mean	589.1	592.4	585.6	520.2	592.4	547.0
	S.D	67.0	69.2	64.4	63.2	65.3	69.4
	≥National min (%)	95.0 %	94.7 %	95.2 %	75 %	96 %	95.8 %
2010	Mean	585.1	591.1	578.8	515.2	588.5	551.4
	S.D	70.4	72.7	67.4	64.7	68.8	70.8
	≥National min (%)	93.1 %	93.3 %	92.9 %	70.4 %	94.3 %	96.1 %
2011	Mean	583.4	589.3	577.3	515.8	586.7	548.5
	S.D	72.1	74.7	68.7	62.2	70.8	72.1
	≥National min (%)	93.0 %	93.0 %	93.0 %	72 %	94.1 %	95.5 %

(Data in both tables adapted from ACARA 2011a)

- At the Year 3 level a higher proportion of females than males score above the national minimum standard NAPLAN score. There is no such consistency at the Year 9 level, with a marginally higher proportion of males performing at or above the minimum level in some years (e.g., 2008, 2010) and a marginally higher proportion of females performing at or above the minimum level in other years (e.g., 2009).

Indigeneity

- Each year, non-Indigenous students do (a lot) better than Indigenous students. From Table 1 it can be seen that Year 5 Indigenous students performed just above the level of Year 3 non-Indigenous students; from Table 2 that Year 9 Indigenous students performed below the level of Year 7 non-Indigenous students.
- In 2011, there was a noticeable increase, compared with the previous years, in the percentage of Indigenous students at Year 3 who performed at or above the national minimum standard. No such increase is apparent at the other Year levels.

Also relevant are the following:

- In 2011, between 240,000 and 250,000 non-Indigenous students sat for the Years 3, 5, 7, and 9 NAPLAN papers. For the Years 3, 5, and 7 papers close to 13,000 Indigenous students participated. A smaller number, about 10,000 sat for the Year 9 paper. Thus at the different Year levels, Indigenous students comprised between 4 and 5 % of the national groups involved in the NAPLAN tests.[3]
- The exemption rates for the two groups are similar: around 2 % for Indigenous students and about 1 % for non-Indigenous students.

These summaries for gender and Indigenous performance outcomes are set against a broader context in the next sections.

Gender

In many countries, including Australia, active concern about gender differences in achievement and participation in mathematics can be traced back to the 1970s. Two reliable findings were given particular prominence: that consistent between-gender differences were invariably dwarfed by much larger within-group differences; and that students who opted out of post compulsory mathematics courses often restricted their longer term educational and career opportunities. These generalizations remain relevant.

[3] The proportion of school students in Australia identified as Aboriginal and/or Torres Strait Islanders has risen from 3.5 % in 2001 to almost 5 % in 2011(http://www.abs.gov.au/ausstats/abs@.nsf/Lookup/4221.0main+features402011).

Evidence of progress towards gender equity more broadly than with respect to mathematics learning specifically has been mapped in many different ways:

> Whereas the challenge of gender equality was once seen as a simple matter of increasing female enrolments, the situation is now more nuanced, and every country, developed and developing alike, faces policy issues relating to gender equality. Girls continue to face discrimination in access to primary education in some countries, and the female edge in tertiary enrolment up through the master's level disappears when it comes to PhDs and careers in research. On the other hand, once girls gain access to education their levels of persistence and attainment often surpass those of males. High repetition and dropout rates among males are significant problems (UNESCO 2012, p. 107).

As can be seen from large scale data bases such as NAPLAN, some gender differences in mathematics performance remain. What explanations for this have been proffered?

Explanatory Models

Over the years a host of, often subtly different, explanatory models for gender differences in mathematics learning outcomes have been proposed. They invariably contain a range of interacting factors—both person-related and environmental. Common to many models is an

> …emphasis on the social environment, the influence of other significant people in that environment, students' reactions to the cultural and more immediate context in which learning takes place, the cultural and personal values placed on that learning and the inclusion of learner-related affective, as well as cognitive, variables (Leder 1992, p. 609).

A comprehensive overview of research concerned with gender differences in mathematics learning is beyond the scope of this paper. Instead, some recent publications, the majority with at least a partial cross-national perspective and published in a variety of outlets, are listed to sketch the range of factors invoked as explanatory or contributing factors for the differences still captured. Included is work in which the need for a repositioning of perspective to examine gender differences, via a different theoretical (often feminist and/or socio-cultural) framework, is prosecuted, as well as several articles in which there are strong attempts to rebut the notion that gender differences persist.

Gender Differences: Possible Explanations

- Kaiser et al. (2012) found, in a large study involving over 1,200 students, that "the perception of mathematics as a male domain is still prevalent among German students, and that this perception is stronger among older students. This is either reinforced by the peer group, parents or teachers" (p. 137).
- Kane and Mertz (2012) concluded "that gender equity and other sociocultural factors, not national income, school type, or religion per se, are the primary determinants of mathematics performance at all levels of boys and girls" (p. 19).

- Stoet and Geary (2012) challenged but ultimately supported the notion of stereotype threat (provided it is carefully operationalized) as an explanation for the higher performance of males in mathematics, particularly at the upper end.
- Wai et al. (2010) examined 30 years of research "on sex differences in cognitive abilities" and focussed particularly on differences in favour of males found in the top 5 %. As well as highlighting the role of sociocultural factors they concluded: "Our findings are likely best explained via frameworks that examine multiple perspectives simultaneously" (p. 8).
- "Traditionally, all societies have given preference to males over females when it comes to educational opportunity, and disparities in educational attainment and literacy rates today reflect patterns which have been shaped by the social and education policies and practices of the past. As a result, virtually all countries face gender disparities of some sort" (UNESCO 2012, p. 21).

Gender Differences: Have They Disappeared?

- Else-Quest (2010) used a meta-analysis of PISA and TIMSS data to examine the efficacy of the gender stratification hypothesis (that is, societal stratification and inequality of opportunity based on gender) as an explanation for the continuing gender gap in mathematics achievement reported in some, but not in other, countries. They concluded that "considerable cross-national variability in the gender gap can be explained by important national characteristics reflecting the status and welfare of women" (p. 125) and that "the magnitude of gender differences in math also depends, in part, upon the quality of the assessment of mathematics achievement" (p. 125).
- Hyde and Mertz (2009) drew on contemporary data from within and beyond the U.S. to explore three major questions: (1) "Do gender differences in mathematics performance exist in the general population? (2) Do gender differences exist among the mathematically talented? (3) Do females exist who possess profound mathematical talent?" (p. 8801). They summarised respectively: (1) Yes, in the U.S. and also in some other countries; (2) Yes, there are more males than females are amongst the highest scoring students, but not consistently in all ethnic groups. Where this occurs, the higher proportion of males is "largely an artefact of changeable sociocultural factors, not (due to) immutable, innate biological differences between the sexes" (p. 8801); and (3) Yes, there are females with profound mathematical talent.

Gender Differences: Looking for New Directions

- Erchick (2012) argued that consideration of conceptual *clusters*, rather than topics in relative isolation, should lead to new questions in as yet fallow ground to be found in the field of gender differences in mathematics. Three clusters are proposed: "Feminism/Gender/Connected Social Constructs; Mathematics/Equity/Social Justice Pedagogies; and Instruction/Perspectives on Mathematics/Testing" (p. 10).

- Jacobsen (2012) is among many of those who argue for a reframing of the deficit model approach to gender differences in which male performance and experience are considered the norm to one recognizing the social construction of gender and accepting that females may learn in different, but not inferior, ways from males. One approach to translating this theoretical perspective into practice is also described.

In some of the publications listed (as well as in others not listed here) gender differences are minimized while in others they are given centre-stage. Collectively, a complex rather than simplistic network of interweaving and sometimes contrasting pressures emerges from this body of work. After four decades of research on gender and mathematics, there is only limited consensus on the size and direction of gender differences in performance in mathematics and stark variation in the explanations put forward to account when differences are found.

The NAPLAN scores summarised in Tables 1 and 2 also require a nuanced rather than uni-dimensional reading. When performance on the NAPLAN test is described in terms of mean scores, the small but consistent gender differences in favour of males mirror those obtained in other large scale tests such as the Trends in International Mathematics and Science Study (TIMSS) and the OECD Programme for International Student Assessment (PISA)[4]. But in terms of another set of NAPLAN achievement criteria, the percentage of students achieving above the minimum national average, the small differences reported generally favour girls in the earlier years of schooling, in each of 2008–2011 at Year 3; for three of the four years (2009–2011) for Years 5 and 7; but in only one year (2009) at the Year 9 level. Clearly, gender differences in performance on the NAPLAN tests are small, consistent or variable, depending on the measuring scale and the method of reporting used.

Assessment: Gender Neutral or not?

That gender differences in mathematics learning may be concealed or revealed by the assessment method used is not a new discovery. Else-Quest et al. (2010) judged that "the magnitude of gender differences in math also depends, in part, upon the quality of the assessment of mathematics achievement" (p. 125). Dowling and

[4] Differences in the samples involved in the three tests are worth noting. NAPLAN is administered to all students in Years 3, 5, 7, and 9. It is best described as a census test. The TIMSS tests, aimed at students in Years 4 and 8, and the PISA tests administered to 15-year-old students, are restricted to "a light sample (of) about 5 % of all Australian students at each year or age level" (Thomson, p. 76).

Burke (2012) pointed to the 2009 General Certificate of Secondary Education examinations in the U.K. as the first occasion in a decade for boys to perform better than girls in an external examination. “This reversal coincided with a change in the form of the examination” (p. 94), they noted.

A now somewhat dated, yet still striking, example of the impact of the format of examinations on apparent gender differences in mathematics achievement is provided by Cox et al. (2004). They tracked gender differences in performance in the high stake, end of Year 12 examinations in Victoria, Australia for the years 1994–1999, a sustained period of stability in the state’s external assessment regime. Student performance in three different mathematics subjects—Further Mathematics (the easiest and most popular of the three mathematics subjects offered at Year 12), Mathematical Methods (a pre-requisite for many tertiary courses), and Specialist Mathematics (the most demanding of the three mathematics subjects)—were among the results inspected. For each of these three subjects there were three different examination components. These were common assessment task (CAT) 1 consisting of a school assessed investigative project or problem, to be completed over several weeks; CAT 2, a strictly timed examination comprising multiple choice and short answer questions; and CAT 3, also a strictly timed examination paper with problems requiring extended answers. Thus CATs 2 and 3 followed the format of traditional timed examinations.

During the period monitored, a student enrolled in a mathematics subject in Year 12 was required to complete three assessment tasks in that subject. A test of general ability was also administered to the Year 12 cohort. These combined requirements provided a unique opportunity to compare the performance of the same group of students on timed and untimed examinations and on papers with items requiring substantially and substantively different responses. In brief:

- Males invariably performed better (had a higher mean score) than females on the mathematics/science/technology component of the general ability test.
- In Further Mathematics, females outperformed males in CAT 1 and in CAT 2 in all of the six years of data considered, and on CAT 3 for five of the six years.
- In Mathematical Methods, females performed better than males in all of the six years on CAT 1; males outperformed females on CAT 2 and CAT 3 for the six years examined.
- In Specialist Mathematics, females performed better than males in all of the six years on CAT 1 and in five of the six years on CAT 3. However males outperformed females on CAT 2 for each of the six years examined.

Thus whether as a group males or females could be considered to be “better” at mathematics depends on which subject or which test component is highlighted. If the least challenging and most popular mathematics subject, Further Mathematics, is referenced then the answer is females. If for all three mathematics subjects the focus is confined to the CAT 1 component, the investigative project or problem assessment task, done partly at school and partly at home, then again the answer is females. But if the focus is on the high stake Mathematical Methods subject, the subject which often serves as a prerequisite for tertiary courses, and on the traditional examination formats of CAT 2 and CAT 3 in that subject, then the answer is males. Collectively these data illustrate that the form of assessment employed can influence which group, males or females, will have the higher mean performance score in mathematics. Would the small but consistent differences found in favour of males' mean performance on the NAPLAN papers disappear if the tests were changed from their traditional strictly timed, multiple choice and short answer format to one resembling the CAT 1 requirements?

Changes to the Year 12 assessment procedures in Victoria were introduced in 2000, seemingly in response to concerns about student and teacher workload and to issues related to the authentication of student work for the teacher-assessed CATs. The changes were described by Forgasz and Leder (2001) as follows:

> For the three VCE mathematics subjects the assessment changes involve the CAT 1 investigative project task being replaced with (generously) timed, classroom based tasks, to be assessed by teachers but with the scores to be moderated by externally set, timed examination results. It is worth recalling that it was on the now replaced format of CAT 1, the investigative project, that females, on average, consistently outperformed males in all three mathematics studies from 1994 to 1999. Is it too cynical to speculate that this consistent pattern of superior female achievement was a tacit factor contributing to the decision to vary the assessment of the CAT 1 task? It is difficult to predict the longer term effects of the new… assessment procedures on students' overall mathematics performance and study scores. Is there likely to be a return to earlier patterns of superior male performance in mathematics? If so, will this satisfy those who are arguing that males are currently the educationally disadvantaged group? (p. 63)

Indigeneity

That there is no ambiguity about the differences in the performance on the NAPLAN tests between Indigenous and non-Indigenous students is clearly apparent from Tables 1 and 2, and widely emphasized elsewhere. Thomson et al. (2011), for example, examined the 2009 PISA data for Australian students and reported a substantial difference between the average performance of Indigenous and non-Indigenous students on the mathematical literacy assessment component. What message is conveyed by the reporting of these differences?

Gutiérrez (2012) has compellingly used the term "gap gazing" to describe preoccupation with performance differences between selected groups of students and has argued convincingly that highlighting such differences can be counter-productive and reinforce stereotyping. "In its most simplistic form, this approach points out there is a problem but fails to offer a solution… (T)hat it is the analytic lens itself that is the problem, not just the absence of a proposed solution" (Gutiérrez 2012, p. 31) should not be ignored.

As mentioned earlier, the results of NAPLAN testings are widely disseminated and described in media outlets. Forgasz and Leder (2011) compared the more nuanced reporting of students' results on these tests in scholarly outlets with the more superficial tone of print media reports. According to these authors "media reports on students' performance in mathematics testing regimes appear to rely heavily on the executive summaries that accompany the full reports of these data… (T)he more detailed and complex analyses undertaken of entire data sets are often omitted" (p. 218). These comments apply equally to the simplified reporting of gender differences, and differences in performance between Indigenous and non-Indigenous students. It is the arguments advanced in the "more superficial tone of the print media reports" that capture the attention of the general public and shape the sociocultural norms and expectations of the broader society. These norms and expectations are, as mentioned above, among the factors identified by Hyde and Mertz (2009) (among others) as contributing to or averting the emergence of gender difference in performance in mathematics.

Unease has been expressed, both nationally and internationally, about the negative impact of high stake, national testing. Common concerns:

> range from the reliability of the tests themselves to their impact on the well-being of children. This impact includes the effect on the nature and quality of the broader learning experiences of children which may result from changes in approaches to learning and teaching, as well as to the structure and nature of the curriculum (Polesel 2012, p. 4).

Disadvantages stemming from blanket reporting of results in large scale examinations have also been widely discussed and selectively elaborated by Berliner (2011). Although his remarks were aimed at indiscriminate and shallow reporting of the PISA results of selected groups of students in the USA, many of his comments are equally applicable to the coverage of performance of Indigenous students on the Australian NAPLAN tests. Three of his concerns seem highly relevant with respect to the portrayal of the numeracy results of Indigenous students: "what was not reported", "social class", and "the rest of the curriculum".

What Was not Reported

Each year the NAPLAN data are published, the rather high proportion of Indigenous students who fail to meet the nationally prescribed minimum numeracy standard attracts the attention of educators and the wider community. As noted by Forgasz and Leder (2011), p. 213:

> The lower performance of Indigenous students, compared with the wider Australian school population, attracted sustained media attention. The discovery that Aboriginal students living in metropolitan areas as a group performed almost as well as their non-Indigenous peers received less media attention than the more startling finding that Aboriginal students living in remote communities had an extremely high failure rate of 70–80 %. 'A combination of low employment and poor social conditions were explanations offered for the distressingly poor performance… their different pass rates are the result of different schooling' (and a high level of absenteeism).

Aggregating data for all Indigenous students overlooks the large diversity within this group, the range of different needs that inevitably accompany such diversity and the fact that there are also Indigenous students who perform at the highest level on the NAPLAN test. Pang et al. (2011) identified how valuable data are lost when the performance of a multi ethnic group is described and treated as a single entity, rather than reportedly separately for each constituent group. “Educational policies and statistical practices in which achievement is measured using the (group) aggregate result in over-generalized findings” (p. 384) and hide, rather than identify, the strengths and needs of the different subgroups. These remarks are highly relevant given the many subgroups within the Indigenous community. Gross reporting of achievement outcomes fails to recognize the substantially different backgrounds, locations, needs, and capabilities of individuals within the broader group.

Social Class

There is much diversity in the home background of Indigenous students. Some live in remote areas; others in urbanized centres with access, inside and outside the home, to the same resources as non-Indigenous students. Social class related differences in performance apply to both Indigenous and non-Indigenous students. Although Indigeneity and family background are among the categories reported separately for group results on the NAPLAN test, there is no explicit information about the interactive effects of these variables on performance. To paraphrase Berliner (2011): the scores of Indigenous students, as a group, are likely to remain low, “not because of the quality of its teachers and administrators, necessarily, but because of the distribution of wealth and poverty and the associated social capital that exist in schools” (p. 83) in different metropolitan and remote communities. In the reporting of NAPLAN data for Indigenous students, the emphasis is disproportionately on those performing below expectations without sufficient recognition of confounding, contributing factors, while high performing Indigenous students remain largely invisible.

The Rest of the Curriculum

Under this heading Berliner (2011) focuses particularly on the narrowing of the curriculum, within and beyond mathematics, when the perceived scope and requirements of a national testing program overshadow other considerations and influence the delivery of educational programs. Although this criticism cannot be ignored with respect to the NAPLAN tests, I want to focus here on another, equally pervasive issue.

In recent years, many special programs for Indigenous students have been devised, and implemented with varying degrees of success. Difficulties associated with achieving a satisfactory synchrony between the intended and experienced curriculum for Indigenous students in remote communities have been discussed by Jorgensen and Perso (2012).

> In the central desert context, the Indigenous people speak their home languages which are shaped by, and also shape, their worldviews. In Pitjantjatjara, for example, the language is quite restricted in terms of number concepts. The lands of the desert are quite stark with few resources so the need for a complex language for number is limited. As such, the counting system is one of 'one, two, three, big mob'. It is rare that a collection of three or more occurs so the need for a more developed number system is not apparent. Even when living in community, the need for number is limited. Few people are aware of their birthdates, and numbers in community are very limited in terms of home numbers or prices in the local store. As such, the immersion in number that is common in urban and regional centres is very limited in remote communities. Therefore, many of the taken for granted assumptions about number that are part of a standard curriculum are limited in this context. This makes teaching many mathematical/number concepts quite challenging as it is not only the teaching of mathematical concepts and processes but a process of induction into a new culture and new worldview (Jorgensen and Perso, pp. 127–128).

Many Indigenous students live and learn in conditions more closely aligned to mainstream educational life in Australia than that depicted for Pitjantjatjara. Nevertheless, this snapshot of the prevailing norms and customs of one community highlights factors that will confound a simplistic interpretation of Indigenous group performance data.

NAPLAN and Mathematics Education Research

Not surprisingly, the introduction of NAPLAN has already fuelled a variety of research projects. An overview of work referring substantively to NAPLAN data and presented at the joint conference in 2011 of the Australian Association for Mathematics Teachers (AAMT) and Mathematics Education Research Group of Australasia (MERGA) is summarized in Table 3. It provides a useful indication of the scope and diversity of these investigations.[5] It is worth noting that the 2011 conference represented the first time the two associations held a fully joint conference. According to Clark et al. (2011) it was a unique opportunity for "practitioners and researchers to discuss key issues and themes in mathematics education, so that all can benefit from the knowledge gained through rigorous research and the wisdom of practice" (p. iii). In addition to "participants from almost every university in Australia and New Zealand, teachers from government and nongovernment schools systems throughout Australia and officers from government Ministries of Education" (Clark. et al. 2011), p. iii, there were authors and presenters from a range of other countries.[6]

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[5] Details are extracted from the published proceedings of this joint conference, comprising 130 papers. The proceedings consisted of two sets of papers: *Research papers* and *Professional papers*, reviewed respectively according to established MERGA and AAMT reviewing processes.

[6] These included Singapore, the United States of America, Papua New Guinea and the United Kingdom.

Table 3 NAPLAN related papers presented at the AAMT-MERGA conference in 2011[a]

Author and paper title	Summary of paper and findings/recommendations
Callingham mathematics assessment: everything old is new again?	Descriptive, rather than incisive, reference was made to the NAPLAN testing program in this presentation. Noted were: the contradiction between teachers generally being urged to use formative assessment and the prominence given to the external measure of numeracy provided by NAPLAN; that no significant change has been captured "across time for any grade group" from 2008 to 2010; and that the NAPLAN "results are used for accountability at the local level". A brief reference is also made to one setting where school based NAPLAN results are used to address elements on which students under-performed
Connolly refining the NAPLAN numeracy construct	An overview is provided of the development of the 2009 and 2010 NAPLAN numeracy test papers. The core content of the test is formally based on the set of nationally agreed curriculum outcomes. Avoided are topics for which there are between state variations in the time of the year they are taught. Items are reviewed multiple times with strong input from key stakeholders. Other factors taken into account in the construction of the test include: item difficulty; cognitive dimension (knowing, applying, and reasoning); item context (abstract or non-abstract); the influence of calculators on content (calculators are not allowed in the Years 3 and 5 papers but at the Year 7 and 9 levels both calculator and non-calculator papers are set); guidelines for item writing; and using accessible language. The Rasch model (Wright, 1980) is used to analyse the test results. This requires not only that certain pre-conditions are met (items are uni-dimensional, locally independent, and uniformly discriminating) but also "allows for sensible comparisons of test scores between different years"
Edmonds-Wathen locating the learner: indigenous language and mathematics education	The author describes the difficulties encountered by Indigenous language speaking students when faced with the typical development of number concepts in the curriculum in the early years of schooling and argues that a different, and group-tailored sequencing of material should be considered. The obstacles created by a "cognitive mismatch between the teacher and student" may fail to gauge accurately the students' understanding of, for example, spatial items and be reflected in low scores on such items on NAPLAN tests—invalidating simplistic comparisons between Indigenous and non-Indigenous students
Helme and Teese how inclusive is year 12 mathematics?	NAPLAN test data are part of a larger pool of material tapped to explore the mathematics learning experiences and expectations of students at schools in the northern suburbs of Melbourne, but are not discussed per se. Nevertheless the authors' conclusions are worth noting: "Perceptions of mathematics classrooms and mathematics teachers, and expectations of success, vary according to subject, (student's) gender and social background"

(continued)

Table 3 (continued)

Author and paper title	Summary of paper and findings/recommendations
Hill Gender differences in NAPLAN mathematics performance	The performance of females and males was compared on items on the Grade 3 and Grade 9 NAPLAN papers for 2008–2010. On each paper, there were some questions on which both groups performed (percentage correct) equally well. When group differences were found they more frequently favoured males than females (e.g., Year 3 paper NAPLAN 2009, no difference on 4 items, females outperformed males on 10 items, males outperformed females on 21 items; Year 9 papers NAPLAN 2010, no difference on 8 items, females outperformed males on 11 items, males outperformed females on 45 items). These trends suggest a "decline in achievement of females as they progress through their schooling"
Hurst connecting with the Australian curriculum: mathematics to integrate learning through the proficiency strands	The scope and demands of NAPLAN tests should not be allowed to dictate the content of the curriculum, nor restrict the instructional strategies used. According to the author, "NAPLAN test scores can greatly assist teachers if they are used appropriately". Rather than expanding on this theme, the author argues that teachers should "use a constructivist approach to teaching mathematics… (with) an emphasis on rich conceptual understanding as opposed to the mere acquisition of procedural knowledge" and provides some examples that support this theme
Morley Victorian Indigenous Children's responses to mathematics NAPLAN Items	Using data from the 2008 Years 5, 7, and 9 NAPLAN papers, "whether children of Indigenous background in Victoria, Australia, have different patterns of mathematical responses from the general population" is explored in this paper. Not surprisingly, both groups perform better on high facility than low facility items. Some advantage in favour of Indigenous students is found on the Space strand of the Year 7 paper but less so on the Year 9 paper. At that level, the Algebra strand appeared to be relatively more difficult for Indigenous students
Nisbet national testing of probability in years 3, 5, 7, and 9 in Australia: a critical analysis	The limits of large scale tests are discussed at some length. Often, Nisbet argues, these tests have "a bias towards mechanical processes, and away from problem solving and creativity". A focus on the probability questions in the 2009 and 2010 NAPLAN numeracy tests for Years 3, 5, 7, and 9 revealed that there were few probability items overall and that only one such item was included in each year level in the 2010 test. Furthermore, Nisbet argued, the scope and aspects of probability probed by the items seemed unacceptably constrained, "with most being multiple-choice items… (and) fundamentally recognition tasks… (to) identify the correct response"

(continued)

Table 3 (continued)

Author and paper title	Summary of paper and findings/recommendations
Pierce and Chick reacting to quantitative data: teachers' perceptions of student achievement reports	The authors use teachers' reactions, to national and school specific information, provided in table and graphical form in a NAPLAN report (usefulness and difficulty of the table and it accompanying annotations and explanations) to gauge the level of teachers' statistical literacy. "Reactions range(d) from those verging on the statistics-phobic… through to deep engagement with the issues". Many teachers preferred the graphical representation, although some welcomed the details provided in the table. Some "reacted strongly about the overwhelming complexity of the data… (and many) expressed uncertainty or confusion over some or all aspects of the data"
Sullivan and Gunningham a strategy for supporting students who have fallen behind in the learning of mathematics	Two items, of different levels of difficulty, from the 2009 NAPLAN Year 9 (no calculator) paper are used to illustrate how poorly some students are performing in mathematics. After this introduction, an out-of-class intervention (the *Getting Ready* intervention) used to prepare students from Years 3 and 8 for work being taught in their next mathematics lesson, is described. No further reference is made to NAPLAN tests
Tomazos improving mathematical flexibility in primary students: what have we learned?	It is often assumed that schools in higher socio-economic areas with students who perform well on NAPLAN tests do not need to provide extra support for their students. Data from a pilot program at such schools revealed not only that procedural approaches were often used when teaching calculation strategies but also that with "relatively little system input, experienced teachers' classroom practices can be changed" to incorporate greater use of flexible calculation strategies. NAPLAN data were again used as a measure in sample selection, but no further reference is made in the paper to NAPLAN tests
Vale, Davidson, Davies, Hooley, Loton, and Weaven using assessment data: does gender make a difference?	Students' performance on NAPLAN tests was among the measures used to determine a student's learning needs and to select students for specifically designed intervention programs. Gender related differences in performance are reported but no further reference is made to NAPLAN tests
White and Anderson teachers' use of national test data to focus numeracy instruction	The authors argue that, without wishing to advocate 'teaching to the test', much can be gained by teachers who use NAPLAN data from their own school to identify students' numeracy needs and develop instructional strategies to combat faulty practices or inadequate understanding. The approach adopted in one school is described. Whether "professional learning support (had) an impact on student learning and on teaching practice" was also examined

[a] To conform with space constraints, the entries in this table are not included separately in the reference list at the end of this paper. All can be found in Clark et al. (2011)

Reference to NAPLAN tests was made in some 10 % of the published papers. As can be seen from Table 3, aspects covered in these papers included issues pertaining to the development of the tests, interpreting the published results of the tests, using test results for curriculum development, and examining the performance of groups of interest, specifically boys and girls and Indigenous students. In some papers reference to NAPLAN data was very much secondary to the core issue explored, for example its (seemingly increasing) use as part of a series of measures to identify a specific group worthy, or in need of, further attention. What could be learnt from the NAPLAN tests about the performance and numeracy needs of high achieving students has, however, not yet attracted research attention. The finding by Pierce and Chick is particularly disturbing. When asked about the statistical and graphical summaries of NAPLAN data relevant to their students the reactions of teachers in their sample ranged "from those verging on the statistics-phobic … through to deep engagement with the issues". The NAPLAN national reports contain much valuable and potentially usable data. But how much of these are actually understood and used constructively?

Final Words

After collating information from some 70 public opinion polls in which questions about the efficacy of national tests were included, Phelps (1998) reported:

> The majorities in favor of more testing, more high-stakes testing, or higher stakes in testing have been large, often very large, and fairly consistent over the years and across polls and surveys and even across respondent groups (with the exception of some producer groups: principals, local administrators, and, occasionally, teachers) (p. 14) .

The data on which Phelps based his conclusions are now somewhat dated. How the Australian public today values national tests, and in particular the NAPLAN testing regime, is a question still waiting to be investigated. When planning future research activities, whether linked to NAPLAN, to gender and mathematics performance, to issues pertaining to Indigenous students, or to the needs of highly able students, the recommendation of Purdie and Buckley (2010) is well worth heeding:

> Although it is important to continue small, contextualised investigations of participation and engagement issues, more large-scale research is called for. Unless this occurs, advancement will be limited because sound policy and generalised practice cannot be extrapolated from findings that are based on small samples drawn from diverse communities (p. 21).

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Early Algebraic Thinking: Epistemological, Semiotic, and Developmental Issues

Luis Radford

Abstract In this article I present some findings of an ongoing 5-year longitudinal research program with young students. The chief goal of the research program is a careful and systematic investigation of the genesis of embodied, non-symbolic algebraic thinking and its progressive transition to culturally evolved forms of symbolic thinking. The investigation draws on a cultural-historical theory of teaching and learning—the theory of objectification—that emphasizes the sensible, embodied, social, and material dimension of human thinking and that articulates a cultural view of development as an unfolding dialectic process between culturally and historically constituted forms of mathematical knowing and semiotically mediated classroom activity.

Keywords Sensuous cognition · Vygotsky · Arithmetic versus algebraic thinking

Introduction

In light of the legendary difficulties that the learning of algebra presents to students, it has been suggested that a progressive introduction to algebra in the early grades may facilitate students' access to more advanced algebraic concepts later on (Carraher and Schliemann 2007). An early development of algebraic thinking may, in particular, ease students' contact with algebraic symbolism (Cai and Knuth 2011).

The theoretical grounding of this idea and its practical implementation remain, however, a matter of controversy. Traditionally, algebra has been taught only after students have had the opportunity to acquire a substantial knowledge of arithmetic.

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That is, arithmetic thinking has been assumed to be a prerequisite for the emergence and development of algebraic thinking. Clearly, an introduction to algebra in the early grades does not conform to such an assumption. Now, if this is so, if algebra needs not to come after arithmetic, the question is: What is the difference and relationship between these two disciplines? Evading these questions does not do us any favours.

In the next section, I briefly discuss the question of the relationship between algebra and arithmetic. Drawing on historical and educational research, I suggest an epistemological distinction between the forms of thinking that are required in both disciplines. Then, I present some findings of a 5-year longitudinal classroom research program where 8-year old students were followed as they moved from Grade 2 to Grade 6. I shall focus in particular on the genesis and development of embodied, non-symbolic algebraic thinking and its progressive transition to cultural forms of symbolic thinking.

Arithmetic and Algebra: Filiations and Ruptures

The question of the filiations and ruptures between arithmetic and algebra was one of the major educational research themes in the 1980s and 1990s. This question was at the heart of several research programs. It was often discussed in various PME's Working Groups and research reports (Bednarz et al. 1996; Sutherland et al. 2001).

Filloy and Rojano's (1989) work points to one of the fundamental breaks between arithmetic and algebra—what they call a *cut*. This *cut* was observed in clinical studies where students faced equations of the form Ax + B = Cx + D. To solve equations of this form, the arithmetic methods of "reversal operations"—which are effective to solve equations of the type Ax + B = D (the students usually subtract B from D and divide by A)—are no longer applicable. The students have to resort to a truly algebraic idea: to *operate* on the unknown. In order to operate on the unknown, or on indeterminate quantities in general (e.g., variables, parameters), one has to think analytically. That is, one has to consider the indeterminate quantities as if they were something known, as if they were specific numbers (see, e.g., Kieran 1989, 1990; Filloy et al. 2007). From a genetic viewpoint, this way of thinking *analytically*—where unknown numbers are treated on a par with known numbers—distinguishes arithmetic from algebra. And it is so characteristic of algebra that French mathematician François Viète (one of the founders of modern symbolic algebra) identified algebra as an *analytic art* (Viète 1983).

A consequence of this difference between arithmetic and algebra is the following. Because of algebra's analytic nature, formulas in algebra are *deduced*. Failing to notice this central analytic characteristic of algebra may lead us to think that the production of formulas in patterns (regardless of how they were produced) is a symptom of algebraic thinking. But as Howe (2005) notes, producing a formula might merely be a question of guessing the formula and trying it. I completely agree with him that there is nothing algebraic in trying and guessing. Try-and-guess strategies are indeed based on arithmetic concepts only.

Epistemological research has also made a contribution to the conversation about the distinction between arithmetic and algebra. This research suggests that the difference between these disciplines cannot be cast in terms of *notations*, as it has often been thought. The alphanumeric algebraic symbolism that we know today is indeed a recent invention. In the west it appeared during the Renaissance, along with other forms of representation, like perspective in painting and space representation, underpinned by changes in modes of production and new forms of labour division. The birth of algebra is not the birth of its modern symbolism. In his *Elements*, Euclid resorted to letters without mobilizing algebraic ideas. Ancient Chinese mathematicians mobilized algebraic ideas to solve systems of equations without using notations. Babylonian scribes used geometric diagrams to think algebraically. As a result, the use of letters in algebra is neither a necessary nor a sufficient condition for thinking algebraically. Naturally, our modern algebraic symbolism allows us to carry out transformations of expressions that may be difficult or impossible with other forms of symbolism. However, as we shall see in a moment, the rejection of the idea that notations are a manifestation of algebraic thinking, opens up new avenues to the investigation of elementary forms of algebraic thinking in young students.

Some Background of the Research

The investigation of young students' algebraic thinking that I report here started in 2007. The decade before, I was interested in investigating adolescent and young adults' algebraic thinking. From 1998 to 2006 I had the opportunity to follow several cohorts of students from Grade 7 until the end of high school. Like many of my colleagues, I started focusing on symbolic algebra, that is, an algebraic activity mediated by alphanumeric signs. One of my goals was to understand the processes students undergo in order to build symbolic algebraic formulas. My working hypothesis was that in order to understand the manner in which students bestow meaning to alphanumeric expressions, we should pay attention to language (Radford 2000). However, during the analysis of hundreds of hours of videotaped lessons, it became apparent that our students were not resorting only to language, but also to gestures, and other sensuous modalities in ways that were far from mere byproducts of interaction. It was clear that gestures and other embodied forms of action were an integral part of the students' signifying process and cognitive functioning. The problem was to come up with suitable and theoretically articulated explanatory principles, in order to provide an interpretation of the students' algebraic thinking that would integrate those embodied elements that the video analyses put into evidence. Although by the early 2000s, some linguists and cognitive psychologists had developed interesting work around the question of embodiment

(Lakoff and Núñez 2000), their accounts were not easy to apply to such complex settings as classrooms; nor were they necessarily taking into account the historical and cultural dimension of knowledge. In the following years, with the help of some students and collaborators, I was able to refine our theoretical approach and reveal non-conventional, embodied forms of algebraic thinking (Radford 2003). In Radford et al. (2007), we reported a passage in which Grade 9 students displayed an amazing array of sensuous modalities to come up with an algebraic formula in a pattern activity. What is amazing in the reported passage is the subtle coordination of words, written signs, drawn figures, gestures, perception, and rhythm. Figure 1 presents an interesting series of gestures that a student makes while trying to perceive a mathematical structure behind the sequence. Focusing on the first term of the sequence (which is shown in the three first pictures of Fig. 1), Mimi, the student, points with her index to the first circle on the top row and says "one;" she moves the finger to the first circle on the bottom row and repeats "one." Then she moves the index to her right and makes a kind of circular indexical gesture to point to the three remaining circles, while saying "plus three." She starts again the same series of gestures, this time pointing to the second term of the sequence (see second term in Pic 4 of Fig. 1), saying now "two, two plus three." She restarts the same series of gestures in dealing with the third term (see third term of the sequence in Fig. 1, Pic 4; we have added dashed lines to the terms of the sequence to indicate the circles that Mimi points to as she makes her gestures). In doing so, Mimi reveals an embodied formula that, instead of being made up of letters, is made up of words and gestures: the formula is displayed *in concreto*: "one, one, plus three; two, two plus three; three, three, plus three." She then applied the formula to Term 10 (which was not drawn and had to be imagined): "you will have 10 dots [i.e., circles] (she makes a gesture on the desk to indicate the position of the circle*s*), 10 dots (she makes a similar gesture), plus 3." The embodied formula rests on a use of variables and functional relations that conform to the requirement of analyticity that, as I suggested previously, is characteristic of algebra. Although the variable 'number of the term' is not represented through a letter, it appears embodied in its surrogates—the particular numbers the variable takes. The formula is then shown as the series of calculations on the instantiated variable. And, as such, the formula is algebraic. Now, our Grade 9 students did use alphanumeric symbolism and built the formula "n + n + 3," which was then transformed into "n × 2 + 3" (Radford et al. 2007). Hence, these Grade 9 students went unproblematically from an embodied form of thinking to a symbolic one.

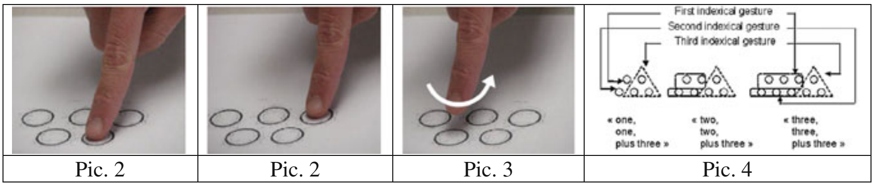

Fig. 1 A Grade 9 student displaying an impressive multimodal coordination of semiotic resources. Recostructed from the video

We came back to other published and unpublished analyses and noticed that the subtle multimodal coordination of senses and signs was a widespread phenomenon in adolescents. Then arose a research question that has kept me busy for the past 6 years: would similar embodied forms of algebraic thinking be accessible to young students? And if yes, how would these embodied forms of thinking develop as the students moved from one grade to the next? As Grade 2 students are still learning to read and write in Ontario, Grade 2 looked like a good place to start. This is how I moved to a primary school and embarked on a new longitudinal research.

Grade 2: Young Students' Non-symbolic Algebraic Thinking

The first generalizing activity in our Grade 2 class was based on the sequence shown in Fig. 2.

We asked the students to extend the sequence up to Term 6. In subsequent questions, we asked them to find out a procedure to determine the number of rectangles in Terms 12 and 25. Figure 3 shows the answers provided by two students: Carlos and James.

Contrary to what we observed in our research with adolescent students, in extending the sequence, most of our Grade 2 students focused on the numerical aspect of the terms only. Counting was the leading activity. Generally speaking, to extend a figural sequence, one needs to grasp a regularity that involves the linkage of two different structures: one *spatial* and the other *numerical*. From the spatial structure emerges a sense of the rectangles' *spatial position*, whereas their numerosity emerges

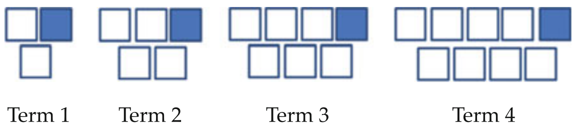

Fig. 2 The first terms of a sequence that Grade 2 students investigated in an algebra lesson

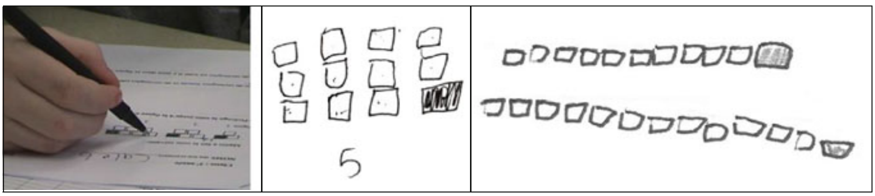

Fig. 3 To the *left*, Carlos, counting aloud, points sequentially to the squares in the top row of Term 3. In the middle, Carlos' drawing of Term 5. To the *right*, James' drawing of Terms 5 (*top*) and 6 (*bottom*)

from a numerical structure. While Carlos attends to the numerical structure in the generalizing activity, the spatial structure is not coherently emphasized. This does not mean that Carlos, James and the other students do not see the figures as composed of two horizontal rows. What this means is that the emphasis on the numerical structure somehow leaves in the background the geometric structure. We could say that the *shape* of the terms of the sequence is used to facilitate the counting process. Thus, as picture 1 in Fig. 3 shows, Carlos always counted the rectangles in a spatial orderly way. The geometric structure, however, does not come to be related to the numerical one in a meaningful and efficient way. It is not surprising within this context, then, that the students encountered difficulties in answering our questions about Terms 12 and 25. Without resorting to an efficient way of counting, the counting process of rectangles one-by-one in remote terms beyond the perceptual field became extremely difficult.

Because of their spatial connotation, it might not be surprising that, in extending the sequences, our young students did not use deictic terms, like "bottom" or "top." In the cases in which the students did succeed in linking the spatial and numerical structures, the spatial structure appeared only ostensibly, i.e., "top" and "bottom" rows were not part of the students' discourse but were made apparent through pointing and actual row counting: they remained secluded in the embodied realm of action and perception. The next day, the teacher discussed the sequence with the students and referred to the rows in an explicit manner to bring to the students' attention the linkage of the numerical and spatial structures. To do so, the teacher drew the first five terms of the sequence on the blackboard and referred to an imaginary student who counted by rows. "This student," she said to the class, "noticed that in Term 1 (she pointed to the name of the term) there is one rectangle on the bottom (and she pointed to the rectangle on the bottom), one on the top (pointing to the rectangle), plus one dark rectangle (pointing to the dark rectangle)." Next, she moved to Term 2 and repeated in a rhythmic manner the same counting process, coordinating the spatial deictics "bottom" and "top," the corresponding spatial rows of the figure, and the number of rectangles therein. To make sure that everyone was following, she started again from Term 1 and, at Term 3, she invited the students to join her in the counting process, going together up to Term 5 (see Fig. 4).

Then, the teacher asked the class about the number of squares in Term 25. Mary raised her hand and answered: "25 on the bottom, 25 on top, plus 1." The class

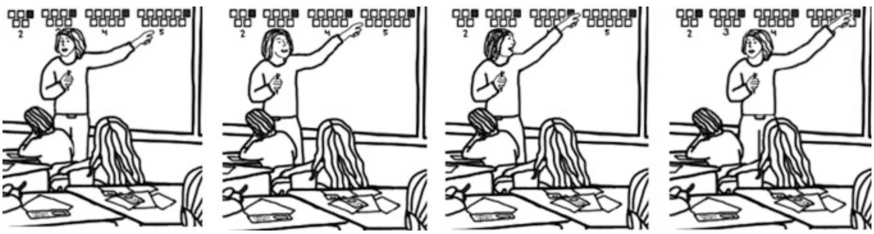

Fig. 4 The teacher and the students counting rhythmically say (see Pic 1) "Term 5", (Pic 2) "5 on the *bottom*", (Pic 3) "5 on *top*", (Pic 4) "plus 1."

Fig. 5 Karl explaining Term 50

spent some time dealing with “remote” terms, such as Terms 50 and 100. Figure 5 shows Karl explaining to the teacher and his group-mates what Term 50 looks like.

In picture 1, Karl moves his arm and his body from left to right in a vigorous manner to indicate the bottom row of Term 50, while saying that there would be 50 white rectangles there. He moves his arm a bit further and repeats the moving arm-gesture to signify the top row of Term 50. Then he makes a semi-circle gesture in the air to signify the dark square.

The students played for a while with remote terms. In Karl’s group, one of the questions revolved around Term 500 and Term 50:

Karl	How about doing 500 plus 500?
Erica	No. Do something simpler
Karl	(*Talking almost at the same time*) 500 plus 500 equals 1000
Erica	plus 1, 1001
Karl	plus 1, equals 1001
Cindy	(*Talking about Term 50*) 50 plus 50, plus 1 equals 101

Schematically speaking, the students’ answer to the question of the number of rectangles in remote particular terms was “x + x + 1” (where x was always a *specific* number). The formula, I argue, is algebraic in nature, even if it is not expressed in standard notations. In this case, indeterminacy and analyticity appear in an intuited form, rather than explicitly. A natural question is: Is this all that Grade 2 students are capable of? In fact, the answer is no. As we shall see in the next section, we were able to create conditions for the emergence of more sophisticated forms of algebraic thinking.

Beyond Intuited Indeterminacy: The Message Problem

On the fifth day of our pattern generalization teaching-learning sequence, the teacher came back to the sequence from the first day (Fig. 2). To recapitulate, she invited some groups to share in front of the class what they had learned about that sequence in light of previous days’ classroom discussions and small group work. Then, she asked a completely new question to the class. She took a box and, in front of the students, put in it several cards, each one having a number: 5, 15, 100, 104, etc. Each one of these numbers represented the number of a term of the sequence shown in

Fig. 2. The teacher invited a student to choose randomly one of the cards and put it into an envelope, making sure that neither the student herself nor the teacher nor anybody else saw the number beforehand. The envelope, the teacher said, was going to be sent to Tristan, a student from another school. The Grade 2 students were invited to send a message that would be put in the envelope along with the card. In the message the students would tell Tristan how to quickly calculate the number of rectangles in the term indicated on the card. The number of the term was hence unknown. Would the students be able to generalize the embodied formula and engage with calculations on this unknown number? In other terms, would our Grade 2 students be able to go beyond intuited indeterminacy and its corresponding elementary form of algebraic thinking? As in the previous days, the students worked in small groups of three. The usual response was to give an example. For instance, Karl suggested: "If the number [on the card] is 50, you do 50, plus 50, plus 1." The teacher commended the students for the idea, but insisted that the number could be something else and asked if there would be another way to say it without resorting to examples. After an intense discussion, the students came up with a suggestion:

Erica	It's the number he has, the same number at the bottom, the same number at the top, plus 1…
Teacher	That is excellent, but don't forget: he doesn't have to draw [the term]. He just has to add… So, how can we say it, using this good idea?
Erica	We can use our calculator to calculate!
Teacher	Ok. And what is he going to do with the calculator?
Erica	He will put the number… (she pretends to be inserting a number into the calculator)… plus the same number, plus 1 (as she speaks, she pretends to be inserting the number again, and the number 1).

Another group suggested "twice the number plus 1." Naturally, the use of the calculator is merely virtual. In the students' real calculator, all inputs are specific numbers. Nevertheless, the calculator helped the students to bring forward the analytic dimension that was apparently missing in the students' explicit formula. Through the virtual use of the calculator, calculations are now performed on this unspecified instance of the variable—the unknown number of the figure.

Let me summarize our Grade 2 students' accomplishments during the first week that they were exposed to algebra. In the beginning, most of our students were dealing with figural sequences like the one in Fig. 1 through a focus on numerosity. Finding out the number of elements (rectangles, in the example here discussed) in remote terms was not easy. The joint counting process in which the teacher and students engaged during the second day helped the students to move to other ways of seeing sequences. The joint counting process made it possible for the students to notice and articulate new forms of mathematical generalization. In particular, they became aware of the fact that the counting process can be based on a *relational idea*: to link the number of the figure to relevant parts of it (e.g. the squares on the bottom row). This requires an altogether new perception of the number of the term and the terms themselves. The terms appear now not as a mere bunch of ordered rectangles but as something susceptible to being decomposed, the decomposed parts bearing potential clues for algebraic relationships to occur. Interestingly enough, historically speaking, the "decomposition" of geometric figures in simpler forms (e.g., straight lines) was systematically developed in the 17th century by Descartes in his *Geometry*, a central book in the development of algebraic ideas. The decomposition of figures permitted the creation of relationships between known and unknown numbers and the carrying out of calculations on them "without making a distinction between known and unknown [parts]" (Descartes 1954, p. 8). Our examples—as well as those reported by other researchers with other Grade 2 students—suggest that the linkage of spatial and numerical structures constitutes an important aspect of the development of algebraic thinking. Such a linkage rests on the cultural transformation in the manner in which sequences can be seen—a transformation that may be termed the *domestication of the eye* (Radford 2010). For the modern mathematician's eye, the complexity behind the perception of simple sequences like the one our Grade 2 students tackled remains in the background, to the extent that to see things as the mathematician's eye does, ends up seeming natural. However, as our results intimate, there is nothing natural there. To successfully attend to what is algebraically meaningful is part of learning to think algebraically. This cultural transformation of the eye is not specific to Grade 2 students. It reappears in other parts of the students' developmental trajectory. It reappears, later on, when students deal with factorization, where discerning structural *syntactic forms* become a pivotal element in recognizing common factors or prototypical expressions.

All in all, the linkage of spatial and numerical structures resulted, as we have seen, in the emergence of an elementary way of algebraic thinking that manifested itself in the embodied constitution of a formula where the variable is expressed through particular instances, which we can schematize as "x + x +1" (where x was always a *specific* number). This formula, I argued on semiotic and epistemological grounds, is genuinely algebraic. That does not mean that all formulas provided by young students are algebraic. To give an example, one of the students suggested that to find out the number of elements in Term 100, you keep adding 2, and 2 and 2 to Term 1 until you get to Term 100. This is an example of arithmetic generalization —not of an algebraic one, as there is no analyticity involved. The "Message Problem" offered the students a possibility to go beyond intuitive indeterminacy and to think, talk, and calculate explicitly on an unknown number. Although several students were able to produce an explicit formula (e.g., "the number plus the number, plus 1" or "twice the number plus 1"), other students produced a formula where the general unknown number was represented through an example. This is what Mason (1996) calls *seeing the general in or through the particular*. Both the explicit formula and the general-through-the-particular formula bear witness to a more sophisticated form of elementary algebraic thinking than the embodied one where the variable and the formula are displayed in action.

Revealing our Grade 2 students' aforementioned elementary, pre-symbolic forms of algebraic thinking responded to our first research question—i.e., whether the embodied forms of thinking that we observed in adolescents are accessible to younger students. Yet, there are differences. Adolescents in general tend to gesture, talk and symbolize in harmonious coordinated manners (often after a period of mismatch between words and gestures (Arzarello and Edwards 2005; Radford 2009a). Our young students, in contrast, tend to gesture with energetic intensity (see e.g. Fig. 5). The energetic intensity may decrease as the students become more and more aware of the variables and the relationship between known and unknown numbers. However, the energetic intensity remains relatively pronounced as compared to what we have seen in adolescents (Radford 2009a, b). This phenomenon may be a token of a problem related to our second research question, namely: How does young students' algebraic thinking *develop*?

Developmental questions are very tricky, as psychologists know very well. It is not enough to collect data year after year and merely compare what students did in Year 1, to what they did in Year 2, etc. Exposing differences *shows* something but does not *explain* anything. I struggled with the question of the development of students' mathematical thinking for about a decade when I was doing research with adolescents, and I have to confess that I was unable to come up with something satisfactory. Yet, my research with adolescents helped me to envision a sensuous and material conception of mathematical cognition (Radford 2009b) that was instrumental in tackling the developmental question. Before going further in my account of what the students did in the following years, I need to dwell on the question of development first.

Thinking and Its Development

In contrast to mental cognitive approaches, thinking, I have suggested (Radford 2009b), is not something that solely happens 'in the head.' Thinking may be considered to be made up of material and ideational components: it *is* made up of (inner and outer) speech, objectified forms of sensuous imagination, gestures, tactility, and our actual actions with cultural artifacts. Thus, in Fig. 5, for instance, Karl *is* thinking *with* and *through* the body in the same way that he *is* thinking *through* and *in* language and the arsenal of conceptual categories it provides for us to notice, highlight, and attend to things, and intend them in certain cultural topical ways. The same can be said of the teacher in Fig. 4. Although it might be argued that the teacher and the student are merely communicating ideas, I would retort that this division between thinking and communicating makes sense only within the context of a conception of the mind as a private space within us, where ideas are created, computed and only then communicated. This computational view of the mind has a long history in our Western idealist and rationalist philosophical traditions. The view that I am sketching here goes against the dualistic assumption of mind versus body or ideal versus material. Thinking appears here as a an ideal-material form of reflection and action, which does not occur solely in the head but also *in* and *through* a sophisticated semiotic coordination of speech, body, gestures, symbols and tools. This is why, during difficult conversations, rather than digging in the head first to find the ideas that we want to express, we hear ourselves thinking as we talk, and realize, at the same time as our interlocutors, what we are thinking about.

Now to say that thinking is made up of (inner and outer) speech, objectified forms of sensuous imagination, gestures, tactility, and our actual actions with cultural artifacts does not mean that thinking is a *collection* of items. If we come back to our examples, Carlos (see Fig. 3, left), while moving the upper part of his body, was resorting to pointing gestures and words to count the rectangles in the first terms of the sequence. Words and gestures were guiding his perceptual activity to deal with the numerosity of the terms. Like Carlos, Karl moved his upper body, made arm- and hand-gestures and resorted to language (Fig. 5). In stating the formula "the number plus the number, plus 1," Erica gestured as if she was pressing keys in the calculator keyboard (Radford 2011). Yet, the relationship between perception, gestures and words is not the same. What it means is that thinking is not a mere collection of items. Thinking is rather a dynamic *unity* of material and ideal components. This is why the same gesture (e.g. an indexical gesture pointing to the rectangles on top of Term 3) may mean something conceptually sophisticated or something very simple. That is, the real significance of a component of thinking can only be recognized by the role such a component plays in the context of the *unity* of which it is a part.

Now I can formulate my developmental question. If thinking is a *systemic unity* of ideational and material components, it would be wrong to study its development by focusing on one of its components only. Thus, the development of algebraic thinking cannot be reduced to the development of its symbolic component (notation use, for instance). The development of algebraic thinking must be studied as a *whole*, by taking into account the interrelated *dialectic* development of its various components (Radford 2012). If in a previous section I talked about the 'domestication of the eye,' this domestication has to be related to the 'domestication of the hand' as well. And, indeed, this is what happened in our Grade 2 class from the second day on. As we recall, the teacher (Fig. 4) made extensive use of gestures and an explicit use of rhythm, and linguistic deictics, followed later by the students, who started using their hands and their eyes in novel ways, opening up new possibilities to use efficient and evolved cultural forms of mathematical generalization that they successfully applied to other sequences with different shapes.

To sum up, it is not only the tactile, the perceptual, or the symbol-use activity that is developmentally modified. In the same way as perception develops, so do speech (e.g., through spatial deictics) and gesture (through rhythm and precision). Perception, speech, gesture, and imagination develop in an interrelated manner. They come to form a new unity of the material-ideational components of thinking, where words, gestures, and signs more generally, are used as means of objectification, or as Vygotsky (Vygotsky 1987), p. 164 put it, "as means of voluntary directing attention, as means of abstracting and isolating features, and as a means of […] synthesizing and symbolising". Within this context, to ask the question of the development of algebraic thinking is to ask about the appearance of new systemic structuring *relationships* between the material-ideational components of thinking (e.g., gesture, inner and outer speech) and the manner in which these relationships are organized and reorganized. It is through these developmental lenses that I studied the data collected in the following years and that I summarize in the rest of this article, focusing on Grades 3 and 4.

Grade 3: Semiotic Contraction

As usual, in Grade 3 the students were presented with generalizing tasks to be tackled in small groups. The first task featured a figural sequence, S_n, having n circles horizontally and $n-1$ vertically, of which the first four terms were given. Contrary to what he did first in Grade 2, from the outset, Carlos perceived the sequence taking advantage of the spatial configuration of its terms. Talking to his teammates about Term 4 he said: "here (pointing to the vertical part) there are four. Like you take all this [i.e., the vertical part] together (he draws a line around), and you take all this [i.e., the horizontal part] together (he draws a line around; see Fig. 6, pic 1). So, we should draw 5 like that (through a vertical gesture he indicates the place where the vertical part should be drawn) and (making a horizontal gesture) 5 like that" (see Fig. 6, pics 2–3).

When the teacher came to see the group, she asked Carlos to sketch for her Term 10, then Term 50. The first answer was given using unspecified deictics and gestures. He quickly said: "10 like this (vertical gesture) and 10 like that" (horizontal gesture). The specific deictic term "vertical" was used in answering the question about Figure 50. He said: "50 on the vertical… and 49…" When the teacher left, the students kept discussing how to write the answer to the question about Term 6. Carlos wrote: "6 vertical and 5 horizontal."

In developmental terms, we see the evolution of the unity of ideational-material components of algebraic thinking. Now, Carlos by himself and with great ease coordinates gestures, perception, and speech. The coordination of these outer components of thinking is much more refined compared to what we observed in Grade 2. This refinement is what we have called a *semiotic contraction* (Radford 2008a), that is, a genetic process in the course of which choices are made between what counts as relevant and irrelevant; it leads to a contraction of previous semiotic

Fig. 6 To the *left*, Term 4 of the given sequence. Middle, Carlos's vertical and horizontal gestures while imagining and talking about the still to be drawn Term 5. To the *right*, Carlos's drawings of Terms 5 and 6

activity, resulting in a more refined linkage of semiotic resources. It entails a deeper level of consciousness and intelligibility of the problem at hand and is a symptom of learning and conceptual development.

Grade 4: The Domestication of the Hand

To check developmental questions, in Grade 4 we gave the students the sequence with which they started in Grade 2 (see Fig. 2). This time, from the outset, Carlos perceived the terms as being divided into two rows. Talking to his teammates and referring to the top row of Term 5, he said as if talking about something banal: "5 white squares, 'cause in Term 1, there is 1 white square (making a quick pointing gesture)… Term 2, 2 [squares] (making another quick pointing gesture); 3, (another quick pointing gesture) 3." He drew the five white squares on the top row of Term 5 and added: "after that you add a dark square." Then, referring to the bottom row of Term 4: "there are 4; there [Term 5] there are 5." When the teacher came to see their work, Carlos and his teammates explained "We looked at Term 2, it's the same thing [i.e., 2 white squares on top]… Term 6 will have 6 white squares."

There was a question in the activity in which the students were required to explain to an imaginary student (Pierre) how to build a big term of the sequence (the "Big Term Problem"). In Grade 2, the students chose systematically a particular term. This time, Carlos wrote: "He needs [to put as many white squares as] the number of the term on top and on the bottom, plus a dark square on top."

The "Message Problem" Again

At the end of the lesson, the students tackled the "Message Problem" again. As opposed to the lengthy process that, in Grade 2, preceded the building of a message without particular examples (Radford 2011), this time the answer was produced quicker:

David The number of the term you calculate twice and add one. That's it!
Carlos (*Rephrasing David's idea*) twice the number plus one

The activity finished with a new challenge. The teacher asked the students to add to the written message a "mathematical formula." After a discussion in Carlos's group concerning the difference between a phrase and a mathematical formula, the students agreed that a formula should include operations only. Carlos's formula is shown in pic 3 of Fig. 7.

From a developmental perspective, we see how Carlos's use of language has been refined. In Grade 2 he was resorting to particular terms (Term 1,000) to answer the same question about the "big term." Here he deals with indeterminacy in an

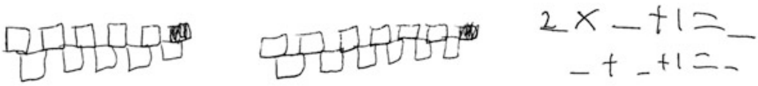

Fig. 7 *Left*, Carlos' drawings of Terms 5 and 6. *Right*, Carlos's formulas

easy way, through the expression "the number of the term." He even goes further and produces two symbolic expressions to calculate the total of squares in the unspecified term (Fig. 7, right). The semiotic activities of perceiving, gesturing, languaging, and symbolizing have developed to a greater extent. They have reached an interrelational refinement and consistency that was not present in Grade 2 and was not fully developed in Grade 3. This cognitive developmental refinement became even more apparent when the teacher led the students to the world of notations, as we shall now see.

The Introduction to Notations

The introduction to notations occurred when the students discussed their answers to homework based on the sequence shown in Fig. 8. The discussion took place right after the general discussion about the "Message Problem" alluded to in the previous sub-section.

The teacher gave the students the opportunity to compare and discuss their answers to the homework by working in small groups. In Carlos' group, the terms of the sequence were perceived as made up of two rows, each one having the same number as the number of the term plus an addition of two squares at the end (see pic 2 in Fig. 8). As Carlos suggests, referring to Term 15, "15 on top, 15 at the bottom, plus 2, that is 32." Or alternatively, as Celia, one of Carlos' teammates, explains, "15 + 1 equals 16, then 16 + 16… which makes 32." After about 10 min of small-group discussion, the teacher encouraged the students to produce a formula like the one that they just provided for the "Message Problem." Then, the class moved to a general discussion where various groups presented their findings. Erica went to the Interactive White Smart Board (ISB) and suggested the following formula: "1 + 1 + 2x__ = __" The teacher asked whether it would be possible to write, instead of the underscores, something else. One student suggested putting an

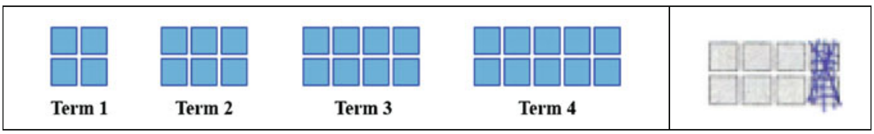

Fig. 8 Pic 1 (*left*), the sequence of the homework. Pic 2 (*right*), Carlos' decomposition of Term 3

interrogation mark. The teacher acknowledged that interrogation mark could also be used, and asked for other ideas. Samantha answered with a question:

Samantha A letter?

Teacher Ah! Could I write one plus one plus two times n? What does n mean?

A student A number…

Teacher Could we write that (i.e., one plus one plus two times n) equals n? (Some students answered yes, others no; talking to Erica who is at the whiteboard) Ok. Write it, write your formula (Erica writes $1 + 1 + 2 \times n = n$)

Carlos No, because n (*meaning the first one*) is not equal to n (*meaning the second one*)

Teacher Ah! Why do you say that n is not equal to n?

Carlos Because if you do 2 times n, that will not equal [the second] n

Teacher Wow!

In order not to rush the students into the world of notations, the teacher decided to delay the question of using a second letter to designate the total. As we shall see, this question will arise in the next activity. In the meantime, the formula was left as $1 + 1 + 2 \times n =$ __.

The next activity started right away. The students were provided with the new activity sheet that featured the sequence shown in Fig. 9. The students were encouraged to come up with as many formulas as possible to determine the number of squares in any term of the sequence.

During the small-group discussion, William offers a way to perceive the terms. Talking to Carlos, and referring to Term 6, which they drew on the activity sheet, William says (talking about the top row): "There are 8 [squares], because 6 + 2 = 8. You see, on the bottom it's always the number of the term, you see?" His utterance is accompanied by a precise two-finger gesture through which he indicates the bottom row (see Fig. 10, left). He continues: "then, on the top, it's always plus 2" (making the gesture shown in Fig. 10, right).

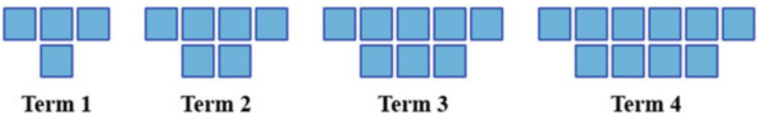

Fig. 9 The featured sequence of the new activity

Fig. 10 William making precise gestures to refer to Term 6

The answer to the "Message Problem" was provided without difficulties. Without hesitation, Carlos said: "Ok. Double the number and add 2." The class moved to a general discussion, which was a space to discuss different forms of perceiving the sequence and of writing a formula. Marianne went to the ISB and suggested that the terms could be imagined as divided into two equal rows and that one square is added to the left and one to the right of the top row. In Fig. 11, referring to Term 3, she points first to the top row (imagined as made up of three squares; see Fig. 11, Pic 1). Then she points to the bottom row (Pic. 2), then to the extra square at the top right (Pic. 3) and to the extra square at the top left (Pic. 4). Celia proposed that a term was the same as the previous one to which two squares are added at the right end. In Fig. 11, Pic 5 and 6, she hides the two rightmost squares in Terms 2 and 3 to show that what remains in each case is the previous term. The developmental sophistication that the perception-gesture-language systemic unity has achieved is very clear.

Then, the students presented their formulas. Carlos presented the following formula: $\underline{N} + \underline{N} + 2 = \_$. The place for the variable in the formula is symbolized with a letter *and* the underscore sign. Letters in Carlos's formula appear timidly drawn, still bearing the vestiges of previous symbolizations (see Fig. 7, right).

The teacher asked if it would be possible to use another letter to designate the result:

Teacher	Well, we started with letters [in your formula]. Maybe we could continue with letters?
Carlos	No!
Teacher	Why not?
Carlos	An r?
Teacher	Why r?
Caleb	The answer (in French, *la réponse*)

Carlos completed the formula as follows: : $\underline{N} + \underline{N} + 2 = \underline{R}$. Other formulas were provided, as shown in Fig. 12:

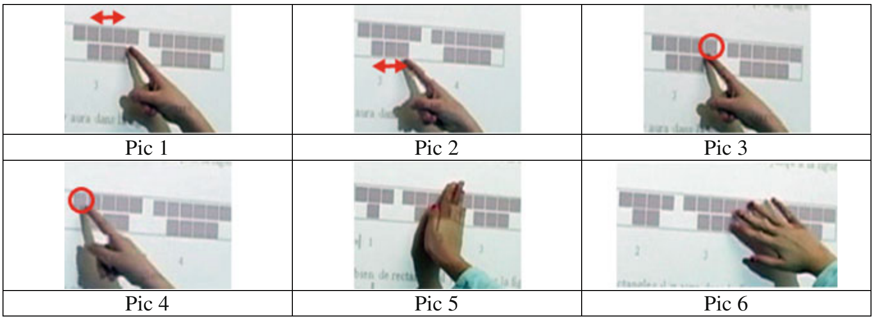

Fig. 11 Marianne's (Pic. 1–4) and Celia's (Pic. 5–6) gestures

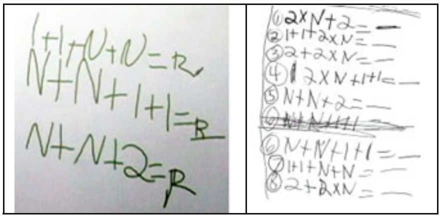

Fig. 12 *Left*, some formulas from the classroom discussion. *Right*, formulas from Erica's group

Synthesis and Concluding Remarks

In the first part of the article I suggested that algebraic thinking cannot be reduced to an activity mediated by notations. As I argued in previous work, a formula to calculate the number of rectangles in sequences like the one presented in Fig. 2, such as "2n + 1," can be attained by arithmetic trial-and-error methods. Algebraic thinking, I suggested, is rather characterized by the *analytic* manner in which it deals with indeterminate numbers. A rigorous video analysis convinced us that students signify indeterminate numbers through recourse to a plethora of semiotic embodied resources that, rather than being merely a by-product of thinking, constitute the very sensible texture of it. From this sensuous perspective on human cognition, it is not difficult to appreciate that 7–8-year-old students can effectively start thinking algebraically. In the second part of the article I dealt with the question of the development of algebraic thinking. Algebraic thinking—like all cultural forms of thinking (e.g., aesthetic, legal, political, artistic)—is a theoretical form that has emerged, evolved and refined in the course of cultural history. It pre-existed in a developed ideal form before the students engaged in our classroom activities. The greatest characteristic of child development consists in how this ideal form exerts a real influence on the child's thinking. But how can this ideal form exert such an influence on the child? Vygotsky's answer is: *under particular conditions of interaction between the ideal form and the child* (1994). In our case, the particular conditions of interaction between algebraic thinking as a historical ideal form and our Grade 2 students were constituted by a sequence of activities that were intentional bearers of this ideal form. Naturally, the students cannot discern the theoretical intention behind our questions, as this cultural ideal form that we call algebraic thinking has still to be encountered and cognized. The lengthy, creative, and gradual processes through which the students encounter, and become acquainted with historically constituted cultural meanings and forms of (in our case algebraic) reasoning and action is what I have termed, following Hegel, *objectification* (Radford 2008b).

The objectification of ideal forms requires a *temporal continuity* and stability of the knowledge that is being objectified. The objectification of ideal forms requires also the mutual emotional and ethical engagement of teacher and students in the joint activity of teaching-learning (Radford and Roth 2011; Roth and Radford 2011).

Drawing on the aforementioned idea of sensuous cognition and development, I suggested that the development of algebraic thinking can be studied in terms of the appearance of new systemic structuring *relationships* between the material-ideational components of thinking (e.g., gesture, inner and outer speech) and the manner in which these relationships are organized and reorganized in the course of the students' engagement in activity. The analysis of our experimental data focused on revealing those relationships and their progressive refinement. We saw how, for instance, the development of perception is consubstantial with the development of gestural and symbolic activity.

The whole story, however, is much more complex. As Vygotsky (1994) argued forcefully development can only be understood if we take into consideration the manner in which the student is actually emotionally experiencing the world. The emotional experience [*perezhivanie*] is, the Russian psychologist contended in a lecture given at the end of his life, the link between the subject and his/her surrounding, between the always changing subject (the perpetual being in the process of becoming) and his/her always conceptually, politically, ideologically moving societal environment. The explicit and meaningful insertion of *perezhivanie* into developmental accounts is, I suppose, still a trickier problem to conceptualize and investigate—an open research problem for sure.

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How We Think: A Theory of Human Decision-Making, with a Focus on Teaching

Alan H. Schoenfeld

Abstract Suppose a person is engaged in a complex activity, such as teaching. What determines what that person does, on a moment-by-moment basis, as he or she engages in that activity? What resources does the person draw upon, and why? What shapes the choices the person makes? I claim that if you know enough about a teacher's knowledge, goals, and beliefs, you can explain every decision he or she makes, in the midst of teaching. In this paper I give examples showing what shapes teachers' decision-making, and explain the theory.

Keywords Decision-making · Teaching · Theory

Introduction

I became a mathematician for the simple reason that I love mathematics. Doing mathematics can be a source of great pleasure: when you come to understand it, the subject fits together beautifully. Here I am not necessarily referring to advanced mathematics. The child who notices that every time she adds two odd numbers the result is even, wonders why, and the figures out the reason why:

> Each odd number is made up of a number of pairs, and one 'extra.' When you add two odd numbers together, the extras make a pair. That means that the sum is made up of pairs, so it's even!

is doing real mathematics. It was that kind of experience that led me into mathematics in the first place.

Sadly, very few people develop this kind of understanding, or this kind of pleasure in doing mathematics. It was this realization, and the thought that it might be possible to do something to change it, that led me into mathematics education.

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For more than 35 years I have pursued the question, “How can we develop deeper understandings of mathematical thinking, problem solving, and teaching, so that we can help more children experience the pleasures of doing mathematics?”

My early work was devoted to mathematical problem solving. I read Pólya’s (1945) book *How to Solve It* early in my mathematical career, and it resonated. Pólya said that mathematicians used a wide range of problem solving strategies, which he called heuristics. When he described them, I recognized them—I used them too! I wondered, though, why I had not explicitly been taught those strategies. The answer, I learned, was that when people tried to teach the strategies described in Pólya’s books, students did not learn to use them effectively. This was disappointing, but it also represented a lovely challenge. Could we understand such problem solving strategies well enough so that we could help students learn to use them effectively?

Thus began a decade’s worth of work in which I tried to develop an understanding of problem solving: What do effective problem solvers do, which enables them to solve difficult problems? What do ineffective problem solvers do, that causes them to fail in their problem solving attempts? What can we do, as teachers, to help students become more effective problem solvers? My answers to those questions, which are summarized very briefly below, were published in my 1985 book *Mathematical Problem Solving*. The book resulted from a decade of simultaneous research on and teaching of problem solving, in which my theoretical ideas were tried in the classroom, and my experience in the classroom gave rise to more theoretical ideas.

Mathematical Problem Solving represented a solid first step in a research agenda. By the time it was written, I knew enough about problem solving to help students become more effective problem solvers. A next, logical goal was to help mathematics teachers to help their students develop deeper understandings of mathematics. In many ways, of course, teaching is an act of problem solving—but it is so much more. The challenge was, could I develop a theoretical understanding of teaching in ways that allowed me to understand how and why teachers make the choices they do, as they teach? Could that understanding then be used to help teachers become more effective? Moreover, to the degree that teaching is typical of knowledge-intensive decision making, could the theoretical descriptions of teaching be used to characterize decision making in other areas as well?

Those questions have been at the core of my research agenda for the past 25 years. My answers to them now exist, in a new book, *How We Think* (Schoenfeld 2010). The purpose of this paper is to illustrate and explain the main ideas in the book. Because my current research has evolved from my earlier problem solving work, I set the stage for the discussion that follows with a brief description of that work—what it showed and, more importantly, the questions that it did not answer. That will allow me to describe what a complete theory should be able to accomplish. I then turn to the main body of this paper, three studies of teaching. In those examples I show how, under certain circumstances, it is possible to model the act of teaching, to the point where one can provide a grounded explanation of every decision that a teacher makes during an extended episode of teaching. Following that, I give some other examples to suggest that the theory is general, and I make a few concluding comments.

The Challenge

Suppose that you are in the middle of some "well practiced" activity, something you have done often so that it is familiar to you. Depending on who you are, it might be

- cooking a meal
- fixing a car
- teaching a class
- doing medical diagnosis or brain surgery.

The challenge is this: If I know "enough" about you, can I explain (i.e., build a cognitive model that explains) every single action you take and every decision you make?

My goal for this paper is to describe an analytic structure that does just that—an analytic structure that explains how and why people act the way they do, on a moment-by-moment basis, in the midst of complex, often social activities such as teaching.

My major claim is this: *People's in-the-moment decision making when they teach, and when they engage in other well practiced, knowledge intensive activities, is a function of their knowledge and resources, goals, and beliefs and orientations. Their decisions and actions can be "captured" (explained and modeled) in detail using only these constructs.*

The main substance of this paper (as in the book) consists of three analyses of teaching, to convey the flavor of the work. Of course, it is no accident that I chose mathematics teaching as the focal area for my analyses. I am, after all, a mathematics educator! But more to the point, teaching is a knowledge intensive, highly interactive, dynamic activity. If it is possible to validate a theory that explains teachers' decision making in a wide range of circumstances, then that theory should serve to explain all well practiced behavior.

Background: Problem Solving

As discussed above, my current work is an outgrowth of my earlier research on mathematical problem solving. Here I want to summarize the core findings of that work, to show how it lays the groundwork for my current research.

My major argument about mathematical problem solving (see Schoenfeld 1985, for detail) was that it is possible to explain someone's success or failure in trying to solve problems on the basis of the following four things:

1. *Knowledge* (*or more broadly, resources*). This is not exactly shocking—but, knowing what knowledge and resources a problem solver has *potentially* at his or her disposal is important.
2. *Problem solving strategies, also known as "heuristics*." We know from Pólya's work that mathematicians use heuristic strategies, "rules of thumb for making

progress when you do not know a direct way to a solution." Faculty pick up these strategies by themselves, through experience. Typically, students don't use them. But, my research showed that students can learn to use them.

3. *"Metacognition," or "Monitoring and self-regulation."* Effective problem solvers plan, and they keep track of how well things are going as they implement their plans. If they seem to be making progress, they continue; if there are difficulties, they re-evaluate and consider alternatives. Ineffective problem solvers (including most students) do not do this. As a result, they can fail to solve problems that they *could* solve. Students can learn to be more effective at these kinds of behaviors.
4. *Beliefs*. Students' beliefs about themselves and the nature of the mathematical enterprise, derived from their experiences with mathematics, shape the knowledge they draw upon during problem solving and the ways they do or do not use that knowledge. For example, students who believe that "all problems can be solved in 5 min or less" will stop working on problems even though, had they persevered, they might have solved them. Students who believe that "proof has nothing to do with discovery or invention" will, in the context of "discovery" problems, make conjectures that contradict results they have just proven. (see Schoenfeld 1985).

In sum: By 1985 we know what "counted" in mathematical problem solving, in the sense that we could explain, post hoc, what accounted for success or failure. As the ensuing 25 years have shown, this applied to all "goal-oriented" or problem solving domains, including mathematics, physics, electronic trouble-shooting, and writing.

BUT… There was a lot that the framework that I have just described did not do. In the research I conducted for *Mathematical Problem Solving*, people worked in isolation on problems that I gave them to solve. Thus: the goals were established (i.e., "solve this problem"); the tasks didn't change while people worked on them; and social interactions and considerations were negligible.

In addition, *Mathematical Problem Solving* offered a *framework*, not a theory. Above and beyond pointing out what is important—which is what a framework does—a theory should provide rigorous explanations of how and why things fit together. That is what my current work is about. What I have been working on for the past 25 years is a theoretical approach that explains how and why people make the choices they do, while working on issues they care about and have some experience with, amidst dynamically changing social environments.

I can think of no better domain to study than teaching. Teaching is knowledge intensive. It calls for instant decision making in a dynamically changing environment. It's highly social. And, if you can model teaching, you can model just about anything! I will argue that if you can model teaching, you can model: shopping; preparing a meal; an ordinary day at work; automobile mechanics; brain surgery (or any other medical practice), and other comparably complex, "well practiced" behaviors. All of these activities involve goal-oriented behavior—drawing on available resources (not the least of which is knowledge) and making decisions in order to achieve outcomes you value.

The goal of my work, and this paper, is to describe a theoretical architecture that explains people's decision-making during such activities.

How Things Work

My main theoretical claim is that goal-oriented "acting in the moment"—including problem solving, tutoring, teaching, cooking, and brain surgery—can be explained and modeled by a theoretical architecture in which the following are represented: Resources (especially knowledge); Goals; Orientations (an abstraction of beliefs, including values, preferences, etc.); and Decision-Making (which can be modeled as a form of subjective cost-benefit analysis). For substantiation, in excruciating detail, please see my book, *How we Think.* To briefly provide substantiation I will provide three examples in what follows. But first, a top-level view of how things work is given in Fig. 1. The basic structure is recursive: Individuals orient to situations and decide (on the basis of beliefs and available resources) how to pursue their goals. If the situation is familiar, they implement familiar routines; if things are

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How Things Work

- An individual enters into a particular context with a specific body of resources, goals, and orientations.
- The individual takes in and orients to the situation. Certain pieces of information and knowledge become salient and are activated.
- Goals are established (or reinforced if they pre-existed).
- Decisions consistent with these goals are made, consciously or unconsciously, regarding what directions to pursue and what resources to use:
  - If the situation is familiar, then the process may be relatively automatic, where the action(s) taken are in essence the access and implementation of scripts, frames, routines, or schemata.
  - If the situation is not familiar or there is something non-routine about it, then decision-making is made by a mechanism that can be modeled by (i.e., is consistent with the results of) using the subjective expected values of available options, given the orientations of the individual.
- Implementation begins.
- Monitoring (whether it is effective or not) takes place on an ongoing basis.
- This process is iterative, down to the level of individual utterances or actions:
  - Routines aimed at particular goals have sub-routines, which have their own subgoals;
  - If a subgoal is satisfied, the individual proceeds to another goal or subgoal;
  - If a goal is achieved, new goals kick in via decision-making;
  - If the process is interrupted or things don't seem to be going well, decision-making kicks into action once again. This may or may not result in a change of goals and/or the pathways used to try to achieve them.

Fig. 1 How things work, in outline. From Schoenfeld (2010), p. 18, with permission unfamiliar or problematic, they reconsider. It may seem surprising, but if you know enough about an individual's resources, goals, and beliefs, this approach allows you to model their behavior (after a huge amount of work!) on a line-by-line basis.

First Teaching Example, Mark Nelson

Mark Nelson is a beginning teacher. In an elementary algebra class, Nelson has worked through problems like, x^5/x^3 = ? Now he has assigned

$$(a)\, m^6/m^2, \quad (b)\, x^3y^7/x^2y^6, \quad \text{and} \quad (c)\, x^5/x^5$$

for the class to work. Nelson expects the students to have little trouble with m^6/m^2 and x^3y^7/x^2y^6, but to be "confused" about x^5/x^5; he plans to "work through" their confusion. Here is what happens.

Nelson calls on students to give answers to the first two examples. He has a straightforward method for doing so:

- He asks the students what they got for the answer, and confirms that it is correct.
- He asks how they got the answer.
- Then he elaborates on their responses.

Thus, for example, when a student says the answer to problem (b) is *xy*, Nelson asks "why did you get *xy*?" When the student says that he subtracted, Nelson asks, "What did you subtract? When the student says "3 minus 2," Nelson elaborates:

> OK. You looked at the *x*'s [pointing to *x*-terms in numerator and denominator] and [pointing to exponents] you subtracted 3 minus 2. That gave you *x* to the first [writes *x* on the board]. And then [points to y terms] you looked at the *y*'s and said [points to the exponents] 7 minus 6, gives you y to the first [writes y on board].

He then asks what to do with x^5/x^5. They expand and "cancel." The board shows $\frac{\not{x}\not{x}\not{x}\not{x}\not{x}}{\not{x}\not{x}\not{x}\not{x}\not{x}}$. Pointing to that expression, he says, "what do I have?" The responses are "zero," "zip," "nada," and "nothing" … not what he wants them to see! He tries various ways to get the students to see that "cancelling" results in a "1", for example,

Nelson: "What's 5/5?"
Students: "1."
Nelson: "But I cancelled. If there's a 1 there [in 5/5], isn't there a 1 there [pointing to the cancelled expression]?"
Students: "No."

Defeated, he slumps at the board while students argue there's "nothing there." He looks as if there is nothing he can say or do that will make sense to the students.

He tries again. He points to the expression $\frac{\cancel{x}\cancel{x}\cancel{x}\cancel{x}\cancel{x}}{\cancel{x}\cancel{x}\cancel{x}\cancel{x}\cancel{x}}$ and asks what the answer is. A student says "x to the zero over 1." Interestingly, Nelson *mis-hears* this as "x to the zero equals 1," which is the correct answer. Relieved, he tells the class,

> "That's right. Get this in your notes: $x^5/x^5 = x^0 = 1$."
> Any number to the zero power equals 1."

To put things simply, this is *very* strange. Nelson certainly knew enough mathematics to be able to explain that if x ≠ 0,

$$\left(\frac{x^5}{x^5}\right) = \left(\frac{x}{x}\right)^5 = 1^5 = 1,$$

but he didn't do so. WHY?

There is a simple answer, although it took us a long time to understand it. The issue has to do with Nelson's beliefs and orientations about teaching. One of Nelson's central beliefs about teaching—the belief that *the ideas you discuss must be generated by the students*—shaped what knowledge he did and did not use.

In the first example above (reducing the fraction x^3y^7/x^2y^6), a student said he had subtracted. The fact that the student mentioned subtraction gave Nelson "permission" to explain, which he did: "OK. You looked at the x's and you subtracted 3 minus 2. That gave you x to the first. And then you looked at the y's and said 7 minus 6, gives you y to the first."

But in the case of example (c), x^5/x^5, he was stymied—when he pointed to the expression $\frac{\cancel{x}\cancel{x}\cancel{x}\cancel{x}\cancel{x}}{\cancel{x}\cancel{x}\cancel{x}\cancel{x}\cancel{x}}$ and asked "what do I have?" the only answers from the students were "zero," "zip," "nada," and "nothing." Nobody said "1." And because of his belief that he had to "build on" what students say, Nelson felt he could not proceed with the explanation. Only later, when he *mis-heard* what a student said, was he able to finish up his explanation.

[Note: This brief explanation may or may not seem convincing. I note that full detail is given in the book, and that Nelson was part of the team that analyzed his videotape. So there is strong evidence that the claims I make here are justified.]

Second Teaching Example, Jim Minstrell

Here too I provide just a very brief description.

Jim Minstrell is an award-winning teacher who is very thoughtful about his teaching. It is the beginning of the school year, and he is teaching an introductory lesson that involves the use of mean, median, and mode. But, the main point of the lesson is that Minstrell wants the students to see that such formulas need to be used *sensibly.*

The previous day eight students measured the width of a table. They obtained these values:

106.8; 107.0; 107.0; 107.5; 107.0; 107.0; 106.5; 106.0 cm.

Minstrell wants the students to discuss the "best number" to represent the width of the table. His plan is for the lesson to have three parts:

1. Which numbers (all or some?) should they use?
2. How should they combine them?
3. With what precision should they report the answer?

Minstrell gave us a tape of the lesson, which we analyzed. The analysis proceeded in stages. We decomposed the lesson into smaller and smaller "episodes," noting for each episode which goals were present, and observing how transitions corresponded to changes in goals. In this way, we decomposed the entire lesson—starting with the lesson as a whole, and ultimately characterizing what happened on a line-by-line basis. See Figs. 2 and 3 (next pages) for an example of analytic detail. Figure 2 shows the whole lesson, and then breaks it into major episodes (lesson

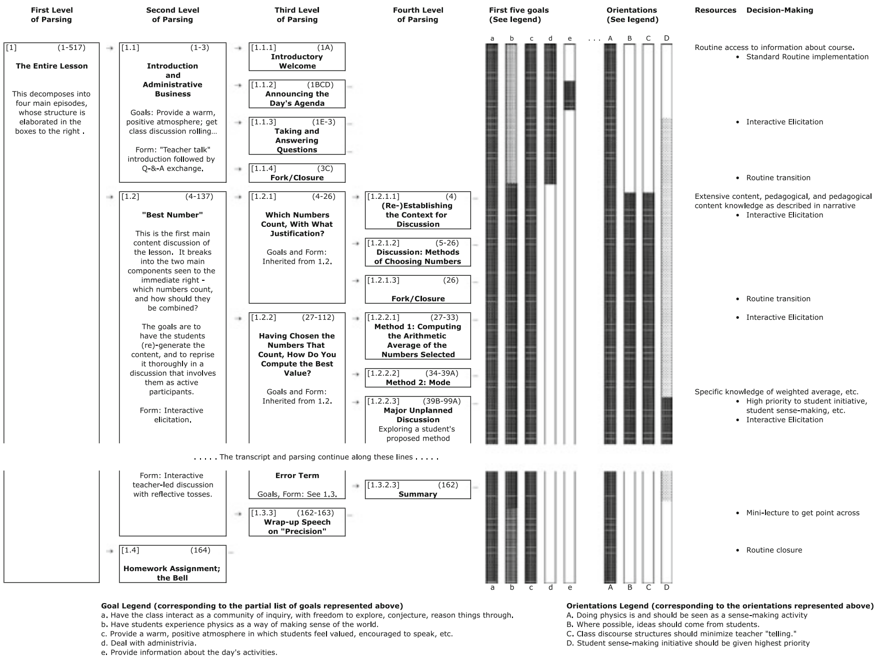

Fig. 2 A "top level" view of Minstrell's lesson, "unfolding" in levels of detail. (With permission, from Schoenfeld 2010, pp. 96–97)

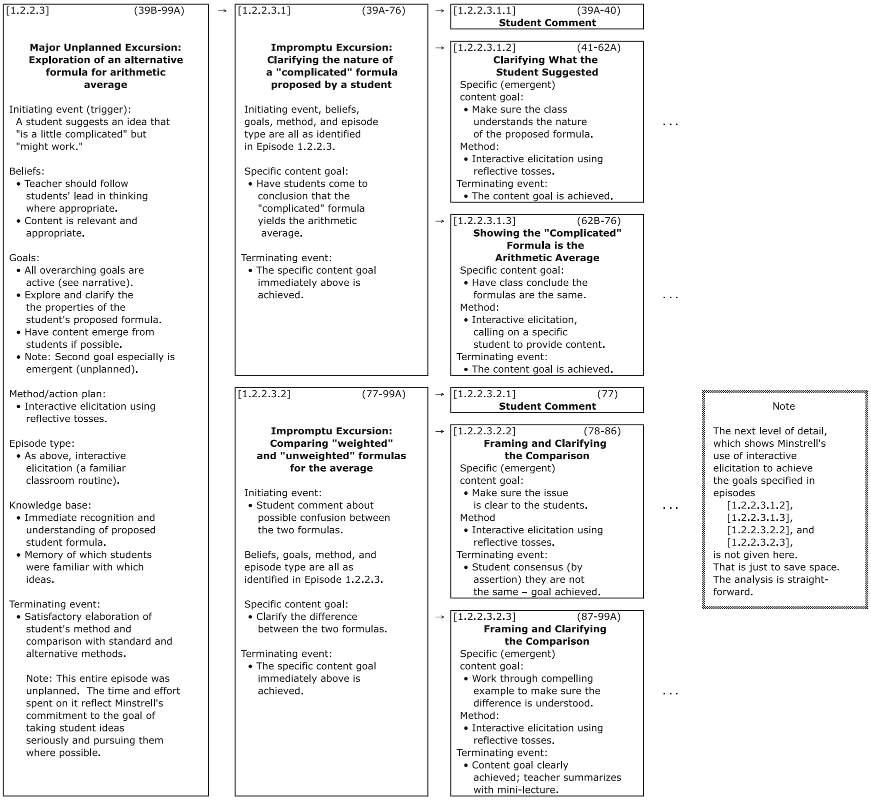

Fig. 3 A more fine-grained parsing of Episode [1.2.2.3]. (From Schoenfeld 2010, pp. 116–117, with permission)

segments), each of which has its own internal structure. Most of the lesson was very simple to analyze in this way.

Minstrell has a flexible "script" for each part of the lesson:

- He will raise the issue;
- He will ask the class for a suggestion;
- He will clarify and pursue the student suggestion by asking questions, inserting some content if necessary.

Once the suggestion has been worked through, he will ask for more suggestions. When students run out of ideas, he may inject more ideas, or move to the next part of the lesson.

In this way, the lesson unfolds naturally, and it is easy to "capture" it—see Fig. 2 for a "top level" summary of how the lesson unfolded. The episodes in the second and third columns, which correspond to an analysis of the lesson as taught, show that Minstrell did cover the big topics as planned.

A line-by-line analysis (see Schoenfeld 1998, 2010) shows that when Minstrell was dealing with expected subject matter, he followed the "script" described above very closely. So, it is easy to model Minstrell's behavior when he is on familiar ground.

But what about unusual events? Remember the data: The eight values the students had obtained for the width of the table were

106.8; 107.0; 107.0; 107.5; 107.0; 107.0; 106.5; 106.0 cm.

As the lesson unfolded, Minstrell asked the students about "a way of getting the best value." (see box 1.2.2 in the third column of Fig. 2.) As the class proceeded, one student mentioned the idea of using the "average" and, when asked by Minstrell, provided a definition. (Box 1.2.2.1 in the fourth column of Fig. 2.) Another student mentioned mode (Box 1.2.2.2). Then, a student said:

> This is a little complicated but I mean it might work. If you see that 107 shows up 4 times, you give it a coefficient of 4, and then 107.5 only shows up one time, you give it a coefficient of one, you add all those up and then you divide by the number of coefficients you have.

This is an unexpected comment, which does not fit directly with Minstrell's flexible script. The question is, can we say what Minstrell would do when something unexpected, like this, arises in the middle of his lesson?

Before proceeding, I want to point out that there is a wide range of responses, which teachers might produce. I have seen responses like all of the following:

> That's a very interesting question. I'll talk to you about it after class.

> Excellent question. I need to get through today's plans so you can do tonight's assigned homework, but I'll discuss it tomorrow.

> That's neat. What you've just described is known as the 'weighted average.' Let me briefly explain how you can work with that…

> Let me write that up as a formula and see what folks think of it.

> Let's make sure we all understand what you've suggested, and then explore it.

So, teachers might do very different things. Is it possible to know what Minstrell will do? According to our model of Minstrell, (1) His fundamental *orientation* toward teaching is that physics is a sense-making activity and that students should experience it as such; (2) One of his major *goals* is to support inquiry and to honor student attempts at figuring things out; (3) His *resource base* includes favored techniques such as "reflective tosses"—asking questions that get students to explain/elaborate on what they said.

Thus, the model predicts that he will pursue the last option—making sure that the students understand the issue that the student has raised (including the ambiguity about how you add the coefficients; do you divide by 5 or 8?) and pursuing it. He will do so by asking the students questions and working with the ideas they produce.

This is, in fact, what Minstrell did. Figure 3 shows how that segment of the lesson evolved. It is an elaboration of Box 1.2.2.3 in Fig. 2.

As noted above, it is possible to model Minstrell's decision. The model shows that, when faced with options such as those listed above, Minstrell is by far most likely to pursue the one I have indicated. The computations take about seven pages of text, so I will spare you the detail! More generally:

We have found that we were able to capture Minstrell's routine decision-making, on a line-by-line basis, by characterizing his knowledge/resources and modeling them as described in Fig. 1, "How Things Work;" and,

We were able to model Minstrell's non-routine decision-making using a form of subjective expected value computation, where we considered the various alternatives and looked at how consistent they were with Minstrell's beliefs and values (his orientations).

In summary, we were able to model every decision Minstrell made during the hour-long class.

Third Teaching Example, Deborah Ball

Some years ago, at a meeting, Deborah Ball showed a video of a third grade classroom lesson she had taught. The lesson was amazing—and it was controversial. In it,

- Third graders argued on solid mathematical grounds;
- The discussion agenda evolved as a function of classroom conversations;
- The teacher seemed at times to play a negligible role, and she made at least one decision that people said was not sensible.

In addition, I had little or no intuition about what happened. Thus, this was a perfect tape to study! There were major differences from cases 1 and 2:

- the students were third graders instead of high school students;
- psychological (developmental) issues differed because of the children's age;
- the "control structure" for the classroom was much more "organic";
- the teacher played a less obvious "directing" role.

The question was, could I model what happened in this lesson? If so, then the theory covered an extremely wide range of examples, which would comprise compelling evidence of its general validity. If not, then I would understand the limits of the theory. (Perhaps, for example, it would only apply to teacher-directed lessons at the high school level.)

Here is what happened during the lesson. Ball's third grade class had been studying combinations of integers, and they had been thinking about the fact that, for example, the sum of two even numbers always seemed to be even. The previous day Ball's students had met with some 4th graders, to discuss the properties of even numbers, odd numbers, and zero. Ball had wanted her students to see that these were complex issues and that even the "big" fourth graders were struggling with them. The day after the meeting (the day of this lesson), Ball started the class by asking what the students thought about the meeting:

- How do they think about that experience?
- How do they think about their own thinking and learning?

Ball had students come up to the board to discuss "what they learned from the meeting." The discussion (a transcript of which is given in full in Schoenfeld 2008, 2010) covered a lot of territory, with Ball seemingly playing a small role as students argued about the properties of zero (is it even? odd? "special"?). For the most part, Ball kept her students focused on the "meta-level" question: what did they learn about their own thinking from the meeting with the fourth graders the previous day?

But then, after a student made a comment, Ball interrupted him to ask a *mathematical* question about the student's understanding. This question, which took almost 3 min to resolve, completely disrupted the flow of the lesson. Many people, when watching the tape of the lesson, call that decision a "mistake." How could Ball, who is a very careful, thoughtful, and experienced teacher, do such a thing? If the decision was arbitrary or capricious in some way, that is a problem for the theory. If highly experienced teachers make arbitrary decisions, it would be impossible to model teachers' decision making in general.

In sum, this part of the lesson seems to unfold without Ball playing a directive role in its development—and she made an unusual decision to interrupt the flow of conversation. Can this be modeled? The answer is yes. A fine-grained analysis reveals that Ball has a "debriefing routine" that consists of asking questions and fleshing out answers. That routine is given in Fig. 4.

In fact, Ball uses that routine five times in the first 6 min of class. Moreover, once you understand Ball's plans for the lesson, her unexpected decision—what has been called her "mistake" by some—can be seen as entirely reasonable and consistent with her agenda. This has been modeled in great detail. For the full analysis, see Schoenfeld 2010; for an analytic diagram showing the full analysis, download Appendix E from my web page, http://www-gse.berkeley.edu/faculty/AHSchoenfeld/AHSchoenfeld.html.

To sum things up: As in the two previous cases, (1) We were able to model Ball's routine decision-making, on a line-by-line basis, by characterizing her knowledge/resources and modeling them as described in Fig. 1. (2) We were able to model Ball's non-routine decision-making as a form of subjective expected value computation.

In short, we were able to model every move Ball made during the lesson segment.

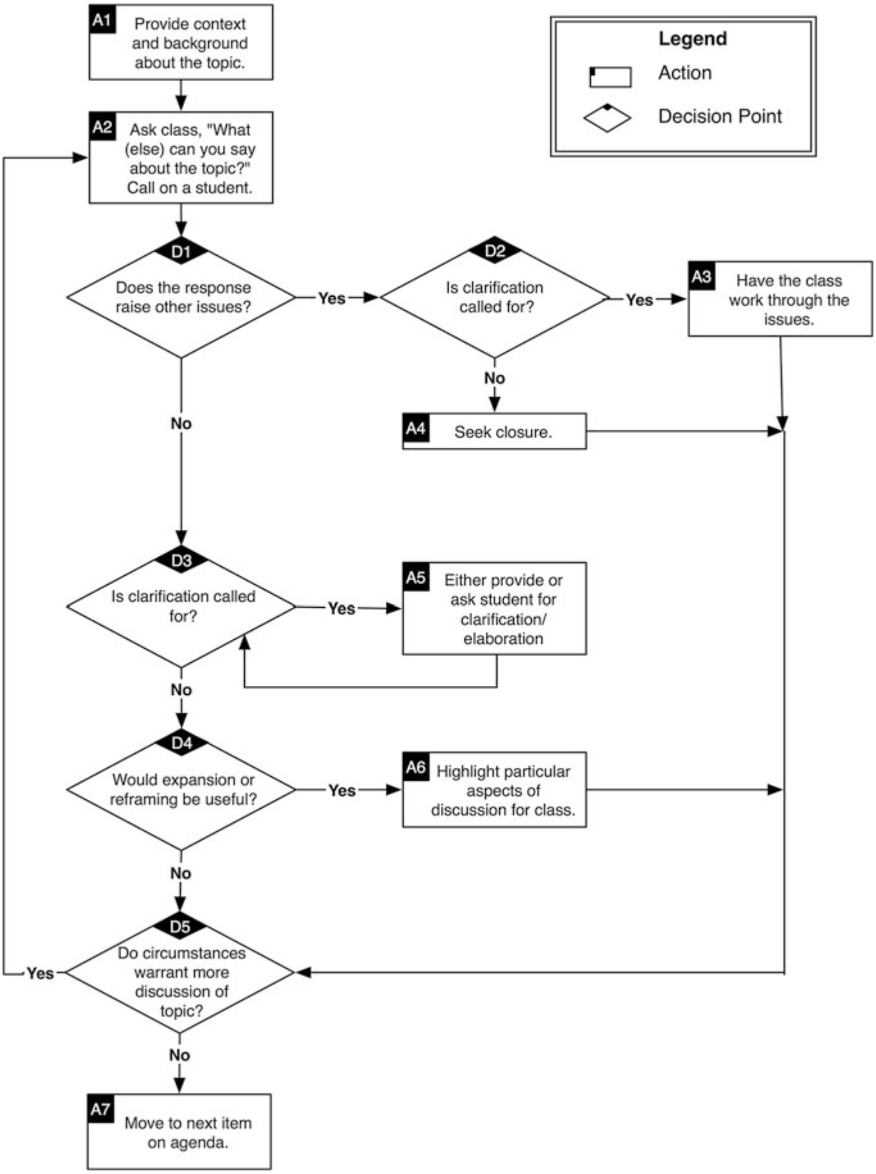

Fig. 4 A flexible, interruptible routine for discussing a topic. (From Schoenfeld 2010, p. 129, with permission)

Yet More Examples

Making Breakfast (or Any Other Meal)

If you look at Fig. 1, you can see that it would be easy to model decision-making during cooking. Usually we have fixed routines for cooking familiar meals. And if something changes (for example, when my daughter asks me to make a fancy breakfast), that calls for a "non-routine" decision, which can also be modeled. Readers might enjoy creating models of their own cooking practices and decision making.

Routine Medical Diagnosis and Practice

To see if my ideas worked outside of the classroom, I asked my doctor if I could tape and analyze one of my office visits with her. She said yes; an analysis of our conversation is given in *How We Think*. The conversation was easy to model, because the doctor follows a straightforward (and flexible) script. Modeling a two-person interaction is a lot easier than modeling a classroom; it is more like modeling a tutoring interaction. When the person being modeled (in this case, the doctor) only has to pay attention to one other person (instead of the 30 children a teacher has to pay attention to), decision-making is comparatively simple—and simple to model.

I should also note that there is a very large artificial intelligence literature on modeling doctors' decision making—there are computer programs that make diagnoses, etc. (The field is well established: see, e.g., Clancey and Shortliffe 1984). So, the idea that it is possible to capture doctors' routine decision making is not new. More recent, and also consistent with my emphasis on beliefs as shaping behavior, there are studies (e.g., Groopman 2007) of how doctors' stereotypes (beliefs and orientations) regarding patient behavior lead them to miss what should be straightforward diagnoses.

Discussion

The approach I have outlined in this paper "covers" routine and non-routine problem solving, routine and non-routine teaching, cooking, and brain surgery—and every other example of "well practiced," knowledge-based behavior that I can think of. All told, I believe it works pretty well as a theory of "how we think."

Readers have the right to ask, why would someone spend 25 years trying to build and test a theory like this? Here is my response.

First, theory building and testing should be central parts of doing research in mathematics education. That is how we make progress.

Second, the more we understand something the better we can make it work; when we understand how something skillful is done we can help others do it. This was the idea behind my problem solving work, where an understanding of problem solving helped me to help students become better problem solvers. I believe that a comparably deep understanding of teachers' decision making can be used to help mathematics teachers become more effective.

Third, this approach has the potential to provide tools for describing developmental trajectories of teachers. Beginning teachers, for example, often struggle with issues of classroom "management"—of creating an orderly classroom environment in which their students can learn productively. While teachers are struggling at this, they have little time or attention to devote to some of the more subtle aspects of expert teaching, such as teaching responsively—listening carefully to what their students say, diagnosing what the students understand and misunderstand, and shaping the lesson so that it helps move the students forward mathematically. The more we understand what teachers understand at particular points in their careers, the more we will be able to provide relevant professional development activities for them. An understanding of teachers' developmental trajectories can help us help teachers get better at helping their students learn. (see Chap. 8 of Schoenfeld 2010, for detail.)

Fourth and finally, it's fun! The challenge of understanding human behavior has proved itself to be every bit as interesting and intellectually rewarding as the challenge of understanding mathematics. It has occupied me for the past 35 years, and I look forward to many more years of explorations. Exploring questions of how teachers' understandings develop, and of how and when one can foster the development of mathematics teachers' expertise, are intellectually challenging. Equally important, addressing them can, over the long run, lead to improvements in mathematics teaching and learning.

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