Part II
Plenary Lectures

The Butterfly Effect

Étienne Ghys

Abstract It is very unusual for a mathematical idea to disseminate into the society at large. An interesting example is chaos theory, popularized by Lorenz's butterfly effect: "does the flap of a butterfly's wings in Brazil set off a tornado in Texas?" A tiny cause can generate big consequences! Can one adequately summarize chaos theory in such a simple minded way? Are mathematicians responsible for the inadequate transmission of their theories outside of their own community? What is the precise message that Lorenz wanted to convey? Some of the main characters of the history of chaos were indeed concerned with the problem of communicating their ideas to other scientists or non-scientists. I'll try to discuss their successes and failures. The education of future mathematicians should include specific training to teach them how to explain mathematics outside their community. This is more and more necessary due to the increasing complexity of mathematics. A necessity and a challenge!

Introduction

In 1972, the meteorologist Edward Lorenz gave a talk at the 139th meeting of the *American Association for the Advancement of Science* entitled "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?". Forty years later, a *google* search "butterfly effect" generates ten million answers. Surprisingly most answers are not related to mathematics or physics and one can find the most improbable websites related to movies, music, popular books, video games, religion, philosophy and even Marxism! It is very unusual that a mathematical idea can disseminate into the general society. One could mention Thom's catastrophe theory in the 1970s, or Mandelbrot's fractals in the 1980s, but these theories remained confined to the scientifically oriented population. On the contrary, *chaos theory*, often

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presented through the butterfly effect, did penetrate the nonscientific population at a very large scale. Unfortunately, this wide diffusion was accompanied with an oversimplification of the main original ideas and one has to admit that the transmission procedure from scientists to nonscientists was a failure. As an example, the successful book *The butterfly effect* by Andy Andrews "reveals the secret of how you can live a life of permanent purpose" and "shows how your everyday actions can make a difference for generations to come" which is not exactly the message of the founding fathers of chaos theory! In Spielberg's movie *Jurassic Park*, Jeff Goldblum introduces himself as a "chaotician" and tries (unsuccessfully) to explain the butterfly effect and unpredictability to the charming Laura Dern; the message is scientifically more accurate but misses the main point. If chaos theory only claimed that the future is unpredictable, would it deserve the name "theory"? After all, it is well known that "Prediction is very difficult, especially the future!".[1] A scientific theory cannot be limited to negative statements and one would be disappointed if Lorenz's message only contained this well known fact.

The purpose of this talk is twofold. On the one hand, I would like to give a very elementary presentation of chaos theory, as a mathematical theory, and to give some general overview on the current research activity in this domain with an emphasis on the role of the so-called *physical measures*. On the other hand, I would like to analyze the historical process of the development of the theory, its successes and failures, focusing in particular on the transmission of ideas between mathematics and physics, or from Science to the general public. This case study might give us some hints to improve the communication of mathematical ideas outside mathematics or scientific circles. The gap between mathematicians and the general population has never been so wide. This may be due to the increasing complexity of mathematics or to the decreasing interest of the population for Science. I believe that the mathematical community has the responsibility of building bridges.

A Brief History of Chaos from Newton to Lorenz

Determinism

One of the main pillars of Science is *determinism*: the possibility of prediction. This is of course not due to a single person but one should probably emphasize the fundamental role of Newton. As he was laying the foundations of differential calculus and unraveling the laws of mechanics, he was offering by the same token a tool enabling predictions. Given a mechanical system, be it the solar system or the collection of molecules in my room, one can write down a differential equation governing the motion. If one knows the present position and velocity of the system, one should

[1] See www.peterpatau.com/2006/12/bohr-leads-berra-but-yogi-closing-gap.html for an interesting discussion of the origin of this quotation.

simply solve a *differential equation* in order to determine the future. Of course, solving a differential equation is not always a simple matter but this implies at least the *principle* of determinism: the present situation determines the future. Laplace summarized this wonderfully in his "Essai philosophique sur les probabilités" (Laplace, 1814):

> We ought then to consider the present state of the universe as the effect of its previous state and as the cause of that which is to follow. An intelligence that, at a given instant, could comprehend all the forces by which nature is animated and the respective situation of the beings that make it up, if moreover it were vast enough to submit these data to analysis, would encompass in the same formula the movements of the greatest bodies of the universe and those of the lightest atoms. For such an intelligence nothing would be uncertain, and the future, like the past, would be open to its eyes.

The fact that this quotation comes from a book on *probability theory* shows that Laplace's view on determinism was far from naïve (Kahane 2008). We lack the "vast intelligence" and we are forced to use probabilities in order to understand dynamical systems.

Sensitivity to Initial Conditions

In his little book "Matter and Motion", Maxwell insists on the sensitivity to initial conditions in physical phenomena (Maxwell, 1876):

> There is a maxim which is often quoted, that 'The same causes will always produce the same effects.' To make this maxim intelligible we must define what we mean by the same causes and the same effects, since it is manifest that no event ever happens more that once, so that the causes and effects cannot be the same in *all* respects. […]
> There is another maxim which must not be confounded with that quoted at the beginning of this article, which asserts 'That like causes produce like effects'. This is only true when small variations in the initial circumstances produce only small variations in the final state of the system. In a great many physical phenomena this condition is satisfied; but there are other cases in which a small initial variation may produce a great change in the final state of the system, as when the displacement of the 'points' causes a railway train to run into another instead of keeping its proper course.

Notice that Maxwell seems to believe that "in great many cases" there is no sensitivity to initial conditions. The question of the frequency of chaos in nature is still at the heart of current research. Note also that Maxwell did not really describe what we would call chaos today. Indeed, if one drops a rock from the top of a mountain, it is clear that the valley where it will end its course can be sensitive to a small variation of the initial position but it is equally clear that the motion cannot be called "chaotic" in any sense of the word: the rock simply goes downwards and eventually stops.

Fear for Chaos

It is usually asserted that chaos was "discovered" by Poincaré in his famous memoir on the 3-body problem (Poincaré 1890). His role is without doubt very important, but maybe not as much as is often claimed. He was not the first to discover sensitivity to initial conditions. However, he certainly realized that some mechanical motions are very intricate, in a way that Maxwell had not imagined. Nevertheless chaos theory cannot be limited to the statement that the dynamics is complicated: any reasonable theory must provide methods allowing some kind of understanding. The following famous quotation of Poincaré illustrates his despair when confronted by the complication of dynamics (Poincaré 1890):

> When we try to represent the figure formed by these two curves and their infinitely many intersections, each corresponding to a doubly asymptotic solution, these intersections form a type of trellis, tissue, or grid with infinitely fine mesh. Neither of the two curves must ever cut across itself again, but it must bend back upon itself in a very complex manner in order to cut across all of the meshes in the grid an infinite number of times. The complexity of this figure is striking, and I shall not even try to draw it. Nothing is more suitable for providing us with an idea of the complex nature of the three-body problem, and of all the problems of dynamics in general […].

One should mention that ten years earlier Poincaré had written a fundamental memoir "Sur les courbes définies par des équations différentielles" laying the foundations of the qualitative theory of dynamical systems (Poincaré 1881). In this paper, he had analyzed in great detail the behavior of the trajectories of a vector field in the plane, i.e. of the solutions of an ordinary differential equation in dimension 2. One of his main results—the Poincaré-Bendixson theorem—implied that such trajectories are very well behaved and converge to an equilibrium point or to a periodic trajectory (or to a so-called "graphic"): nothing chaotic in dimension 2! In his 1890 paper, he was dealing with differential equations in dimension 3 and he must have been puzzled—and scared—when he realized the complexity of the picture.

Taming Chaos

Hadamard wrote a fundamental paper on the dynamical behavior of geodesics on negatively curved surfaces (Hadamard, 1898). He first observes that "a tiny change of direction of a geodesic […] is sufficient to cause any variation of the final shape of the curve" but he goes much further and creates the main concepts of the so-called "symbolic dynamics". This enables him to prove positive statements, giving a fairly precise description of the behavior of geodesics. Of course, Hadamard is perfectly aware of the fact that geodesics on a surface define a very primitive mechanical system and that it is not clear at all that natural phenomena could have a similar behavior. He concludes his paper in a cautious way:

> Will the circumstances we have just described occur in other problems of mechanics? In particular, will they appear in the motion of celestial bodies? We are unable to make such an assertion. However, it is likely that the results obtained for these difficult cases will be analogous to the preceding ones, at least in their degree of complexity. […]
> Certainly, if a system moves under the action of given forces and its initial conditions have given values *in the mathematical sense*, its future motion and behavior are exactly known. But, in astronomical problems, the situation is quite different: the constants defining the motion are only *physically* known, that is with some errors; their sizes get reduced along the progresses of our observing devices, but these errors can never completely vanish.

So far, the idea that some physical systems could be complicated and sensitive to small variations of the initial conditions—making predictions *impossible in practice*—remained hidden in very confidential mathematical papers known to a very small number of scientists. One should keep in mind that by the turn of the century, physics was triumphant and the general opinion was that Science would eventually explain everything. The revolutionary idea that there is a strong conceptual limitation to predictability was simply unacceptable to most scientists.

Popularization

However, at least two scientists realized that this idea is relevant in Science and tried—unsuccessfully—to advertize it outside mathematics and physics, in "popular books".

In his widely circulated book *Science and Method*, Poincaré expresses the dependence to initial conditions in a very clear way. The formulation is very close to the butterfly slogan and even includes a devastating cyclone (Poincaré 1908):

> Why have meteorologists such difficulty in predicting the weather with any certainty? Why is it that showers and even storms seem to come by chance, so that many people think it quite natural to pray for rain or fine weather, though they would consider it ridiculous to ask for an eclipse by prayer? We see that great disturbances are generally produced in regions where the atmosphere is in unstable equilibrium. The meteorologists see very well that the equilibrium is unstable, that a cyclone will be formed somewhere, but exactly where they are not in a position to say; a tenth of a degree more or less at any given point, and the cyclone will burst here and not there, and extend its ravages over districts it would otherwise have spared. If they had been aware of this tenth of a degree they could have known it beforehand, but the observations were neither sufficiently comprehensive nor sufficiently precise, and that is the reason why it all seems due to the intervention of chance.

In 1908 Poincaré was less scared by chaos than in 1890. He was no longer considering chaos as an obstacle to a global understanding of the dynamics, at least from the probabilistic viewpoint. Reading Poincaré's papers of this period, with today's understanding of the theory, one realizes that he had indeed discovered the role of what is called today *physical measures* (to be discussed later) which are at the heart of the current approach. Unfortunately, none of his contemporaries could grasp the idea—or maybe he did not formulate it in a suitable way—and one had to wait for seventy years before the idea could be re-discovered!

> You are asking me to predict future phenomena. If, quite unluckily, I happened to know the laws of these phenomena, I could achieve this goal only at the price of inextricable computations, and should renounce to answer you; but since I am lucky enough to ignore these laws, I will answer you straight away. And the most astonishing is that my answer will be correct.

Another attempt to advertize these ideas outside mathematics and physics was made by Duhem (1906) in his book *The aim and structure of physical theory*. His purpose was to popularize Hadamard's paper and he used simple words and very efficient "slogans":

> Imagine the forehead of a bull, with the protuberances from which the horns and ears start, and with the collars hollowed out between these protuberances; but elongate these horns and ears without limit so that they extend to infinity; then you will have one of the surfaces we wish to study. On such a surface geodesics may show many different aspects. There are, first of all, geodesics which close on themselves. There are some also which are never infinitely distant from their starting point even though they never exactly pass through it again; some turn continually around the right horn, others around the left horn, or right ear, or left ear; others, more complicated, alternate, in accordance with certain rules, the turns they describe around one horn with the turns they describe around the other horn, or around one of the ears. Finally, on the forehead of our bull with his unlimited horns and ears there will be geodesics going to infinity, some mounting the right horn, others mounting the left horn, and still others following the right or left ear. […] If, therefore, a material point is thrown on the surface studied starting from a geometrically given position with a geometrically given velocity, mathematical deduction can determine the trajectory of this point and tell whether this path goes to infinity or not. But, for the physicist, this deduction is forever unutilizable. When, indeed, the data are no longer known geometrically, but are determined by physical procedures as precise as we may suppose, the question put remains and will always remain unanswered.

Unfortunately the time was not ripe. Scientists were not ready for the message… Poincaré and Duhem were not heard. The theory went into a coma. Not completely though, since Birkhoff continued the work of Poincaré in a strictly mathematical way, with no attempts to develop a school, and with no applications to natural sciences. One should mention that Poincaré's work had also some posterity in the Soviet Union but this was more related to the 1881 "non chaotic" theory of limit cycles (Aubin and Dahan Dalmedico 2002).

Later I will describe Lorenz's fundamental article which bears the technical title "Deterministic non periodic flow", and was largely unnoticed by mathematicians for about ten years (Lorenz, 1963). Lorenz gave a lecture entitled "Predictability: does the flap of a butterfly's wings in Brazil set off a tornado in Texas?" which was the starting point of the famous butterfly effect (Lorenz, 1972).

> If a single flap of a butterfly's wing can be instrumental in generating a tornado, so all the previous and subsequent flaps of its wings, as can the flaps of the wings of the millions of other butterflies, not to mention the activities of innumerable more powerful creatures, including our own species.
> If a flap of a butterfly's wing can be instrumental in generating a tornado, it can equally well be instrumental in preventing a tornado.

This is not really different from Poincaré's "a tenth of a degree more or less at any given point, and the cyclone will burst here and not there". However, meanwhile, physics (and mathematics) had gone through several revolutions and non-predictability had become an acceptable idea. More importantly, the world had also gone through several (more important) revolutions. The message "each one of us can change the world[2]" was received as a sign of individual freedom. This is probably the explanation of the success of the butterfly effect in popular culture. It would be interesting to describe how Lorenz's talk reached the general population. One should certainly mention the best seller *Chaos: making a new science* (Gleick 1987) (which was a finalist for the Pulitzer Prize). One should not minimize the importance of such books. One should also emphasize that Lorenz himself published a wonderful popular book The essence of chaos in 1993. Note that the two main characters of the theory, Poincaré and Lorenz, wrote popular books to make their researches accessible to a wide audience.

Lorenz's 1963 Paper

Lorenz's article is wonderful (Lorenz 1963). At first unnoticed, it eventually became one of the most cited papers in scientific literature (more than 6,000 citations since 1963 and about 400 each year in recent years). For a few years, Lorenz had been studying simplified models describing the motion of the atmosphere in terms of ordinary differential equations depending on a small number of variables. For instance, in 1960 he had described a system that can be explicitly solved using elliptic functions: solutions were *quasiperiodic* in time (Lorenz 1960). His article (Lorenz 1962) analyzes a differential equation in a space of dimension 12, in which he numerically detects a sensitive dependence to initial conditions. His 1963 paper lead him to fame.

> In this study we shall work with systems of deterministic equations which are idealizations of hydrodynamical systems.

After all, the atmosphere is made of finitely many particles, so one indeed needs to solve an ordinary differential equation in a huge dimensional space. Of course, such equations are intractable, and one must treat them as partial differential equations. In turn, the latter must be discretized on a finite grid, leading to new ordinary differential equations depending on fewer variables, and probably more useful than the original ones.

The bibliography in Lorenz's article includes one article of Poincaré, but not the right one! He cites the early 1881 "non chaotic" memoir dealing with 2 dimensional dynamics. Lorenz seems indeed to have overlooked the Poincaré's papers that we have discussed above. Another bibliographic reference is a book by Birkhoff (1927)

[2] Subtitle of a book by Bill Clinton (2007).

on dynamical systems. Again, this is not "the right" reference since the "significant" papers on chaos by Birkhoff were published later. On the occasion of the 1991 Kyoto prize, Lorenz gave a lecture entitled "A scientist by choice" in which he discusses his relationship with mathematics (Lorenz 1991). In 1938 he was a graduate student in Harvard and was working under the guidance of… Birkhoff "on a problem in mathematical physics". However he seems unaware of the fact that Birkhoff was indeed the best follower of Poincaré. A missed opportunity? On the other hand, Lorenz mentions that Birkhoff "was noted for having formulated a theory of aesthetics".

Lorenz considers the phenomenon of *convection*. A thin layer of a viscous fluid is placed between two horizontal planes, set at two different temperatures, and one wants to describe the resulting motion. The higher parts of the fluid are colder, therefore denser; they have thus a tendency to go down due to gravity, and are then heated when they reach the lower regions. The resulting circulation of the fluid is complex. Physicists are very familiar with the Bénard and Rayleigh experiments. Assuming the solutions are periodic in space, expanding in Fourier series and truncating these series to keep only a small number of terms, Salzman had just obtained an ordinary differential equation describing the evolution. Drastically simplifying this equation, Lorenz obtained "his" differential equation:

$$\frac{dx}{dt} = \sigma(x+y); \; \frac{dy}{dt} = -xz + rz - y; \; \frac{dz}{dt} = xy - bz.$$

Here x represents the intensity of the convection, y represents the temperature difference between the ascending and descending currents, and z is proportional to the "distortion" of the vertical temperature profile from linearity, a positive value indicating that the strongest gradients occur near the boundaries. Obviously, one should not seek in this equation a faithful representation of the physical phenomenon. The constant σ is the *Prandtl number*. Guided by physical considerations, Lorenz was lead to choose the numerical values $r = 28$, $\sigma = 10$, $b = 8/3$. It was a good choice, and these values remain traditional today. He could then numerically solve these equations, and observe a few trajectories. The electronic computer Royal McBee LGP-30 was rather primitive: according to Lorenz, it computed (only!) 1,000 times faster than by hand. The anecdote is well known (Lorenz 1991):

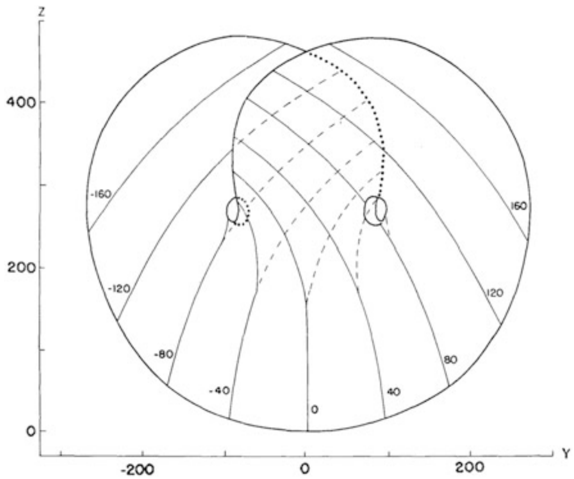
> I started the computer again and went out for a cup of coffee. When I returned about an hour later, after the computer had generated about two months of data, I found that the new solution did not agree with the original one. […] I realized that if the real atmosphere behaved in the same manner as the model, long-range weather prediction would be impossible, since most real weather elements were certainly not measured accurately to three decimal places.

Let us introduce some basic terminology and notation. For simplicity we shall only deal with ordinary differential equations in $\mathbb{R}^n$ of the form $\frac{dx}{dt} = X(x)$ where x is now a point in $\mathbb{R}^n$ and X is a vector field in $\mathbb{R}^n$. We shall assume that X is transversal to some large sphere, say $\|x\| = R$, pointing inwards, which means that the scalar product $x.X(x)$ is negative on this sphere. Denote by B the ball $\|x\| \leq R$. For any point x in B, there is a unique solution of the differential equation with initial condition x and defined for all $t \geq 0$. Denote this solution by $\phi^t(x)$. The purpose of the theory of *dynamical systems* is to understand the asymptotic behavior of these trajectories when t tends to infinity. With this terminology, one says that X is *sensitive to initial conditions* if there exists some $\delta > 0$ such that for every $\epsilon > 0$ one can find two points x, x' in B with $\|x - x'\| < \epsilon$ and some time $t > 0$ such that $\|\phi^t(x) - \phi^t(x')\| < \delta$.

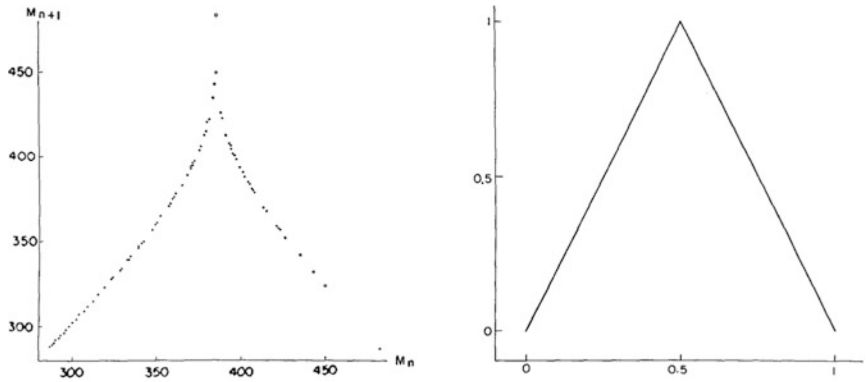
Lorenz's observations go much further than the fact that "his" differential equation is sensitive to initial conditions. He notices that these unstable trajectories seem to accumulate on a complicated compact set, which is itself *insensitive* to initial conditions and he describes this limit set in a remarkably precise way. There exists some compact set K in the ball such that for almost every initial condition x, the trajectory of x accumulates precisely on K. This attracting set K (now called the *Lorenz attractor*) approximately resembles a surface presenting a "double" line along which two leaves merge.

> Thus within the limits of accuracy of the printed values, the trajectory is confined to a pair of surfaces which appear to merge in the lower portion. […] It would seem, then, that the two surfaces merely appear to merge, and remain distinct surfaces. […] Continuing this process for another circuit, we see that there are really eight surfaces, etc., and we finally conclude that there is an infinite complex of surfaces, each extremely close to one or the other of the two merging surfaces.

Lorenz (1963)



Starting from an initial condition, the trajectory rapidly approaches this "two dimensional object" and then travels "on" this "surface". The trajectory turns around the two holes, left or right, in a seemingly random way. Notice the analogy with Hadamard's geodesics turning around the horns of a bull. Besides, Lorenz studies how trajectories come back to the "branching line" where the two surfaces merge, which can be parameterized by some interval [0,1]. Obviously, this interval is not very well defined, since the two merging surfaces do not really come in contact, although they coincide "within the limits of accuracy of the printed values". Starting from a point on this "interval", one can follow the future trajectory and observe its first return onto the interval. This defines a two to one map from the interval to itself. Indeed, in order to go back in time and track the past trajectory of a point in [0,1], one should be able to select one of the two surfaces attached to the interval. On the figure the two different past trajectories seem to emanate from the "same point" of the interval. Of course, if there are two past trajectories starting from "one" point, there should be four, then eight, etc., which is what Lorenz expresses in the above quotation. Numerically, the first return map is featured on the left part of Figure, extracted from the original paper.

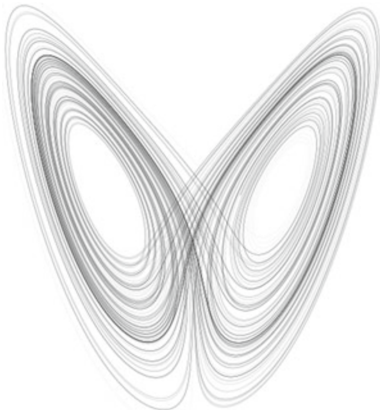


Working by analogy, Lorenz compares this map to the (much simpler) following one: $f(x) = 2x$ if $0 \leq x \leq \frac{1}{2}$ and $f(x) = 2 - 2x$ if $\frac{1}{2} \leq x \leq 1$ (right part of the Figure). Nowadays the chaotic behavior of this "tent map" is well known, but this was much less classical in 1963. In particular, the periodic points of f are exactly the rational numbers with odd denominators, which are dense in [0,1]. Lorenz does not hesitate to claim that the same property applies to the iterations of the "true" return map. The periodic trajectories of the Lorenz attractor are "therefore" dense in K. What an intuition! Finally, he concludes with a lucid question on the relevance of his model for the atmosphere.

> There remains the question as to whether our results really apply to the atmosphere. One does not usually regard the atmosphere as either deterministic or finite, and the lack of periodicity is not a mathematical certainty, since the atmosphere has not been observed forever.

To summarize, this remarkable article contains the first example of a physically relevant dynamical system presenting all the characteristics of chaos. *Individual trajectories are unstable* but their asymptotic behavior seems to be *insensitive* to initial conditions: they converge to the *same* attractor. None of the above assertions are justified, at least in the mathematical sense. How frustrating!

Surprisingly, an important question is not addressed in Lorenz's article. The observed behavior happens to be *robust*: if one slightly perturbs the differential equation, for instance by modifying the values of the parameters, or by adding small terms, then the new differential equation will feature the same type of attractor with the general aspect of a branched surface. This property would be rigorously established much later by Guckhenheimer and Williams.



The Lorenz attractor looks like a butterfly

Meanwhile, Mathematicians…

Lack of Communication Between Mathematicians and Physicists?

Mathematicians did not notice Lorenz's paper for more than ten years. The mathematical activity in dynamical systems during this period followed an independent and parallel path, under the lead of Smale. How can one understand this lack of communication between Lorenz—the MIT meteorologist—and Smale—the Berkeley mathematician? Obviously, during the 1960s the scientific community had already reached such a size that it was impossible for a single person to master mathematics and physics; the time of Poincaré was over. No bridge between different sciences was available. Mathematicians had no access to the *Journal of Atmospheric Sciences.*[3]

[3] In order to find an excuse for not having noticed Lorenz paper, a famous mathematician told me that Lorenz had published in "some obscure journal"!.

Smale's Axiom A

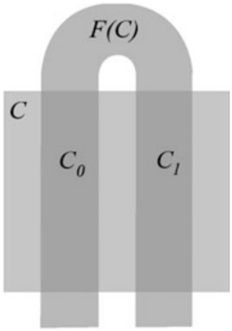
In 1959 Smale had obtained remarkable results in topology, around the Poincaré conjecture in higher dimension. The main tool was Morse theory describing the gradient of a (generic) function. The dynamics of such a gradient is far from chaotic: trajectories go uphill and converge to some equilibrium point. Smale initiated a grandiose program aiming at a qualitative description of the trajectories of a *generic* vector field (on compact manifolds). His first attempt was amazingly naïve (Smale 1960). He conjectured that a generic vector field has a finite number of equilibrium points, a finite number of periodic trajectories, and that every trajectory converges in the future (and in the past) towards an equilibrium or a periodic trajectory. He was therefore proposing that chaos does not exist! Poincaré, Hadamard or Birkhoff had already published counterexamples many years earlier! Looking back at this period, Smale wrote (1998a, b):

> It is astounding how important scientific ideas can get lost, even when they are aired by leading scientific mathematicians of the preceding decades.

Smale realized soon *by himself* [4] that the dynamics of a generic vector field is likely to be much more complicated than he had expected. He constructed a counterexample to his own conjecture (Smale 1961). The famous *horseshoe* is a simple example of a dynamical system admitting *an infinite number of periodic trajectories in a stable way*.

In order to describe this example, I should explain a classical construction (due to Poincaré). Suppose we start with a vector field X (in a ball in $\mathbb{R}^n$, as above). It may happen that one can find some $n-1$ dimensional disc D, which is transverse to X and which is such that the trajectory of every point x in D intersects D infinitely often. In such a situation, one can define a map $F : D \to D$ which associates to each point x in D the next intersection of its trajectory with D. For obvious reasons, this map is called the *first return map*. Clearly the description of the dynamics of X reduces to the description of the iterates of F. Conversely, in many cases, one can construct a vector field from a map F. It is often easier to draw pictures in D since it is one dimension lower than B. In Smale's example, D has dimension 2 and corresponds to a vector field in dimension 3, like in Lorenz's example. The map F is called a *horseshoe map* since the image $F(C)$ of a square C does look like a horseshoe as in the picture.

[4] As if obeying Goethe's dictum "Was du ererbt von deinen Vätern hast, erwirb es, um es zu besitzen" ("That which you have inherited from your fathers, earn it in order to possess it.").



The infinite intersection $\cap_{-\infty}^{+\infty} F^i(C)$ is a nonempty compact set $K \subset D$, and the restriction of F to K is a homeomorphism. The intersection $C \cap F(C)$ consists of two connected components C_0 and C_1. Smale shows that one can choose F in such a way that for every bi-infinite sequence a_i(with $a_i = 0$ or 1), there exists a unique point x in K such that $F^i(x) \in C_i$ for every i. In particular, periodic points of F correspond to periodic sequences a_i; they are dense in K.

More importantly, Smale shows that his example is *structurally stable*. Let us come back to a vector field X defined in some ball in $\mathbb{R}^n$ and transversal to the boundary. One says that X is *structurally stable* if every vector field X' which is close enough to X (say in the C^1 topology) is topologically conjugate to X: there is a homeomorphism h of B sending trajectories of X to trajectories of X'. Andronov and Pontryagin (1937) had introduced this concept in 1937 but in a very simple context, certainly not in the presence of an infinite number of periodic trajectories. The proof that the horseshoe map defines a structurally stable vector field is rather elementary. It is based on the fact that a map F' from D to itself close enough to F is also described by the same infinite sequences a_i.

Smale published this result in the proceedings of a workshop organized in the Soviet Union in 1961. Anosov tells us about this "revolution" in Anosov (2006).

> The world turned upside down for me, and a new life began, having read Smale's announcement of 'a structurally stable homeomorphism with an infinite number of periodic points', while standing in line to register for a conference in Kiev in 1961. The article is written in a lively, witty, and often jocular style and is full of captivating observations. […] [Smale] felt like a god who is to create a universe in which certain phenomena would occur.

Afterwards the theory progressed at a fast pace. Smale quickly generalized the horseshoe; see for instance (Smale 1966). Anosov proved in 1962 that the geodesic flow on a manifold of negative curvature is structurally stable (Anosov 1962)[5]. For this purpose, he created the concept of what is known today as *Anosov flows*. Starting from the known examples of structurally stable systems, Smale cooked up in 1965 the fundamental concept of dynamical systems satisfying the *Axiom A* and conjectured that these systems are *generic and structurally stable.* Smale's (1967)

[5] Surprisingly, he does not seem to be aware of Hadamard's work. It would not be difficult to deduce Anosov's theorem from Hadamard's paper.

article “Differential dynamical systems” represents an important step for the theory of dynamical systems (Smale 1967), a “masterpiece of mathematical literature” according to Ruelle. But, already in 1966, Abraham and Smale found a counter-example to this second conjecture of Smale: Axiom A systems are indeed structurally stable but they are not generic (Smale 1966, Abraham and Smale 1968).

Lorenz’s Equation Enters the Scene

Lorenz’s equation pops up in mathematics in the middle of the 1970s. According to Guckenheimer, Yorke mentioned to Smale and his students the existence of Lorenz’s equation, which did not fit well with their approach. The well-known 1971 paper by Ruelle and Takens (1971) still proposed Axiom A systems as models for turbulence, but in 1975 Ruelle observed that “Lorenz’s work was unfortunately overlooked” (Ruelle 1976a). Guckenheimer and Lanford were among the first people to have shown some interest in this equation (from a mathematical point of view) (Guckenheimer 1976; Lanford 1977). Mathematicians quickly adopted this new object which turned out to be a natural counterexample to Smale’s conjecture on the genericity of Axiom A systems. It is impossible to give an exhaustive account of all their work. By 1982 an entire book was devoted to the Lorenz’s equation, although it mostly consisted of a list of open problems for mathematicians (Sparrow 1982).

Bowen’s review article is interesting at several levels (Bowen, 1978). Smale’s theory of Axiom A systems had become solid and, although difficult open questions remained, one had a rather good understanding of their dynamics. A few “dark swans” had appeared in the landscape, like Lorenz’s examples, destroying the naïve belief in the genericity of Axiom A systems. However mathematicians were trying to weaken the definition of Axiom A in order to leave space to the newcomer Lorenz. Nowadays, Axiom A systems seem to occupy a much smaller place than one thought at the end of the 1970s. The Axiom A paradigm had to abandon its dominant position… According to (Anosov 2006):

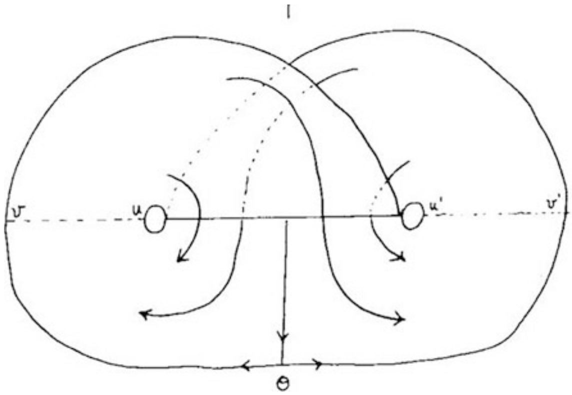
> Thus the grandiose hopes of the 1960s were not confirmed, just as the earlier naive conjectures were not confirmed.

For a more detailed description of the “hyperbolic history” one can also read the introduction of (Hasselblatt 2002), or (Ghys 2010). See also “What is… a horse-shoe” by one of the main actors of the field (Shub 2005).

Lorenz’s Butterfly as Seen by Mathematicians

In order to understand Lorenz’s butterfly from a mathematical point of view, Guckhenheimer and Williams (1979) introduced a “geometrical model” in 1979. Remember that Lorenz had observed that “his” dynamics seems to be related to the iterates of a map f from an interval to itself, even though this interval and this map were only defined "within the limits of accuracy of the printed values". The main idea of Guckenheimer and Williams is to *start* from a map f of the interval and to construct some vector field in 3-space whose behavior "looks like" the observed behavior of the original Lorenz equation. The question of knowing if the constructed vector field, called the *geometric Lorenz model*, is actually related to the original Lorenz equation was not considered as important. After all, the original Lorenz equation was a crude approximation of a physical problem and it was unclear whether it was connected with reality, and moreover mathematicians in this group were not really concerned with reality!

The following figure is reprinted from[6] (Guckenheimer and Williams 1979)



This is a *branched surface* Σ embedded in space. One can define some dynamical system f^t $(t \geq 0)$ on Σ whose trajectories are sketched on the figure: a point in Σ has a future but has no past because of the two leaves which merge along an interval. The first return map on this interval is the given map f from the interval to itself. The dynamics of f^t *is* easy to understand: the trajectories turn on the surface, either on the left or on the right wing, according to the location of the iterates of the original map f. So far, this construction does not yield a vector field. Guckhenheimer and Williams construct a vector field $X(f)$ in some ball B in $\mathbb{R}^3$, transversal to the boundary sphere, whose dynamics mimics f^t. More precisely, denote by $\phi^t(x)$ the trajectories of $X(f)$ and by Λ the intersection $\cap_{t \geq 0} \phi^t(B)$, so that for every point x in B, the accumulation points of the trajectory $\phi^t(x)$ are contained in Λ. The vector field $X(f)$ is such that Λ is very close to Σ and that the trajectories $\phi^t(x)$ shadow f^t. In other words, for every point x in Λ, there is a point x' in Σ such that $\phi^t(x)$ and $f^t(x')$ stay at a very small distance for all positive times $t \geq 0$. This vector field $X(f)$ is not unique but is well defined *up to topological equivalence*, i.e. up to some homeomorphism sending trajectories to trajectories. This justifies Lorenz's intuition,

[6] Incidentally, this figure shows that the quality of an article does not depend on that of its illustrations.

according to which the attractor Λ behaves like a branched surface. Moreover, every vector field in B which is close to $X(f)$ is topologically conjugate to some $X(f')$ for some map f' of the interval which is close to f. Furthermore, they construct explicitly a two-parameter family of maps $f_{(a,b)}$ which represent all possible topological equivalence classes. In summary, *up to topological equivalence*, *the vector fields in the neighborhood of* $X(f)$ *depend on two parameters and are Lorenz like*. This is the robustness property mentioned above.

Hence, the open set in the space of vector fields of the form $X(f)$ does not contain *any* structurally stable vector field. If Smale had known Lorenz's example earlier, he would have saved time! Lorenz's equation does not satisfy Axiom A and cannot be approximated by an Axiom A system. Therefore any theory describing generic dynamical systems should incorporate Lorenz's equation.

As we have mentioned, the geometric models for the Lorenz attractor have been *inspired* by the original Lorenz equation, but it wasn't clear whether the Lorenz equation indeed behaves like a geometric model. Smale chose this question as one of the "mathematical problems for the next century" in 1998. The problem was positively solved in Tucker (2002). For a brief description of the method used by Tucker, see for instance (Viana 2000).

The Concept of Physical SRB Measures

Poincaré

The main method to tackle the sensitivity to initial conditions uses *probabilities*. This is not a new idea. As mentioned earlier, Laplace realized that solving differential equations requires a "vast intelligence" that we don't have… and suggested developing probability theory in order to get some meaningful information. In his "Science and method", Poincaré gives a much more precise statement. Here is an extract of the chapter on "chance":

> When small differences in the causes produce great differences in the effects, why are the effects distributed according to the laws of chance? Suppose a difference of an inch in the cause produces a difference of a mile in the effect. If I am to win in case the integer part of the effect is an even number of miles, my probability of winning will be ½. Why is this? Because, in order that it should be so, the integer part of the cause must be an even number of inches. Now, according to all appearance, the probability that the cause will vary between certain limits is proportional to the distance of those limits, provided that distance is very small.

This chapter contains much more information about Poincaré's visionary idea and one can even read some proofs between the lines… In modern terminology, Poincaré considers a vector field X in a ball B in $\mathbb{R}^n$, as before. Instead of considering a single point x and trying to describe the limiting behavior of $\phi^t(x)$, he suggests choosing some probability distribution μ in the ball B and to study its evolution $\phi^t_\star \mu$ under the dynamics. He then gives some arguments showing that if μ has a *continuous* density, and if there is “a strong sensitivity to initial conditions”, the family of measures $\phi^t_\star \mu$ should converge to some limit ν which is *independent of the initial distribution* μ.[7] Even though individual trajectories are sensitive to initial conditions, the asymptotic *distribution* of trajectories is independent of the initial distribution, assuming that this initial distribution has a continuous density. Amazingly, none of his contemporaries realized that this was a fundamental contribution. This may be due to the fact that Poincaré did not write this idea in a formalized mathematical paper but in a popular book. One would have to wait for about seventy years before this idea could surface again.

Lorenz

We have seen that the 1972 conference of Lorenz on the butterfly emphasized the sensitivity to initial conditions and that this idea eventually reached the general public. However, this conference went much further:

> More generally, I am proposing that over the years minuscule disturbances neither increase nor decrease the frequency of occurrence of various weather events such as tornados; the most they may do is to modify the sequence in which these events occur.

This is the real message that Lorenz wanted to convey: the *statistical* description of a dynamical system could be *insensitive* to initial conditions. Unfortunately, this idea is more complicated to explain and did not become as famous as the “easy” idea of sensitivity to initial conditions.

Sinai, Ruelle, Bowen

Mathematicians also (re)discovered this idea in the early 1970s, gave precise definitions and proved theorems. A probability measure ν in the ball B, invariant by ϕ^t, is an *SRB measure* (for Sinai-Ruelle-Bowen), also called a physical measure, if, for each *continuous* function $u : B \to \mathbb{R}$, the set of points x such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T u(\phi^t(x))dt = \int_B u d\nu$$

[7] I may be exaggerating because of my excessive worship of Poincaré, but it seems to me that, in modern terminology, Poincaré explains that the limiting probability ν is absolutely continuous on instable manifolds and may not be continuous on stable manifolds.

has *nonzero Lebesgue measure*. This set of points is called the *basin* of v and denoted by *B(v)*. Sinai, Ruelle and Bowen (Sinai 1972; Ruelle 1976b; Bowen 1978) proved that this concept is indeed relevant in the case of Axiom A dynamics. If X is such a vector field in some ball B, there is a finite number of SRB measures $v_1,\ldots,v_k$ such that the corresponding basins $B(v_1),\ldots,B(v_k)$ cover B, up to a Lebesgue negligible set. Of course, the proof of this important theorem is far from easy but its general structure follows the lines sketched in Poincaré paper…

In summary, the existence of SRB measures is the right answer to the "malediction" of the sensitivity to initial conditions. In the words of Lorenz, "the frequency of occurrence of various weather events such as tornados" could be insensitive to initial conditions. If for example the ball B represents the phase space of the atmosphere and $u : B \to \mathbb{R}$ denotes the temperature at a specific point on the Earth, the average $\frac{1}{T}\int_0^T u(\phi^t(x))dt$ simply represents the average temperature in the time interval [0,T]. If there is an SRB measure, this average converges to $\int u dv$, *independently* of the initial position x (at least in the basin of v). The task of the forecaster changed radically: instead of guessing the position of $\phi^t(x)$ for a large t, he or she tries to estimate an SRB measure. *This is a positive statement about chaos as it gives a new way of understanding the word "prevision". It is unfortunate that such an important idea did not reach the general population*. Poor Brazilian butterflies! They are now unable to change the fate of the world!

The quest for the weakest conditions that guarantee the existence of SRB measures is summarized in the book (Bonatti et al. 2005). This question is fundamental since, as we will see, one hopes that "almost all" dynamical systems admit SRB measures.

The geometric Lorenz models are not Axiom A systems, hence are not covered by the works of Sinai, Ruelle and Bowen. However, it turns out that *the Lorenz attractor supports a unique SRB measure* (Bunimovich 1983; Pesin 1992). Lorenz was right!

Palis

The history of dynamical systems seems to be a long sequence of hopes… quickly abandoned. A non chaotic world, replaced by a world consisting of Axiom A systems, in turn destroyed by an abundance of examples like Lorenz's model. Yet, mathematicians are usually optimists, and they do not hesitate to remodel the world according to their present dreams, hoping that their view will not become obsolete too soon. Palis (1995, 2005, 2008) proposed such a vision in a series of three articles. He formulated a set of conjectures describing the dynamics of "almost all" vector fields. These conjectures are necessarily technical, and it would not be useful to describe them in detail here. I will only sketch their general spirit.

The first difficulty—which is not specific to this domain—is to give a meaning to "almost all" dynamics. The initial idea from the 1960s was to describe an *open* *dense* set in the space of dynamical systems, or at least, a countable intersection of open dense sets, in order to use *Baire genericity*. Yet, this notion has proved to be too strict. Palis uses a concept of "prevalence" whose definition is technical but which is close in spirit to the concept of "full Lebesgue measure". Palis *finiteness conjecture* asserts that in the space of vector fields on a given ball *B*, *the existence of a finite number of SRB measures whose basins cover almost all the ball is a prevalent property*.

Currently, the Lorenz attractor serves as a model displaying phenomena that are believed be characteristic of "typical chaos", at least in the framework of mathematical chaos. Even so, the relevance of the Lorenz model to describe meteorological phenomena remains largely open (Robert 2001).

Communicating Mathematical Ideas?

In Poincaré's time, the total number of research mathematicians in the world was probably of the order of 500. Even in such a small world, even with the expository talent of Poincaré as a writer, we have seen that some important ideas could not reach the scientific community. The transmission of ideas in the theory of chaos, from Poincaré to Palis has not been efficient. In the 1960s we have seen that the Lorenz equation took ten years to cross America from the east coast to the west coast, and from physics to mathematics. Of course, the number of scientists had increased a lot. In our 21st century, the size of the mathematical community is even bigger ($\sim$50,000 research mathematicians?) and the physical community is much bigger. Nowadays, the risk is not only that a good idea could take ten years to go from physics to mathematics: there could be tiny subdomains of mathematics that do not communicate at all. Indeed, very specialized parts of mathematics that look tiny for outsiders turn out to be of a respectable size, say of the order of 500, and can transform into "scientific bubbles". As Lovász (2006) writes in his "Trends in Mathematics: How they could Change Education?":

> A larger structure is never just a scaled-up version of the smaller. In larger and more complex animals an increasingly large fraction of the body is devoted to 'overhead': the transportation of material and the coordination of the function of various parts. In larger and more complex societies an increasingly large fraction of the resources is devoted to non-productive activities like transportation information processing, education or recreation. We have to realize and accept that a larger and larger part of our mathematical activity will be devoted to communication.

Of course, this comment does not only apply to mathematics but to Science in general and to the society at large. Nowadays, very few university curricula include courses on communication aimed at mathematicians. We need to train mediators who can transport information at all levels. Some will be able to connect two different areas of mathematics, some will link mathematics and other sciences, and some others will be able to communicate with the general public. It is important that we consider this kind of activity as a genuine part of scientific research and that it could attract our most talented students, at an early stage of their career. We should not only rely on journalists for this task and we should prepare some of our colleagues for this noble purpose. We have to work together and to improve mathematical communication. We should never forget that a mathematical giant like Poincaré took very seriously his popular essays and books, written for many different audiences.

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Whither the Mathematics/Didactics Interconnection? Evolution and Challenges of a Kaleidoscopic Relationship as Seen from an ICMI Perspective

Bernard R. Hodgson

Abstract I wish in this lecture to reflect on the links between mathematics and didactics of mathematics, each being considered as a scientific discipline in its own right. Such a discussion extends quite naturally to the professional communities connected to these domains, mathematicians in the first instance and mathematics educators (didacticians) and teachers in the other. The framework I mainly use to support my reflections is that offered by the International Commission on Mathematical Instruction (ICMI), a body established more than a century ago and which has played, and still plays, a crucial role at the interface between mathematics and didactics of mathematics. I also stress the specificity and complementarity of the roles incumbent upon mathematicians and upon didacticians, and discuss possible ways of fostering their collaboration and making it more productive.

Keywords Mathematics · Didactics of mathematics · Mathematicians · Mathematics educators · ICMI

Introduction

I wish in this lecture to reflect on the links between mathematics on the one hand, and the didactics of mathematics on the other, each being considered as a scientific discipline in its own right. From that perspective, mathematics is a domain with a very long history, while didactics of mathematics, or mathematical education as it is predominantly called by Anglophones, is of a much more recent vintage. Such a discussion extends quite naturally to the professional communities connected to these domains, mathematicians in the first instance, and mathematics educators

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(didacticians) and teachers in the other. The general framework I mainly use to support my reflections is that offered by the International Commission on Mathematical Instruction (ICMI), a body established more than a century ago and which has played, and still plays, a crucial role at the interface between mathematics and its teaching, between mathematics and didactics of mathematics.

As shown notably by the history of ICMI, there is a long tradition of eminent mathematicians being professionally involved in educational matters, including with regard to primary or secondary education. But the emergence, during the last decades of the previous century, of didactics of mathematics as an internationally recognized academic discipline has had among its effects an increase of the gap between mathematicians and mathematical educators, culturally and otherwise. Both mathematics and didactics depend for their development on research, founded in each case upon specific paradigms eventually hindering the fluidity of the communication between the two groups. While most professional mathematicians are involved not only in the creation or application of mathematics but also in its teaching, only a small number of them actually pay substantial attention to what recent research in education tells about the difficulties intrinsic to the learning of mathematics at various levels. And the development of didactics of mathematics, as a field both of practice and of research with distinctive concepts and vocabulary, amplifies to a certain extent the opaqueness of its results to the outsider. At the same time some suspicion may have developed within the mathematical education community about the role and importance of mathematicians in education. Such a situation may be reinforced at times by somewhat naive views expressed by some mathematicians in educational debates, as well as by the fact that, in opposition to the early days of didactics of mathematics, a larger proportion of didacticians nowadays, including teacher educators, have had little contact with higher mathematics, say, at the graduate level or even at the advanced undergraduate level.

I mainly base my discussion both upon my 11-year experience as ICMI Secretary-General (1999–2009) and on various elements stemming from activities organised by or under the auspices of ICMI, for instance ICME congresses or ICMI Studies, as well as on episodes from its history. I consider different contexts where mathematics and mathematical education interact and the way these contexts have evolved over the years. In connection with the complexity of educational issues related to both the teaching and the learning of mathematics, I also stress the specificity and complementarity of the roles incumbent upon mathematicians and upon mathematical educators, and examine possible ways of fostering their collaboration and making it more productive, notably in the context of ICMI activities.

Linguistic Prolegomena

Before embarking on my topic per se, it may be helpful to pay attention to some expressions appearing in the title of this lecture, so to make my use of these as clear as possible.

"Whither" or "Wither"

In spite of my patronymic, I share with the majority of the people in this audience the fact that English is not my mother tongue. Besides regretting any inconvenience stemming from my "French English", I need to point to potential problems provoked by the use of a certain vocabulary representing not only a substantial elocutionary challenge for non-native English speakers like me, but that moreover is usually not part of daily discourse. Such is possibly the case with the "whither" in my title. I do not know if many of you had to look into a dictionary for its exact meaning. I definitely did, when I first met this interrogative adverb. If my memory serves me well, my first encounter with this intriguing word—or at least the first time it really caught my attention—was in the title of one of the concluding chapters ("Whither mathematics?") of a thought-provoking book by Kline (1980) about the nature and role of mathematics. I met it again many years later through the plenary lecture "Whither mathematics education?" presented by Anna Sierpinska at ICME-8, in 1996 (Sierpinska 1998). I remember being fascinated by the idea of the likely future of a given matter being concealed in that single word "whither". And this is precisely what I have in mind in this talk about the mathematics/didactics links.

But depending on one's pronunciation of today's *lingua franca*, non-trivial difficulties may arise when using this word. You will have noted the two aitches ("h") in "whither", thus allowing to distinguish (at least visually!) this word from its neighbour "wither", a verb with a totally different meaning. But how is this difference to be communicated orally? I clearly was myself the source of some confusion recently when discussing with a former ICMI officer the topic of the present lecture. Quite obviously I then dropped the first aitch, either inadvertently or by a lack of capacity of rendering it orally in a proper way. "Why are you proposing such a strong title for your talk? was then wondering my colleague. Why do you insist on the possibility that the interconnection between mathematics and didactics may be drying, waning, decaying?" Such is not at all the message I aim at conveying in this lecture, and this is why the initial aitch is so important. As a matter of fact, I am concerned with quite the opposite: how to ensure that this crucial aitch never gets dropped!

Through the Kaleidoscope

Those of you aware of my long-term involvement in the mathematical preparation of primary school teachers will possibly be familiar with my deep interest for the kaleidoscope, a "philosophical toy" invented—and named[1]—in the early 19th

[1] The name "kaleidoscope" was coined by Brewster from the Greek words "kalos", *beautiful*, "eidos", *aspect*, and "skopein", *to see*. With a typical poetical flavour, the Chinese name for this instrument, 万花筒 ("wàn huā tǒng"), can be translated literally as *ten thousand flowers cylinder*,

century by the Scottish physicist Sir David Brewster.[2] This instrument, so simple yet so fertile, is in my opinion a wonderful "attention-catcher" eventually leading to scientific thinking, as it fascinates people of all ages through the richness and beauty of the images created by the interplay of mirrors.[3] It is in my opinion an ideal vehicle for putting teachers in contact with geometry, both practical and theoretical. The kaleidoscope has regularly been part of my teaching with primary school teachers for more than three decades (Hodgson 1987), and I still see as an important personal experience for teachers to explore the explosion of images provoked by the actual interaction of physical mirrors, notwithstanding the virtual possibilities offered by the computer (Graf and Hodgson 1990). A thorough theoretical understanding of the mathematical principles underlying the kaleidoscope is a challenge fully appropriate for primary school student teachers, and I am deeply convinced that the mastery of such a mathematical "micro-theory" can have a positive impact on their perception of mathematics and their personal relation to it (Hodgson 2004).

My mention of the kaleidoscope in the context of this talk is more than a mere wink to a mathematical pet subject of mine offering such a fecund pedagogical environment. I use in my title the kaleidoscope as a metaphor in order to suggest the changing nature of the mathematics/didactics relationship, like the stunning, if not unpredictable, alterations provoked on the image generated by a kaleidoscope by even a small shaking of the glass pieces inside the device. The history of ICMI, for instance, vividly illustrates the evolution over the past century of the links between mathematics and didactics, as well as the communities supporting these fields. But more to my point, the complexity and richness of kaleidoscopic rosettes can also serve as an analogy to the potential fruitfulness not only of the connections between mathematics and didactics as scholarly domains, but also of the collaboration between mathematicians and didacticians.

What?—and Who?

I now wish to comment on the mathematics/didactics tandem on which this talk is based. There is possibly no need to expand on the concept of *mathematics* in itself,

(Footnote 1 continued)
or more appropriately, *cylinder with myriads of flowers*. In a similar vein, the Korean name, 만화경 ("mân hwa gyong"), can be translated as *ten thousand brightnesses mirrors*, again suggesting the proliferation of a myriad of images. Quite interestingly, the word "myriad", used in English to convey the idea of an extremely large number, originally designated a unit of ten thousand in classical Greek numeration.

[2] Brewster commented about his instrument that "it was impossible not to perceive that it would prove of the highest service in all the ornamental arts, and would, at the same time, become a popular instrument for the purposes of rational amusement." (Brewster 1819, p. 7).

[3] This fascination for the kaleidoscope has possible been rendered no better than by the famous French writer André Gide (1869–1951), 1947 Nobel laureate in literature, in his autobiographical *Si le grain ne meurt* (cf. Graf and Hodgson 1990, p. 42).

except to stress that I am concerned here with mathematics as both a body of knowledge and an academic discipline implemented as a subject-matter in given teaching and learning environments, at different levels of educational systems all around the world.[4] The word *didactics* is slightly more difficult to circumscribe. I have in mind of course didactics *of mathematics*, rather than a kind of general-purpose didactics. I am aware that in English the adjective didactic may come with a pejorative connotation,[5] and that the noun didactics could be interpreted with the somewhat restricted meaning of "the science and art of teaching"[6]—see also Kilpatrick (2003) for similar linguistic comments. Consequently the expression *mathematics education* has become the one typically used among Anglophone circles to designate the scholarly domain that has developed, especially in the second part of the previous century, in relation to the teaching and learning of mathematics.

It is not my intent to enter here into fine discussions about the respective merits or limitations of expressions such as *didactics of mathematics* and *mathematics education*, and to examine their exact scope. Nor do I wish to focus on the specific case of the so-called French school of "didactique des mathématiques"—I refer those interested for instance to the analysis offered by Kilpatrick (2003, 2012). Still I will mostly use here the expression didactics of mathematics (rather than the more frequent mathematics education), partly because of my own linguistic bias, and partly because of a kind of general agreement, especially among some of the European countries, that seems to be emerging about its use, even in English.[7] In doing so, I am in line with the description proposed by Winsløw (2007), where didactics is understood as "the study of the teaching and learning of specific knowledge, usually within a disciplinary domain" (p. 534). In the same paper, Winsløw stresses how in some European contexts. "[d]idactics is regarded as a continuation of the study of the scientific discipline, in much the same way as the study of its history and philosophy" (p. 524).

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[4] Dossey (1992) offers an overview of various conceptions of mathematics, including in an historical perspective, and discusses "their current and potential impact on the nature and course of mathematics education" (p. 30). See also Kilpatrick (2008, pp. 29–31), for helpful nuances about the question "What is mathematics?" with regard to educational contexts, in particular in connection with the idea of mathematics then becoming a domain of practice.

[5] As is witnessed for instance by the following definition: "in the manner of a teacher, particularly so as to treat someone in a patronizing way", from the *New Oxford American Dictionary* (2nd edition, 2005, electronic version included in the Mac environment).

[6] According to the *Oxford English Dictionary* (online version), this seems to be a typical 19th-century vision. It is in that sense for instance that the word "didactics" is used in the title of one of the sections on the programme of the International Congress of Mathematicians held in Cambridge in 1912—cf. Hobson and Love (1913), Section IV, *Philosophy, History and Didactics*.

[7] It may be of interest to note that as early as 1968, Hans Georg Steiner was using (in English) the expression "didactics of mathematics" to designate the "new discipline" that, he claimed, had to be established to support what he saw as "new possibilities for mathematics teaching and learning" (cf. Steiner 1968, pp. 425–426). He presented this new discipline as "separate from the 'methodology of mathematics teaching'" (p. 426).

Another facet of the mathematics/didactics dichotomy concerns the actors involved in those fields. This is also far from easy to describe, as the context is intrinsically complex and can vary considerably from one country to the other—and even within a single country—, due to economical, social and cultural factors, as well as local traditions. This is why the local educational structures in which these people are to be found (vg, schools, colleges, universities, teacher education institutes, etc., not to speak of research centres and suchlike) come in a variety of forms. That said, I will now try to briefly identify, but without any pretention to exhaustiveness, what may be considered as typical working environments and structural frameworks for the colleagues I have in mind.

One obvious category of actors is that of the *mathematicians*, that is, people whose main interest is with mathematics as a body of knowledge and eventually contributing to its development through research.[8] To borrow from the title of a well-known math book from the time of my graduate studies (Mac Lane 1971), they are "working mathematicians", active in the field. The vast majority of these people, and especially those in the academia, will belong to a mathematics unit (department, etc.) and be involved in some form of teaching, from courses to math majors to large classes of engineers or graduate courses and seminars with a handful of students. Because of such teaching duties, they are undoubtedly "educators", although one could think that for a number of them, educational activities do not represent their main professional concern and would even have a possibly limited impact on the evolution of their career (promotion, etc.). Still there seems to be a growing number of faculty members in mathematics department developing a *bona fide* interest for educational matters, notably at the tertiary level. A crucial issue then becomes how they can find in the community the kind of support needed for their educational endeavour. I shall say a few words about this later.

Among the mathematicians is a subset of specific interest to this talk, and to which I myself belong: those whose teaching is substantially targeted at the mathematical education of teachers, both of primary and of secondary school. I have discussed in (Hodgson 2001) the importance of this specific contribution of mathematicians[9]—a contribution, I maintain, that should be considered as an intrinsic part of the "mission" of a mathematics department.

But mathematicians are of course not the only players involved in the preparation of mathematics schoolteachers. Another group of teacher educators of prime importance will typically be found in faculties of education (or of educational sciences). While many of them would call themselves *mathematics educators*, I prefer to use here the expression *didacticians*, in line with the preceding

[8] While I fully adhere with the statement made by IMU president Ingrid Daubechies, in her ICME-12 opening address, that the term "mathematicians" should be construed as including, for instance, participants at an ICME congress, I am using this word, for the purpose of my talk, in a slightly more restrictive (and customary) sense.

[9] "Mathematicians have a major and unique role to play in the education of teachers—they are neither the sole nor the main contributors to this complex process, but their participation is essential." (Hodgson 2001, p. 501).

comments.[10] Besides the graduate supervision of future didacticians or the development of their own research programme, a large portion of the teaching time of didacticians, at the undergraduate level, would mostly be devoted to the education of primary and secondary school teachers. One possible distinction between their contribution to the education of teachers and that of the mathematicians may be the extent to which emphasis is placed on the challenges encountered in the actual teaching and learning of some mathematical topic. This is to be contrasted with the attention mathematicians may give to the mastery of a given mathematical content, both in itself and as a potential piece of mathematics to be taught, as well as its place in the "global mathematical landscape", for instance when seen from an advanced standpoint *à la Klein* (see Klein 1932).

The actual "location" of didacticians inside the academic environment can vary a lot, but they often belong to a faculty of education. A specific case I wish to stress is when didactics of mathematics is attached, as an academic domain, to the same administrative unit (vg, a given university department) to which mathematics belongs[11]—a context that may be seen as related to the comments of Winsløw quoted above. Such a situation is far from being the general rule—and I would not want to push it as an ideal universal model—, but it clearly offers an interesting potential for fostering the links between mathematicians and didacticians, and eventually improving mutual understanding and respect.

More generally, there is an obvious need for a community and a forum where mathematicians and didacticians can meet in connection to issues, general or specific, related to the teaching and learning of mathematics. An interesting context to that effect is that offered by ICMI.

A Glimpse into the History of ICMI

The International Commission on Mathematical Instruction (ICMI) celebrated in 2008 its centennial, an event that stimulated the publication of a number of papers dealing with various aspects of its history. Detailed information about the origins of the Commission and its evolution over the years can be found for instance in Bass (2008b), Furinghetti et al. (2008) and Schubring (2008), three papers appearing in the proceedings of the ICMI centennial symposium. Other papers of a historical nature include Furinghetti (2003) and Schubring (2003), written on the occasion the

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[10] My reluctance to speak of "mathematics educators" in that context also stems from the fact that in my opinion, expressions such as "mathematics educators" or "teacher educators" should not be construed as belonging exclusively to or denoting specifically either the community of didacticians or that of mathematicians: as stressed earlier, we are *all* educators, but of course with our own specific ways of addressing educational issues.

[11] As a concrete example, I mention that the position in "didactique des mathématiques" created in 1999 at Université Paris Diderot (a scientific university of international research fame) and first occupied by former ICMI president Michèle Artigue is attached to the mathematics department.

centennial of *L'Enseignement Mathématique*—the journal which since the inception of ICMI has been its official organ—, as well as Hodgson (2009). The survey of Howson (1984) was prepared on the occasion of the 75th anniversary of ICMI. Many ICMI-related sections are found in Lehto (1998), a book about the history of the International Mathematical Union (IMU), the organization to which ICMI owes its legal existence.

The beginnings of ICMI can be seen as resting upon the assumption that mathematicians have a role to play in issues related to school mathematics—at least at the secondary level. Its establishment resulted from a resolution adopted at the Fourth International Congress of Mathematicians held in Rome in 1908 and appointing a commission, under the presidency of the eminent German mathematician Felix Klein, with the mandate of instigating "a comparative study of the methods and plans of teaching mathematics at secondary schools" (Lehto 1998, p. 13). This resolution can be seen as addressing concerns present at the turn of the twentieth century in educational debates and provoked by the spreading of mass education combined with a greater sensitivity towards internationalism that stimulated the need for self-reflection, comparison and communication. Still today, the formal definition of ICMI's global mission and framework for action points to the importance of connecting its educational enterprises with the community of mathematicians as represented by IMU. For instance the Terms of reference of ICMI state that "ICMI shall be charged with the conduct of the activities of IMU bearing on mathematical or scientific education". More details are provided below on the recent and current links between ICMI and IMU.

A sharp distinction is manifest between the "old ICMI's tradition" (Furinghetti 2008, p. 49) of publishing national reports and international analyses of school curricula, as done abundantly in its early years,[12] and the activities of ICMI after its rebirth[13] in 1952, at a time when the international mathematical community was being reorganized, as a permanent commission of the then newly established IMU. Furinghetti (2008) stresses how at that latter time "the developments of society and schools were making the mere study and comparison of curricula and programs (…) inadequate to face the complexity of the educational problems" (p. 49). Highlighting the use of the "new expression 'didactical research'" in the title of a short lecture presented at the 1954 International Congress of Mathematicians, she presents this as a sign of an emerging shift about mathematics education, from a "national business" mainly concerned with curricular comparisons to a "personal business" centred on learners and teachers (Furinghetti 2008, pp. 49–50). The 1950s also saw the development of a new community, the *Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques*

[12] Fehr (1920–1921, p. 339) indicated for instance that between 1908 and 1920, ICMI, jointly with eighteen of the countries it gathered, had produced 187 volumes containing 310 reports, for a total of 13,565 pages.

[13] This rebirth followed a hiatus in ICMI activities around the two World Wars. Like most international scientific organizations of that time, ICMI was deeply affected by the ongoing international tensions.

(CIEAEM /International Commission for the Study and Improvement of Mathematics Teaching, ICSIMT), where the importance of reflecting on the students themselves as well as on the teaching processes and classroom interactions was strongly emphasised, in contrast to educational work typical of the time.

Such deep changes were the reflection of the emergence of a new sensitivity with regard to educational issues. As a result, a context arose propitious not only to the development of new approaches to study the teaching and learning of mathematics, but also to the eventual birth of a new academic discipline, gradually accepted and recognized as such, namely didactics of mathematics (i.e., mathematics education in usual parlance). ICMI itself was at times strongly influenced by these changes—Furinghetti et al. (2008) speak of a "Renaissance" of ICMI under the influence of events from the 1950s and 1960s. But ICMI also accompanied the evolution of didactics of mathematics, and at times even fostered it, thus contributing significantly to its acceptance as a *bona fide* academic domain.

This was particularly true during the ICMI presidency of Hans Freudenthal from 1967 to 1970. This particular moment was definitely a turning point in the renewal of ICMI, principally because of two major events that then occurred, essentially at Freudenthal's personal initiative, and that proved to have a considerable long-term impact: the establishment in 1968 of an international research journal in didactics of mathematics (*Educational Studies in Mathematics*, *ESM*), and the launching in 1969 of a new series of international congresses (the International Congress on Mathematical Education, ICME), the twelfth of which we are now celebrating in Seoul.

Bass (2008b) uses the expressions "Klein era" and "Freudenthal era" (from the names of the first and eighth presidents of ICMI) to designate two pivotal segments structuring the life of ICMI up to its 100th anniversary and corresponding more or less to its first two half-centuries: from ICMI beginnings in 1908 up to World War II, and from ICMI rebirth in 1952 to its centennial celebration. Of central interest to my lecture is the distinction Bass introduces about the actors then involved in ICMI circles. While those of the first period were mostly "mathematicians with a substantial, but peripheral interest in education, of whom Felix Klein was by far the most notable example, plus some secondary teachers of high mathematical culture" (Bass 2008b, p. 9), the majority of the players in the Freudenthal era are professional researchers in the teaching and learning of mathematics, i.e., didacticians. Bass also adds that "[i]n this period we see also the first significant examples of research mathematicians becoming professionally engaged with mathematics education even at the scholarly level" (Bass 2008b, p. 10), and suggests Freudenthal as a outstanding example of such a phenomenon—but of course the name of Hyman Bass himself provides an eloquent example of a more recent nature. A thorny question, in that connection, is the extent to which the growing specificity of the main actors of the Freudenthal era may create a widening distance with the "working mathematician" with regard to educational issues.

As discussed in Hodgson (2009), the presidency of Freudenthal resulted in what might be rightly seen as "years of abundance" for ICMI, in the sense that the scope and impact of its actions expanded considerably. Not only were the newly established *ESM* and ICMEs highly successful, but also new elements were gradually added to the mission of ICMI. To name a few, ICMI introduced in the mid-1970s a notion of Affiliated Study Groups, serving specific segments of a community becoming more and more diverse.[14] There was also a regular collaboration between ICMI and UNESCO, contributing in particular to outreach actions of ICMI towards developing countries. And later, in the mid-1980s, the very successful program of ICMI Studies was initiated. Still this deep evolution of ICMI, notably through the influence of Freudenthal himself, did not happen without some tensions with IMU, in particular as it was often the case that IMU faced decisions that were *faits accomplis*, taken without any consultation between the Executive Committees of ICMI and IMU—such had been the case for instance with the launching of the first ICME congress.[15]

Another moment of tension between IMU and ICMI happened in connection with the program of the section on the Teaching and Popularisation of mathematics at the 1998 International Congress of Mathematicians.[16] As a consequence, the first Executive Committee of ICMI on which I served, under the presidency of Hyman Bass, had to deal with an episode of misunderstanding, and even mistrust, between the communities of mathematicians and didacticians as represented by IMU and ICMI. I will come back to this episode later in this lecture and contrast it with the very positive climate of collaboration and mutual respect between these two bodies that now prevails.

This overview of the history of ICMI may help appreciate the origins of didactics of mathematics as an academic domain, as well as its evolution over the years. One can also see the changing profile of both the main actors involved in the reflections about the teaching and learning of mathematics and the communities gathering them, notably via the two main bodies under consideration in the context I am discussing, ICMI and IMU.

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[14] HPM and PME, the first two Study Groups affiliated to ICMI, both in 1976, are typical of the development of several specific strands in didactics of mathematics that has happened during the last 35 years or so. The affiliation in 1994 of WFNMC, whose action is centered on mathematical competitions, is linked to an interest of a number of mathematicians concerning the identification and nurturing of mathematical talents. In their survey of international organizations in mathematics education, Hodgson et al. (2013) contrast the mere three international bodies established up to the early 1960s (ICMI—1908, CIEAEM—1950 and CIAEM—1961) with the proliferation since the mid-1970s, each new body corresponding to a particular component of the mathematics education landscape. They comment that "[t]he presence of such subcommunities wanting to become institutionalized within the mathematical education world can be interpreted as a sign of the vitality of the field and the diversity of its global community" (p. 935).

[15] The interested reader will find in Lehto (1998) and Hodgson (2009) more information about this episode of tension between IMU and ICMI resulting from Freudenthal's initiatives.

[16] Comments on this episode and its context, notably with respect to the so-called 'Math War' in the USA, can be found in Artigue (2008, p. 189). See also Hodgson (2009, pp. 85–86), and in particular endnote 5, p. 94.

Some Challenges that Mathematicians and Didacticians Are Facing

I commented above on the fact that both mathematicians and didacticians have a specific contribution to bring to educational issues, and in particular to the preparation of mathematical schoolteachers. In a sense they are more or less compelled to collaborate—at least in principle. But that is easier said than done.

One point at stake, in the case of mathematicians, is the extent to which they are willing to fully acknowledge education as part of their real responsibilities. But there are encouraging signs on that account. For instance more and more national societies of mathematicians, most of which are typically centred on research in mathematics, now devote a non-negligible part of their energy and activities to educational issues, very often with a genuine concern. A striking example, to take one close to my personal environment, is given by the American Mathematical Society, definitely an outstanding research-supporting body, but with pertinent and well-focused actions about educational matters. In a similar vein, one could think of the European Mathematical Society, whose Education Committee has launched in 2011 a series of articles in the *Newsletter* of the EMS under the general label *'Solid findings' in mathematics education*. The 'solid findings' papers are designed as "brief syntheses of research on topics of international importance" (Education Committee of the EMS, 2011 p. 47) which aim at presenting to an audience of non-specialists (especially mathematicians and mathematics teachers) what current research may tell us about how to improve the teaching and learning of a given mathematical topic. The message conveyed by such societies is very clear concerning the place that mathematicians may or should occupy with regard to educational matters, and even debates.[17] The message is also clear, consequently, about the responsibilities of a math department in this connection with respect to the inclusion of education as part of its mission. But transferring this into the daily life of the department is far from trivial.

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[17] In his ICME-10 plenary lecture concerning the educational involvement of mathematicians, Bass (2008a) makes an important *caveat*:

> I choose specifically to focus on the involvement of *research mathematicians*, in part to dispel two common myths. First, it is a common *belief among mathematicians* that attention to education is a kind of pasturage for mathematicians in scientific decline. My examples include scholars of substantial stature in our profession, and in highly productive stages of their mathematical careers. Second, many *educators have questioned* the relevance of contributions made by research mathematicians, whose experience and knowledge is so remote from the concerns and realities of school mathematics education. I will argue that the knowledge, practices, and habits of mind, of research mathematicians are not only relevant to school mathematics education, but that this mathematical sensibility and perspective is essential for maintaining the mathematical balance and integrity of the educational process—in curriculum development, teacher education, assessment, etc. (pp. 42–43).

I dream of a day when it would be normal for a university math department to open a tenure-track position in mathematics but with a very strong educational emphasis, vg with regard to the preparation of schoolteachers or the development of innovative teaching approaches for very large undergraduate classes. Some of this already exists in some places,[18] but at a much too modest level altogether.

But an immediate concern follows: what about promotion to a higher academic rank? Would a significant involvement in education by a mathematician be judged by his peers as a valuable academic activity, on a par, say, with mathematical research or supervising graduate students? Many indicators point to the fact that this may remain for some time a major challenge that university administrations will be facing. But there are signs that mentalities may be changing.[19] Still it would probably be naive to expect a young mathematician recently hired by a math department to devote much time and energy to education matters, unless the position occupied would be very explicit on that account.

In a survey of the ICMI program of actions as seen from a Canadian perspective that I presented at a meeting of the Canadian Mathematics Education Study Group (Hodgson 2011), I suggested as a major challenge for the Canadian community the question of the actual involvement of individual mathematicians—especially the young ones—in educational matters and in activities of a group such as CMESG. The same challenge also exists, at the international level, with regard to the participation of mathematicians in activities of ICMI. What percentage of the people in the present audience, for instance, would consider themselves first and foremost as "working mathematicians"?

That said, past implications of mathematicians in educational matters have not been always optimal, to say the least. The level of rigor typically shown by mathematicians in their own research work is sometimes less perceptible when they come to express opinions about educational matters, sometimes on the basis of extremely naive observations or opinions. Bass and Hodgson (2004) comment for instance that "mathematicians sometimes lack a sufficient knowledge and/or appreciation of the complex nature of the problems in mathematics education" (p. 640). A particularly eloquent episode on that account is probably that of the Math War.[20] In her presidential closing talk at the ICMI Centennial symposium, Artigue (2008) describes not only the role of ICMI at the interface of mathematics and mathematics education, as announced in the title of her paper, but also at the interface of the communities of mathematicians and didacticians. She speaks of the

[18] As a concrete example, the mathematics department to which I belong has currently two such positions for mathematicians, one established as early as in the mid-1970s for the mathematical education of primary school teachers, and the other (mid-1990s) for secondary teachers.

[19] I have witnessed, over the past decade or so, a few successful cases of promotion for tenure or for full professorship concerning mathematicians with a career strongly focused on education and belonging to renowned research-oriented math departments.

[20] Bass (2008a) notes about the expression "Math War" that it is "an unfortunate term coined in the U.S. to describe the conflicts between mathematicians and educators over the content, goals, and pedagogy of the curriculum" (p. 42).

tensions that arose in the 1990s between those communities because "the supposed influence of mathematics educators was considered by some mathematicians as an important, if not the major, source of the observed difficulties in mathematics education, leading to such extremes as the so-called Math War in the USA" (p. 189).

Such a perception by mathematicians connects to a comment from Winsløw (2007), when he contrasts the necessary close ties he sees didactics having with the discipline, and the reality of the "[i]nstitutional policies and tradition" that imposes a distance between mathematicians and didactics (p. 533). He adds that "[t]he hesitancy of mathematicians to admit the need or worth of didactics could perhaps also be interpreted as an instance of a more general scepticism, among mathematicians, with respect to educational research." (p. 534)

But another side of the coin is related to the fact that didactics of mathematics has grown over the past decades into a fully-fledged academic domain, so that it has developed its specific paradigms, concepts, vocabulary. An unavoidable and obvious consequence is an increase of the communication gap between mathematicians and didacticians. Issues connected to the teaching and learning of mathematics can no more be approached with mere naive views or ideas—fortunately, one may say! But even mathematicians with a genuine interest in education feel a greater distance, as communication has become less transparent. A body of knowledge has now been developed, which must be grasped to a certain extent by mathematicians wishing to be part of the ongoing reflections.[21] Mathematicians will of course be familiar with this phenomenon internally, from one branch of mathematics to the other, but they may not be sensitive to its importance when it comes to educational contexts, if they have somehow developed the conviction that educational matters could be addressed seriously even through a very rudimentary approach. There is a responsibility for mathematicians here to keep abreast of recent didactical developments. But maybe more to my point, there is a responsibility for didacticians to make their work accessible without imposing unnecessary jargon or constructs. I believe more needs to be done on that account.

I would like to conclude this part of my talk with a comment of a possibly sensitive nature concerning the education of didacticians and the prerequisites they

[21] It is of interest to note, in that connection, that without denying the importance for mathematicians of gaining competency with respect to current developments in didactical research, some networks are developing that allow mathematicians to discuss educational issues and develop familiarity with ongoing work in less 'threatening' contexts, so to say. Such is the case for instance of Delta, an informal collaboration network among Southern Hemisphere countries that has developed since the end of the 1990s. In their survey of international organizations in mathematics education, Hodgson et al. (2013) write: "A central idea of Delta is to provide a forum in which mathematicians feel comfortable in discussing issues related to tertiary mathematics teaching and learning without being intimidated by what some may consider educational jargon or constructs. Many participants at the conferences are thus mathematicians wishing to report about a teaching experience or experiment that would normally not classify as bona fide research in mathematics education, but may still be helpful in inspiring those who want to reflect on their teaching" (p. 927).

should meet to be recognized as such. To make my case clear, I have in mind here the mathematical prerequisites. This issue is even more difficult to circumscribe as it does vary considerably from one country to the next.

As a starting vantage point, let me stress that the majority of the didacticians of my generation, if not all, had a substantial education in mathematics before switching to didactics of mathematics. The reason is simply that graduate studies in mathematics education are still, in most places, of a somewhat recent vintage. So it would not be so uncommon for a didactician of my age to have first done a certain amount of studies in mathematics, even at the graduate level. Today, with the development of didactics of mathematics as an autonomous academic field, the situation has changed substantially. While in many countries the road to didactics of mathematics is still intertwined with an important mathematical component, often of an advanced nature, I am aware of contexts where such is not the case, contexts where someone could be called a didactician of mathematics while having a rather limited experience of undergraduate mathematics, if any, even of the level of basic calculus or linear algebra. I must say that I really see problems with such a possibility. I do not wish here, of course, to express any opinion that may be received as offensive or as a personal criticism by any individual. It is more the "system" allowing this to happen that I want to comment on.

A didactician with no personal direct experience of mathematics at a somewhat advanced level will in my opinion lack a global "vision of the mathematical landscape" that I see as crucial, some aspects of it will escape his or her expertise. I am not at all suggesting here that all didacticians of mathematics should have followed loads of graduate math courses or experienced highly specialized mathematics research. But to take a concrete example, a deep understanding of basic number systems is clearly facilitated when these are considered as steps on the road towards the real numbers, the basic context for elementary analysis.

The present context does not allow me here to enter into fine discussions about the mathematical background that I would hope didacticians to have experienced. In a certain way, as may be the case with the mathematical education of teachers, rather than a simple matter of "doing more math", it is a matter of doing more math that may prove to be significant in order to allow the development of a deep intuition of the mathematical objects one is bound to meet in didactical situations.

Paying attention to this aspect is clearly a good way of facilitating communication between mathematicians and didacticians, as well as helping to foster mutual respect and understanding, unquestionably a vital ingredient in my opinion.

ICMI at the Dawn of Its Second Century

In this final section I examine selected actions recently launched by ICMI that may offer ways of fostering the collaboration between mathematicians and didacticians, and making it more productive. I am not proposing these undertakings as representing a kind of "ideal future" for mathematics or for didactics, nor for their interconnection. But these may be considered as pointing to possible models for concrete joint efforts bringing together the two communities discussed in this paper.

A common feature of the three projects that I discuss below is that they have been launched jointly by ICMI and its mother organization IMU. They thus represent meeting grounds for mathematicians and didacticians as they are represented by these two bodies. It is appropriate from that perspective to go back to the time of the beginnings of the term of office of the first ICMI Executive Committee under the presidency of Hyman Bass. I have already alluded earlier in this paper to two previous events that had provoked not only tensions between ICMI and IMU as bodies, but also between the two communities of mathematicians and didacticians: the so-called Math War in the USA and the turmoil resulting from the setting up of the program of the section on Teaching and Popularization of mathematics at the 1998 ICM. To use the words of Artigue (2008) in her description of the resulting context, “tension was at its maximum” (p. 189). She also comments that when the 1999–2002 ICMI Executive started its term of office, the situation had evolved so badly that “[v]oices asking ICMI to take its independence from a mother institution that expressed such mistrust were becoming stronger and stronger” (p. 189). But she finally concludes:

> Retrospectively this crisis was beneficial. It obliged the ICMI EC to deeply reflect about the nature of ICMI and what we wanted ICMI to be. This led us to reaffirm the strength of the epistemological links between mathematics and mathematics education (…). At the same time, we were convinced that making these links productive needed combined efforts from IMU and ICMI; the relationships could not stay as they were. (p. 190)

Conscious and explicit efforts were thus made by the IMU and ICMI Executives to improve the situation. I have described in Hodgson (2008, 2009) some of these efforts, which started with the (re)establishment of regular contacts between the two ECs, and especially between the presidents and secretaries [-general], and eventually resulted in the mounting of joint IMU/ICMI projects. Consequently, “after certain periods of dormancy and at times profound distance” (Hodgson 2008, p. 200), the IMU/ICMI relations were entering a time of welcomed harmony and intense collaboration. Concrete examples of such collaboration are given in Hodgson (2009, p. 87).

It should be mentioned, *en passant*, that a stunning outcome of this reinvigorated relationship, totally unexpected at the time of the 1998 crisis, is the “dramatic and historic change in the governance of ICMI” (Hodgson 2009, p. 87) represented by the fact that since 2008, the election of its Executive occurs at its own General Assembly (such as the one held just prior to this congress), rather than at the IMU GA, as was the case earlier. Such a development is a strong evidence of the maturity not only of the field represented by ICMI, but also of the relationship of ICMI with the organization to which it owes its legal existence.[22] More comments on this quite extraordinary episode can be found in Hodgson (2009).

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[22] In that connection, the following comment made by IMU President László Lovász in his report to the 2010 IMU General Assembly may be of interest: “The IMU has a Commission, the ICMI, to

I now describe briefly three recent projects organized jointly by ICMI and IMU. I believe these suggest that concrete actions bringing together mathematicians and didacticians may contribute to resolve the issue of the mathematics/didactics interconnection. Additional information on these projects is to be found on the ICMI website.

The "Pipeline" Issue

Already in 2004, IMU approached ICMI, its education commission, expressing concerns in connection with a perceived decline in the numbers and quality of students choosing to pursue mathematics study at the university level and requesting the collaboration of ICMI to better understand this situation. The ensuing discussions pointed to another related phenomenon that needed to be investigated, namely the apparently inadequate supply of mathematically qualified students choosing to become mathematics teachers in the schools. IMU invited ICMI to partner in this undertaking, and take responsibility for its design.

Eventually the project (coined "Pipeline") was connected to, and became an extension of, the work of one of the Survey Teams for ICME-11, on the topic of "Recruitment, entrance and retention of students to university mathematical studies in different countries". It aimed at gathering data about different countries as well as promoting better understanding of the situation internationally. It was decided to focus on eight pilot countries for reasons of manageability (Australia, Finland, France, Korea, New Zealand, Portugal, UK, and USA), and to centre the study around four crucial transition points:

- From school to undergraduate program
- From undergraduate program to teacher education (and to school teaching)
- From undergraduate program to higher degrees in mathematics
- From higher degrees to the workforce

The final report of the Pipeline project was presented in a panel at the last International Congress of Mathematicians held in 2010 in Hyderabad, India. The resulting picture[23] is that there may not be a worldwide crisis in the numbers of mathematically gifted students, but that there is a crisis in some of the pilot countries. The numbers of such students in universities is susceptible to changes in school curricula and examination systems.

(Footnote 22 continued)
deal with math education. The [IMU] General Assembly in 2006 gave a larger degree of autonomy to this Commission, including separate elections for their officials. I would say that this did not loosen the connections between IMU and ICMI, to the contrary, I feel that we have developed an excellent working relationship." (Lovász 2010, p. 13).

[23] From ICMI quadrennial report of activities 2006–2009 submitted to the 2010 IMU General Assembly [cf. *Bulletin of the International Mathematical Union 58* (2010, p. 100)].

ICMI from Klein to Klein

It was at the first meeting of the 2007–2009 ICMI Executive Committee, under the presidency of Michèle Artigue, and in the context of a discussion about worthy projects that would bind the communities of mathematicians and didacticians, that the so-called Klein project was first mentioned. The ICMI EC saw it as a valuable undertaking to revisit the vision of ICMI first President, Felix Klein, in his milestone book *Elementary mathematics from an advanced standpoint*, published a century earlier and based on his lectures to secondary teachers. Klein's aim was on the one hand to help prospective and new teachers connect their university mathematics education with school mathematics and thus overcome the "double discontinuity" which they face when going from secondary school to university, and then back to school as a teacher (cf. Klein 1932, p. 1). But more generally Klein wanted to allow mathematics teachers to better appreciate the recent evolution in mathematics itself and make connections between the school mathematics curricula and research mathematics. This is in line with the view that a fundamental contribution of mathematicians to the reflections on teaching is by providing teachers with access to recent advances in mathematics and to conceptual clarifications (cf. Artigue 2010).

The reflections of the ICMI EC on this project were pursued in conjunction with the IMU EC and a Design Team responsible for the project was jointly appointed in 2008. The Klein project has already provoked a lot of very positive reactions from mathematicians, didacticians and teachers, and it is expected to have a triple output: a book simultaneously published in several languages, a resource DVD for teachers, and a wiki-based web-site continually updated and intended as a vehicle for the people who may wish to contribute to the project in an ongoing way.[24]

Capacity and Networking

The history of ICMI shows a long tradition of outreach initiatives with regard to developing countries. But this prime responsibility of our community has received a renewed attention recently. In her reviews of challenges now facing ICMI, Artigue (2008) stresses the importance, for the successful integration of colleagues from developing countries into the ICMI network, of developing new relationships between "centers and peripheries". She thus points to a necessary evolution from the traditional "North-South" model towards "more balanced views and relationships" (Artigue 2008, p. 195).

The Capacity and Networking Project (CANP) was developed by ICMI with this spirit in mind. It aims at enhancing mathematics education at all levels in developing countries by supporting the educational capacity of those responsible for the

[24] More information on the project and its evolution can be found at www.kleinproject.org.

preparation of mathematics teachers, and creating sustained and effective networks of teachers, mathematics educators and mathematicians in a given region. CANP was officially launched in 2011 jointly by IMU and ICMI, in conjunction with UNESCO. A prerequisite for the acceptability of a given proposal is some evidence of existing collaboration between local mathematicians and mathematics educators.

Each CANP program is based on a two-week workshop of about forty participants, half from the host country and half from regional neighbours. It is primarily aimed at mathematics teacher educators, but also includes mathematicians, researchers, policy-makers, and key teachers. Three CANP actions have already taken place or been announced: Mali (2011), Costa Rica (2012) and Cambodia (2013).

Conclusion

This lecture has centred on the specificity and complementarity of the contributions brought by mathematicians and didacticians of mathematics to the reflections on the teaching and learning of mathematics. Another more encompassing approach would be to consider the general framework of the sciences to which research in the didactics of mathematics is connected because of its interdisciplinary nature. The importance of "defining and strengthening the relations to the supporting sciences" is discussed in Blomhøj (2008), where emphasis is placed on the need for mathematics education research "to benefit from new developments in the supporting disciplines" (p. 173). In particular the author stresses that "[o]n a more political level the relationships to the supporting disciplines are very important for the integration of mathematics education research in academia and thereby for the institutionalisation of our research field" (Blomhøj 2008, p. 173). Mathematics appears of course as a fundamental *cas de figure* on that account.

The issue of the mathematics/didactics interconnection is clearly a very vast one and my focus in this talk was to look at it from the vantage point of the International Commission on Mathematical Instruction, through both its history and its current actions. In a survey paper aiming at encouraging mathematicians' participation to the ICME-10 congress, Bass and Hodgson (2004) have raised the question: "So how are mathematics and mathematics education, as domains of knowledge and as communities of practice, now linked, and what could be the most natural and productive kinds of connections?" Their comment was that "ICMI represents one historical, and still evolving, response to those questions at the international level" (p. 640). To borrow from the beautiful title of Artigue (2008), ICMI was, and is still there, at the interface between mathematics and mathematics education.

In his reaction to Kilpatrick's paper (2008) on the development of mathematics education as an academic field, Dorier (2008) mentions the multiple types of cooperation that mathematics education has developed with other academic fields "because the development of research shows that the complexity of the reality of education needs to be tackled from different viewpoints" (p. 45). Emphasizing the importance for mathematics education, amidst this diversity, "to put forward the specificities of its objects, methods, and epistemology" (p. 45) in comparison to other fields connected to educational issues, he notes the following:

> In that sense, the relation [of mathematics education] to mathematics is essential, and the role of ICMI is thus vital in order to maintain and develop in all its variety an academic field specific to mathematics education that maintains a privileged relation with the mathematical community at large. (p. 45)

But seeing as a risk that mathematics education may fail to develop as a fully-fledged autonomous academic domain and be absorbed in related fields, Dorier concludes that "[a] barrier against this possible dilution remains the attachment of mathematics education to mathematics that ICMI can guarantee while encouraging cooperative work with other academic fields connected to education" (p. 45). That describes in a very fitting way the framework I was proposing in this talk to reflect on the links, past and future, between mathematics and didactics and between the main communities that support these domains.

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Mathematics Education in the National Curriculum—with Some Reflections on Liberal Education

Lee Don-Hee

Abstract Mathematics has been recognized and justified to be placed in the prime core of the formal curriculum for general education. In this paper, however, some reflections are made on the national curriculum together with mathematics education in accordance with the tradition of "liberal education." Liberal education is education for liberal men. The basic education of liberal human being is the discipline of his rational powers and the cultivation of his intellect. It has sustained its meaning and value to be different from the vocational training for the purpose of earning one's living. But John Dewey differently contends that the vocational training may claim a pertinent candidate to the position playing a role in cultivating the human mind, the intellect (or intelligence). For Dewey, important is not the content of teaching but rather the intelligence in its operation.Intelligence is "equipped" with some properties that are functionally related to the properties of the problematic situation, which they take on the character of "method." A kind of mental process, "a methodic process," connecting problematic situations and resolved consequences is what Dewey qualified to be "reflective thinking," where the intelligence keeps itself alive and activating for its full operation. Then, we would have two different, but closely related tasks. One is (i) the self-habituation of methodic activity; and the other is (ii) the nurturing of children in methods. The curricular device is bound to gratify a variety of different needs and motives. No matter how worth studying mathematics may be, it can never be learnt unless the body of learning materials are so organized that students may cope with its degree of difficulty settled for the teaching purpose. Then contents must be appropriately selected and efficiently programmed on the part of learners. Learnability is prior to the academic loftiness at least in educational situations.

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Why Should We Teach Mathematics?

For the first nine school years of the elementary and secondary education in Korea, mathematics is one of the major subject-matters of which the national curriculum formally consists. In several partial amendments, mathematics has outdone other competing subjects in the official process of allocating the weekly teaching hours. In spite of the fact that mathematics fails to draw students' favor and popularity, it has been recognized and justified to be placed in the prime core of the formal curriculum for general education, both elementary and secondary.

Mathematical study itself has occupied integral part of the human civilization as well as intellectual life. As a matter of fact, mathematics has been taught as the core subject-matter in the history of school curriculum everywhere in any civilized part of the world. Its system of knowledge, together with its language and method, has been shared among the world intellectual communities more than any other discipline, probably more than any other human undertaking.

Now, however, I would like to raise an unexpected question: Why should we teach relevantly mathematics to all the young people at elementary and secondary levels of education? And, does it really deserve attention as a competitive power in curriculum development?

From the standpoint of social utility, among different points of view, there may be four reasons, at least, why we should teach mathematics in the school. First, to raise mathematical specialists; second, to meet needs of mathematical knowledge required for the advanced level of professional services; third, to promote problem-solving abilities, namely those of logical or formal reasoning; and fourth, to help people to be familiar with basic mathematical knowledge necessitated for the ordinary daily life.

It may be realistically the case that there must be those who devote themselves to study the highly advanced and outstanding mathematics in any civilized society; that mathematical knowledge must be applied to a variety of professional services; that mathematics by its own nature shows us how to make our thinking logically valid and how to solve efficiently complicated problems encountered in our daily life; and that even basic rules or ideas of mathematics help us to see the complexity of the world in organized forms by virtue of its symbolic power.

But it seems to be necessitated to recognize that only a limited number of mathematicians and professionals are in need of training at higher levels, some basic parts of which are already embedded in the national curriculum for the upper or even lower secondary education. A greater part of students say that mathematics is too unintelligible for them to learn, and that it gives them toilsome and boring time in the class room situations. You cannot teach students anything if they are not able, and not willing, to learn it properly. And your instructional device cannot work in teaching mathematics if they extremely hate and stubbornly refuse to learn it at their own will.

In order to see why we should teach mathematics in the school, and what kinds of mathematization should be experienced, I, as a student of philosophy of education and, to be sure, a rank outsider, here would like to say something, perhaps what an ordinary consumer of education experiences with reflections on the national curriculum together with mathematics education.

Here, I would like to make some reflections in accordance with the tradition of "liberal education." I believe that the question should be answered in terms of values and implications of liberal education. For it represents, in its very nature of meaning, authentic communications between the human mind and the cultural tradition. But I do not try to make mathematics education fitted into an orthodox admittedly dominant in its tradition, but rather to discuss about how we should understand the idea of liberal education in its consideration with teaching mathematics.

In the Tradition of Liberal Education

Liberal education is education for liberal men. Originally, as Leo Strauss mentioned, a liberal man was a man who possessed a privilege to behave in a manner becoming a free man, as distinguished from a slave (Strauss 1968, p. 10). A slave is also a human being who lives yet for another human being, his master; he has no life of his own. The master, on the other hand, has all his time for himself, that is, for the pursuits becoming him in the world, with its meaning, of his social and intellectual life.

Nowadays, in the democratic society, however, we may say that a liberal man is a man, a rational being, who is to live under his own will, not other's. By education, one becomes, and maintains oneself, a liberal man in the genuine sense. The basic education of liberal human being is the discipline of his rational powers and the cultivation of his intellect. Historically, it is believed that this discipline can be achieved by the liberal arts, basically the communicational arts, namely reading, speaking, writing, listening, reckoning, and reasoning. The three R's (reading, writing, and reckoning), which always signified the formal discipline, are qualified for the essence of liberal or general education.

In the tradition of liberal education, numeracy, together with literacy, has been integral part of human abilities for the societal life civilized more or less so as to engage in liberal education. Plato especially points out that the mathematical studies develop the soul in two ways: In the first place, they provoke reflection and bring out all the contradictions that lie hid in ordinary opinions based on mere sense-knowledge; in the second place, they take him part of the road towards the good which is the goal of all learning and all life (Boyd and King 1975, pp. 34–35).

In his master-work, the Republic, Plato discusses an educational scheme to show how the ideal State might be created out of programs cultivating the mind of the youth. Up to seventeen or eighteen, the children, assumed to be the future rulers, were all to devote themselves to gymnastics and music. After 2 years of physical training, the youth who had proved themselves capable of more advanced studies were to work at the mathematical sciences–arithmetic, geometry, astronomy, and harmonics (the mathematical theory of music) from twenty to thirty. Finally a select group who had shown distinction both of mind and character throughout the whole course of their previous training were to spend 5 years in the study of dialect (or philosophy), the science of the good (ideas), before taking their place in the ranks of the "guardians."

There may be at least two different conceptions of liberal education: one is intellectualistic while the other is pragmatic. Among others I want to mention here Mortimer J. Adler as an intellectualist who stands against a pragmatist John Dewey. The idea of liberal education itself was genetically aristocratic, for the truly free man who can live in a manner becoming a free man is the man of leisure. But liberal education is not simply entitled to a kind of program for the free man in a political sense, but also understood differently so as to mention a certain principle overriding activities cultivating the human intelligence and creativity.

For liberal education, Adler maintains that the human reason may be at first trained in its proper operations by the communicational arts, since man is a social animal as well as a rational one and his intellectual life is lived in a community which can exist only through the communication of men. The intellect cannot be accomplished merely by the three R's, but, in addition, through furnishing it with knowledge and wisdom, acquainting it with truth, and giving it a mastery of ideas. At this point, he suggests that the other basic feature of liberal education appears, namely the great books, that is, the master productions in all fields, philosophy, science, history, and belles-lettres. These constitute the cultural tradition by which the intellects of each generation must first be cultivated.

Mortimer J. Adler says:

> … If there is philosophical wisdom as well as scientific knowledge, if the former consists on insights and ideas that change little from time to time, and if even the latter has many abiding concepts and a relatively constant method, lf the great works of literature as well as of philosophy touch upon the permanent moral problems of mankind and express the universal convictions of men involved in moral conflict–if these things are so, then the great books of ancient and medieval, as well as modern, times are repository of knowledge and wisdom, a tradition of culture which must initiate each new generation. (Adler 1939)

In Adler's conception, liberal education is a kind of program which provides the youth with communicational arts (reading, writing, speaking, reckoning etc.), and thereafter with the intellectual mediator for the constant intercourse between them and the greatest minds in the cultural tradition. Liberal education is learning for its own sake or for the sake of all those self-rewarding activities which include the political, aesthetic, and speculative. It differentiates itself from vocational training which no one should have to take without compensation, and which is just preparatory to work for the sake of earning. (Adler 1951)

Intelligence, Method, and Methodic

Now, we may ask again "what for liberal education?" It is education to cultivate the human intellect, and thus to liberate the human mind. It has sustained its meaning and value to be different from the vocational training which is confined to learning skills for the purpose of earning one's living. Traditionally, it is literate education of a certain kind: some sort of education in letters or through letters, as tools for developing the intellect. It has been conceived to be a kind of program for teaching the youth in subjects, namely liberal arts, and studying the great books reminding oneself of human excellence, of human greatness.

As John Dewey differently contends, however, that there seems to be no relevant reason why we are to confine ourselves to literate education for teaching in so-called liberal arts. Even the vocational training may claim a pertinent candidate to the position playing a role in cultivating the human mind, the intellect (or intelligence). For this qualification, of course, vocational training is also availed of the capacity as efficiently a tool to be utilized as the traditional program in liberal arts.

Dewey says as follows:

> … Instead of trying to split schools into two kinds, one of a trade type for children whom it is assumed are to be employees and one of a liberal type for the children of the well-to-do, it will aim at such a reorganization of existing schools as will give all pupils a genuine respect for useful work, an ability to render service, and a contempt for social parasites whether they are called tramps or leaders of 'society.'…
>
> … It will indeed make much of developing motor and skills, but not of a routine or automatic type. It will rather utilize active and manual pursuits as the means of developing constructive, intentive and creative power of mind… the individual may be able to make his own choices and his own adjustments, and be master, so far as in him lies, of his own economic fate… So far as method is concerned, such a conception of industrial education will prize freedom more than docility; initiative more than automatic skills; insight and understanding more than capacity to recite lessons or to execute tasks under the direction of others… (Dewey 1917)

For Dewey, what must be important is not whether the content of teaching consists of letters or non-letters for developing the mind, indeed the mind of the liberal man, but rather whether "the human intelligence" can work properly in its operation. Intelligence can work to solve the problem situation, trifling or serious, that we encounter in our daily life, such as conflict with neighbors, discord within the family, crises of confidence in business and the like. We need a social intelligence to solve the problem situation, such as deep economic depression, state security risk, vicious inflationary spiral, chronic rebellion, and the like. Academically, a variety of disciplines, theoretical or practical, are products of intelligence managing to work out of the problem situation where academics struggle with a systematic body of highly complicated ideas and matters. Mathematics is a structure of resolutions painstaking with forms of mathematical intelligence.

Intelligence does not operate vacuously: It is "equipped" with some properties that are functionally related to the properties of the problematic situation. When these properties are systematically distinguished, formulated and organized so as to apply to the problematic situation, they take on the character of "method." Method then is not outside of or divorced from material. Method may be philosophical, literary, scientific, mathematical, or technological. Dewey writes, "The fact that the material of a science is organized is evidence that it has already been subjected to intelligence; it has been methodized, so to say" (Dewey 1916, p. 165). Method then is a logical description of intelligence in operation. Indeed, intelligence and method are synonymous.

In this consideration, mathematics as a discipline or a subject-matter may be admittedly said to be a sort of human product subjected to intelligence, thus its material content methodized in such a way that it has characteristically differentiate itself in its properties from other modes of human works.

In the educational discourses, we often refer to “intelligence” as the prime human ability among others to be developed in teaching or training programs. “Habit” is also referred to as objective pertinent to educational activities. But “intelligence” or “abilities” mostly includes those which are characteristically cognitive and self-directive, whereas “habits” mostly represents those which mainly pertain to physical and routine actions. This is the reason why the vocational training makes itself mistakenly different in its mode of learning from the traditional conception of liberal education.

A general theory that accounts for habits and intelligence and their various relationships becomes a matter of our concern. The question here, of course, is what kind of action is both habitual and intelligent: And the problem is to distinguish the appropriate kind of situations for the use of the terms, “habitual” and “intelligent,” respectively, to be employed.

If method is a logical description of intelligence in operation, and indeed intelligence and method are synonymous, then we may ask: Could methods be habituated? Could they become habitual? These questions have to do neither with the possibility of forming the habit of adopting methods nor with the evolution of a method into habit. Rather, these questions have to do with the possibility of habituating “methodic” activities. But the habit of methodic activity could still be understood as a habit of translating methods into the pursuit of an end. This sense of “methodic habit” implies a habit of reproduction. The intelligence that has served methods is secondary to the intelligence functioning in methodic activities. For the former intelligence is not activating while the latter intelligence is. Furthermore, the powers that methods may execute are not necessarily powers of intelligence, nor are they human powers. What we actually look for is the habit of methodizing or controlling problem-situations, of pursuing methods, and of utilizing methodized patterns in the pursuit of an end, that is, a methodic habit.

A kind of mental process connecting problematic situations and resolved consequences is what Dewey qualified to be the process of “reflective thinking,” where the intelligence keeps itself alive and activating for its full operation.

Dewey’s conception of reflective thinking is in somewhat temporal terms, different from a methodological account featuring formal properties. Dewey is not providing a formula, but a temporal account of the activity in which the formula does its work. Dewey’s theory of reflective thinking should therefore not be understood to rule out the adoption of ready-made methodic formulae—those which have been already methodized. Indeed, he cautioned us that we ignore these at our peril. He argues that things as methodized represent the office of intelligence, in projection, in pursuit, and in the control of new experience. In short, methodic habits are valued habits of intelligence. They are embodied in, or are cases of, intelligence.

Dewey claims that reflective thinking should be an educational aim since it carries with itself qualities significant as educational values (1933, p. 17). In the first place, it emancipates us from merely impulsive and merely routine activity. It enables us to direct our activities with foresight and to plan according to ends-in-view, or purposes. Secondly, it enables us to develop and arrange "artificial" signs to remind us by representing in advance not only "existential" consequences but also ways of securing and avoiding them. Thirdly, since an object to which we react is not a mere thing, but rather a thing having a definite significance or meaning, reflective thinking enriches the object with further meanings as we compare a thing or event as it was before with what it is after. Intelligent mastery over the object is obtained through thinking.

Here we would have two different, but closely related tasks. One is (i) the self-habituation of methodic activity; and the other is (ii) the nurturing of children in methods. The first is based on the assumption that methodic activity, reflective thinking or the problem-solving process, is not only a methodological mechanism for teaching knowledge of substances (or subject-matters) emerging in the system of educational values, but it is also something to teach. This means that methodic activity is not merely a means serving in the pursuit of various educational objectives, but also is itself a candidate for being an educational objective.

But the second task should not be understood as the fact that the educative process is methodic because it is a process of applying the method supposedly common to all disciplines. That is, if all disciplines are cases of method then they should display common properties—common formal properties. The ends of various disciplines differ in form and substance; the means differ in the force of applicability of their theories. But each is an affair of controlled means and ends. To say again, mathematics is a discipline of mathematical method as well as mathematical intelligence.

Methods are symbolic expressions of what is performed in the process of controlled activity. They represent among other matters the material involved. But substantial materials, for example, problems, issues, situations, events, or reports, are not always of a single type in their mode of placement in the means-end relating.

The objective common to all sciences, including mathematics, is assertion making, conclusion drawing, proposition forming, and possibly theory structuring. Each science is a kind of knowledge forging, hypothesis testing, prediction constructing, and so on. Thus, we have physiological and biological knowledge, economic and astronomical facts, geological and biological hypotheses, historical and anthropological reports, and philosophical and mathematical arguments. We sometimes call all these bodies of knowledge–meaning, of course, the fruits of the inquiries of these sciences. All are equally sciences and human achievements according to scientific method. But each is different in the sense that each is proceeding with different problems, materials, concepts, and terminology. Thus it seems unlikely that one can learn a method in a specific type of problematic situation without experience in the conduct of dealing in and with that situation.

The implication that the educative process is a process of nurturing children in methods leads to a theoretical corollary that the object of educational research or inquiry may be found in the universe of methods. This is to claim that if there is a universe of discourse, a block of conceptual equipment which more adequately deals with the tasks of educators, it must be the universe of discourse about methods or methodic activities in which intelligence is to be embedded. Mathematics is also methodic in its nature.

Conditions of Mathematics for Liberal Education in the National Curriculum

Last February, I found an interesting column in one of the issues of the daily newspaper Joong Ang Ilbo that is published in Seoul. It was titled "No Easy and Interesting Math Learning" written by a professor of mathematics, named Yong-Jin Song of In-Ha University. To partly translate into English as follows:

> … Mathematics for today has become a systematic discipline which has grown up sophisticated by virtue of great geniuses intelligence in the human history, and thus it must be difficult in its nature for ordinary people to learn. You cannot make yourself master of its hard and tough contents without taking a well-planned course of learning. Mathematics is the supreme product of human intellect such that even its fundamental level requests you to undergo a well-organized training which is somewhat intensive to some extent. To be sure, there exists no mathematics that is easy and, at the same time, interesting; but perhaps rather there may exist such a kind of mathematics that is both difficult and interesting. Many a thing is popular and interesting because of its difficulty: playing a game of go, golf, soccer, computer or the like. (Song 2012)

Professor Song, however, does not mention what kind of mathematics to be taught. Of course, he may presuppose the possibility such that its contents be organized in accordance with the condition of learners describable in terms of age, experience, motivation and cultural orientation. Nevertheless, he seems to assume that the mathematics may be enjoyed exclusively by those who are intellectually equivalent to appreciate of its value. It seems to me that he assumes there exist "the (one and only one in kind) mathematics" which schools should teach to all young people.

If he believes, as Karl Mannheim opposes the possibility of a sociology of mathematics, and as Pythagoreans and Platonists believe, that mathematical truths are eternal objects, not culturally relative (Restivo and Collins 1982), then he may be right in the assertion that we should not be concerned with the degree of difficulty in mathematics education.

But, as Oswald Spengler says that there is no mathematic but only mathematics, we define mathematics as methodic products, we discussed earlier, of intelligence in operation for the struggle with problematic situations. Mathematics is a particular mode of experience, distinguishable from other disciplines and arts, and the character of mathematical inquiries vary with cultural toils, and with problematic motives and interests. There may be quite a few different ways of inclusion and exclusion in organizing contents for different orientations.

And if learning values are appreciated, and academic needs are gratified by the experience of mathematical difficulty, the very difficulty of "the mathematics as such," then it confines to a very limited number of people who can intrinsically enjoy the subject-matter, what is called "mathematics," just as a very limited number of few people, the professional or ardent players, enjoy the game, go, golf, and soccer. The difficulty provides no legitimate ground that mathematics may outdo other competing subjects in the competitive process of allocating the weekly teaching hours, and that it can claim to be the prime core of the national curriculum for general education.

We cultivate the human mind (intellect or intelligence, whatever) by the instrumentality of mathematics in association with other teaching-learning programs, that take care of, and improve, the native faculties of the mind. Therefore, mathematics is to deserve a core subject-matter among those worthwhile to teach for liberal education, the finished product of which is a cultured human being. Mathematics, which cultivates and thus liberates the human mind, consists of intrinsic values, that is, those which are good in itself. We do not necessarily enforce it to demonstrate any practical utilities, that is, extrinsic values which are instrumentally good for something other than itself. Even its applied ramifications may be so organized as to materialize their cultivating and liberating powers to the maximum extend. Even in non-academic activities where mathematics is subsidiary, they must be planned to methodically activate the human potentials of creativity and productivity.

Probably, of course, an outstanding group in mathematics can enjoy its intrinsic value at the highly advanced level. And the well-trained professional proficiency in teaching may open up new path into a more sophisticated realm as a benefit to ambitious students. To them mathematics becomes not any more a painstaking burden, but rather an enjoyable game.

The curricular device is bound to gratify a variety of different needs and motives. No matter how worth studying mathematics may be, it can never be learnt unless the body of learning materials are so organized that students may cope with its degree of difficulty settled for the teaching purpose. Then contents must be appropriately selected and efficiently programmed on the part of learners. Learnability is prior to the academic loftiness in educational situations. You cannot enjoy what you are not learnable. The variety may avail with us widely open learning opportunities where many a different mathematical need may be gratified.

In sofar as mathematical education is concerned, we may justifiably say that learning opportunity in its genuine sense be available to the learners, if and only if it is not the case that its course of study is too unequivalent for the students to carry out in the regular school activities. Especially, it is true of the national curriculum system which is assumed to be compulsory to all youngsters.

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Quality Teaching of Mathematical Modelling: What Do We Know, What Can We Do?

Werner Blum

Introduction

The topic of this paper is mathematical modelling or—as it is often, more broadly, called—*applications and modelling*. This has been an important topic in mathematics education during the last few decades, beginning in particular with Henry Pollak's survey lecture (Pollak 1979) at ICME-3, Karlsruhe 1976 (my first ICME). By using the term "applications and modelling", both the products and the processes in the interplay between the real world and mathematics are addressed. In this paper, I will try to summarize some important aspects, in particular, concerning the *teaching* of applications and modelling. For obvious reasons, I have to restrict myself and hence omit some important aspects, such as gender issues or the question of how to embed applications and modelling in curricula and lessons. My paper is mainly a *survey*, only occasionally I can go into depth. I will concentrate on the *secondary school* level. I hope it will become clear that we have made considerable *progress* in the field during the last few decades, both theoretically and empirically, although still a lot remains to be done. For those who would like to find more on this topic I would refer to ICMI Study 14 on Modelling and Applications in Mathematics Education (Blum et al. 2007) where one can also find a short history of the field. Further, I would refer to the Proceedings of the ICTMA conference series (the International Conferences on the Teaching of Mathematical Modelling and Applications), held biennially since 1983. One can see how dynamically the filed develops by only looking at the number of papers in these books (see the last two Volumes: Kaiser et al. 2011, and Stillman et al. 2013).

In this paper, I will switch between theoretical aspects (Parts 2 and 4) and empirical aspects (Parts 3, 5–7). Part 8 is on teacher education, and I will start and close with concrete examples (Parts 1 and 9).

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Two Introductory Real World Examples

I work and live in Kassel, the city where, every five years, the "documentas" take place, the world's most important exhibitions for contemporary art (in 2012 with 850.000 visitors). Each documenta leaves some of its exhibits in the city. One of those is Claes Oldenburg's oversized pick-axe from documenta-7, 1982 (see Fig. 1).

The story that Oldenburg invented and Kassel people like to continue to tell is that Hercules, the landmark of Kassel (see Fig. 2), has thrown this pick-axe from his place, in the mountain park Wilhelmshöhe above Kassel, to the Fulda river. I will come back to this story in the final part of my paper.

Fig. 1 Oldenburg's pick-axe in Kassel

Fig. 2 The Kassel Hercules

The first question is: *How tall would a giant have to be for this pick-axe to fit to him? Would it fit to the Kassel Hercules himself?*

Following Pollak's famous characterization of modelling "Here is a situation—think about it" (Pollak 1969), we begin with comparing the pick-axe and a normal person. Using proportionality, we find that this pick-axe is about 13 m long. A normal pick-axe measures about 1 m. So, using again a proportional model, we find that a suitable giant would be about 25 m tall or, better perhaps, something between 20 and 30 m. The Kassel Hercules measures only 9 m, so he seems a bit too small for this pick-axe, unfortunately.

$$\boxed{\begin{array}{l}\mathbf{x : \ 1.80 \approx 13 : \ 1} \\ \mathbf{x \approx 23.40}\end{array}}$$

A second example from documenta in Kassel: During documenta everything is more expensive in downtown Kassel. A Hercules T-shirt, for instance, as a Kassel souvenir, costs 15.99 € downtown, whereas in the shopping mall dez which is not far away, the same T-shirt costs only 12.99 €. The second question is: *Is it worthwhile to drive to dez in order to buy this T-shirt there?*

We will solve this problem in several steps (the same steps that we have applied also in the pick-axe example without noticing it).

Step 1: We construct a mental model of the situation (Fig. 3).
Step 2: We simplify and structure this mental model by assuming that we go by car, that our car consumes 10 l/100 km in the city, that the gas costs 1.599 €/l and that the distance we have to drive from downtown to the mall is 5 km.

Fig. 3 Mental map of the situation

Step 3: We construct a suitable mathematical model by mathematizing these concepts and relations:

C-downtown = 15.99 €
C-dez = 12.99 € + 2 · d · a · b, d: distance, a: consumption, b: gas price
C-dez < C-downtown?

Step 4: We work mathematically by calculating **C-dez ≈ 12.99 € + 1.60 € = 14.59 €** and by comparing: Yes, C-dez < C-downtown!
Step 5: We interpret this mathematical result in the real world: It is indeed by 1.40 € cheaper to drive to the shopping mall!
Step 6: We validate our result: Does it really make sense to drive 10 km in order to save 1.40 €? What about using this time instead to see more of Kassel's beauties? What about the risk of an accident or the air pollution caused by our trip? So perhaps we will refine our model and start again, or we will simply decide against that simple mathematical solution.
Step 7: In the end, we write down the whole solution.

This seven-step-process is one of the many schemas for the modelling process (Fig. 4, see Blum and Leiß 2007a).

Here are a few more such schemas (Fig. 5).

All these schemas have their specific strength and weaknesses, depending on the respective purposes. For cognitive analyses, this seven-step-model seems particularly helpful. It is a blend of models from applied mathematics (Pollak 1979; Burghes 1986), linguistics (Kintsch and Greeno 1985) and cognitive psychology (Staub and Reusser 1995).

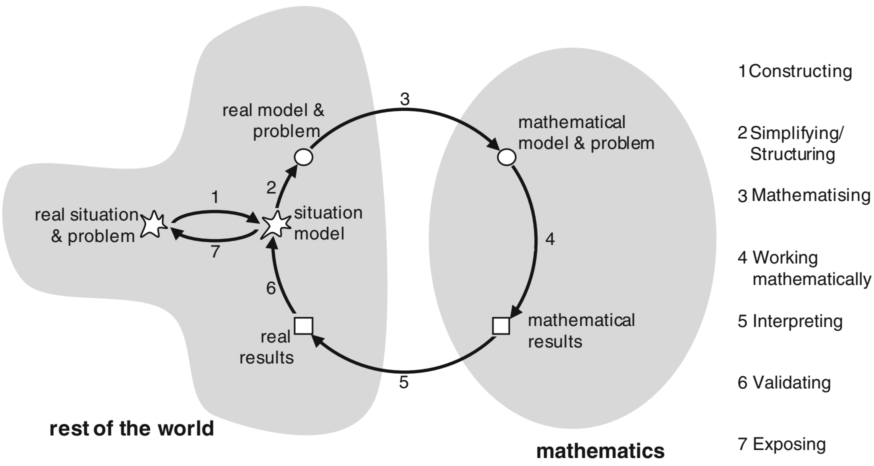

Fig. 4 Seven step modelling schema

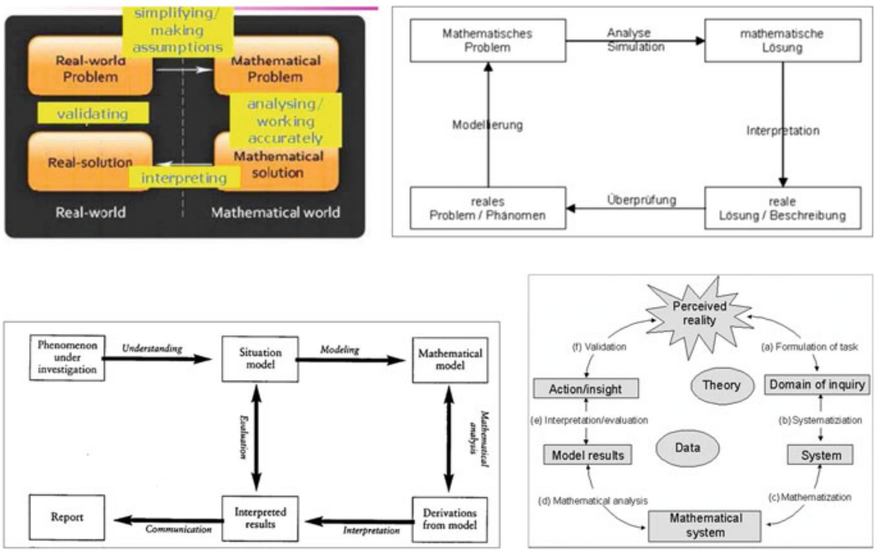

Fig. 5 Modelling schemas

Mathematical Modelling Competency

Here comes some theory. The topic of this paper is the teaching and learning of mathematics in the context of relations between mathematic and the extra-mathematical world. The latter is often called reality or the real world, or better, in the words of Pollak (1979), the "rest of the world", including nature, culture, society, or everyday life. The process of solving real world problems by means of mathematics can, from a cognitive point of view, be described by the schema from Fig. 4. If need be, one has to go round the loop several times. A key concept here is the concept of a model. A *mathematical model* is a deliberately simplified and formalized image of some part of the real world, formally speaking: a triple (D, M, f) consisting of a domain D of the real world, a subset M of the mathematical world and a mapping from D to M (Niss et al. 2007). Among the purposes of models are not only describing and explaining ("descriptive models") but also predicting and even creating parts of the real world ("normative models").

In the language of competencies according to Niss and colleagues (see Niss 2003), the ability to carry out those steps corresponds to certain *competencies* or *sub-competencies* such as understanding a given real world situation or interpreting mathematical results in relation to a situation (Blomhøj and Jensen 2007; Maaß 2006; Kaiser 2007; Turner et al. 2013). Cognitively speaking, an individual's competency is his/her ability to carry out certain actions in a well-aimed way. *Modelling competency* in a comprehensive sense means the ability to construct and to use or apply mathematical models by carrying out appropriate steps as well as to analyse or to compare given models (Blum et al. 2007). It is this comprehensive idea of modelling that will be used in the following.

The Niss competencies are also the conceptual basis for the PISA study and for the heart of PISA, *mathematical literacy* (see, e.g., OECD 2013, p. 23 ff). In large parts, PISA items require some modelling in a broad sense. An important source for the PISA philosophy was Hans Freudenthal's view of "mathematical concepts, structures and ideas as tools to organise the phenomena of the physical, social and mental world" (Freudenthal 1983). It is an open question whether this spirit of PISA will be preserved also in future PISA cycles.

Students' Modelling Activities

Mathematical modelling is a *cognitively demanding* activity since several competencies involved, also non-mathematical ones, extra-mathematical knowledge is required, mathematical knowledge and, in particular for translations, conceptual ideas (in German: "Grundvorstellungen") are necessary (e.g., in the examples in part 1, ideas about proportional functions), and appropriate beliefs and attitude are required, especially for more complex modelling activities.

These cognitive demands are responsible for *empirical difficulty*. Modelling is indeed rather difficult for students (see, for instance, Houston and Neill 2003, or Frejd and Ärlebäck 2011). Figure 6 shows the PISA task "Rock Concert".

The correct solution is C. In the OECD, only 26 % of all 15-year-olds have solved this task correctly, in Finland, one of the top performing countries, only 37 %, and in Korea, another top performing country, even only 21 %. The PISA Mathematics Expert Group has shown that the empirical difficulty of PISA

Fig. 6 PISA task "Rock Concert"

ROCK CONCERT

For a rock concert a rectangular field of size 100 m by 50 m was reserved for the audience. The concert was completely sold out and the field was full with all the fans standing.

Which one of the following is likely to be the best estimate of the total number of people attending the concert?

A 2 000
B 5 000
C 20 000
D 50 000
E 100 000

mathematics tasks can indeed be substantially explained by the competencies needed to solve these tasks (see Turner et al. 2013).

Several studies have shown that each step in the modelling process (see Fig. 4) is a potential cognitive barrier for students, a potential "blockage" or "red flag situation" (Goos 2002; Galbraith and Stillman 2006; Stillman 2011). "The weakest link in their modelling chain will set the limits on what they can do" (Treilibs et al. 1980).

Here are some remarks to *step 1* "Understanding the situation and constructing a situation model". Many students get stuck already here. This is not only or even not primarily a cognitive deficiency. For, many students around the world have learned, as part of the hidden curriculum, that they can survive without the effort of careful reading and understanding given contextual tasks. Instead, they successfully follow a substitute strategy for word problems: "*Ignore the context, just extract all data from the text and calculate something according to a familiar schema*" (see, e.g., Nesher 1980; Baruk 1985; Schoenfeld 1991; Lave 1992; Reusser and Stebler 1997; Verschaffel et al. 2000; Xin et al. 2007; de Bock, Verschaffel et al. 2010). Schoenfeld and Verschaffel speak of the "suspension of sense-making" when playing the "word problem game". This strategy even becomes more popular with age, and in the school context it may indeed make a lot of sense to follow this strategy in order to pass tests and to survive. This is empirically well documented, in very many countries. Here is a well-known example (Verschaffel et al. 2000):

450 soldiers must be bussed to their training site. Each army bus can hold 36 soldiers. How many busses are needed?

Popular answers are "12 busses remainder 18" or "12.5 busses". Another example of a calculation without imagining the situation clearly is:

An orchestra needs 40 min for Beethoven's 6th symphony. How long will it take for Beethoven's 9th symphony?

The popular answer is 60 min. In the PISA task "Rock Concert" (see Fig. 6), the by far most attractive distractor (49 %) was no. 2, the one that follows exactly the substitute strategy: $50 \cdot 100 = 5{,}000$.

Step 2 "Simplifying and structuring" is a source of difficulties as well. In particular, learners are afraid of making *assumptions* by themselves.

Step 6 "Validating" is mostly not present at all in students' solutions. Here (Fig. 7) is a solution of the pick-axe task.

The answer 254.84 m for the giant's height is, first, ridiculously accurate (rounding off is a rare event in mathematics classrooms) and, second, obviously much too big. However, students normally do not validate their solutions, it seems to be part of the "contract didactique": Checking the correctness and suitability of a solution is exclusively the teacher' responsibility!

I would like to mention a few other important empirical results concerning students' dealing with modelling tasks. Several studies have shown (Matos and Carreira 1997; Leiß 2007; Borromeo Ferri 2011; Schukajlow 2011; Sol et al. 2011): If students are dealing with modelling tasks independently, the process is normally *non-linear* according to one of those ideal-typical loops but rather characterized by jumps forth and back, by omissions or mini-loops. Borromeo Ferri (2007) speaks of

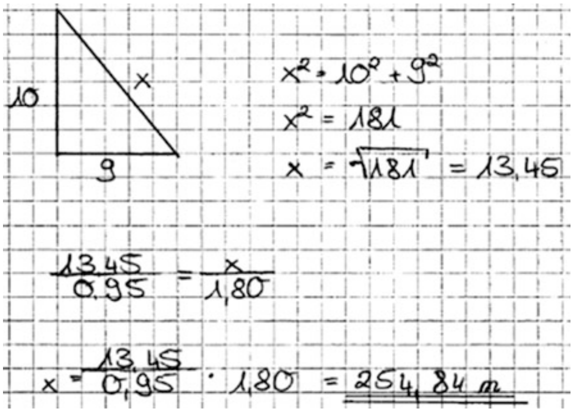

Fig. 7 A student solution of the pick-axe task

"individual modelling routes" which are determined by individual knowledge and preferences such as individual thinking styles.

Another well-documented observation is that students normally do not have *strategies* available for solving real world problems. More generally, students usually do not reflect upon their activities and, closely related to that, are not able to *transfer* their knowledge and skills from one context or task to a different context or task, even if there are structural similarities. For instance, in one of our projects, grade 9 students dealt in a lesson with the "Filling up" task (see Blum and Leiß 2006) which is quite analogous to the T-Shirt task from part 1. The question is whether it is worthwhile for a certain Mrs. Stone to drive from her hometown Trier across the nearby border of Luxemburg, where the gas is cheaper, in order to fill up her car there. In the following test, the students had to solve very similar tasks, among others whether it is worthwhile to drive to a nearby strawberry field in order to pick the berries for a cake instead of buying them in a supermarket, or whether it is worthwhile to use cloth-diapers instead of disposable ones. For many students, these were totally new challenges, now about strawberries and diapers instead of cars. The PISA study also demonstrates every three years how difficult it is for 15-year-olds to transfer their school knowledge to real world problem situations.

The phenomena just described are, as is well-known, special instances of *situated cognition*, or in the words of Jürgen Baumert: Every learning topic carries with it the "indices" referring to its learning context. This is particularly relevant for learning in the field of relations between the real world and mathematics (DeCorte et al. 1996; Niss 1999). Actually, when we report on empirical results about "modelling competency" we have to write this construct with several indices, especially referring to the mathematical topics and the extra-mathematical contexts involved. The question is even: Is there a "general modelling competency" at all? Much more research is necessary into how and how far the desired transfer can be achieved. I will come back to this aspect in parts 5 and 7.

Aims and Perspectives of Modelling

We come back to theoretical aspects of modelling. Modelling is a cognitively demanding activity, so why should learners have to deal with such activities? Why is it not sufficient to learn pure mathematics in order to achieve the aims of mathematics as a school subject? Mathematics is, as we know, a compulsory subject at school for the following reasons (see, e.g., Niss 1996): Mathematics as

- a powerful tool for better understanding and mastering present or future real world situations,
- a tool to develop general mathematical competencies,
- an important part of culture and society, and a world of its own.

The basis for that are general educational goals such as the ability to take part in social life as an independent and responsible citizen.

On this background, we can distinguish between four groups of *justifications* for the inclusion of applications and modelling in curricula and everyday teaching (see, e.g., Blum and Niss 1991; Blum 2011):

1. "*pragmatic*" justification: In order to understand and master real world situations, suitable applications and modelling examples have to be explicitly treated; we cannot expect any transfer from intra-mathematical activities.
2. "*formative*" justification: Competencies can be advanced also by engaging in modelling activities; in particular, modelling competency can only be advanced in this way, and argumentation competency can be advanced by "reality-related proofs" (Blum 1998).
3. "*cultural*" justification: Relations to the extra-mathematical world are indispensable for an adequate picture of mathematics as a science in a comprehensive sense.
4. "*psychological*" justification: Real world examples may contribute to raise students' interest in mathematics, to motivate or structure mathematical content, to better understand it and to retain it longer.

We can see a certain *duality* here (Niss et al. 2007): Whereas the first aspect deals with mathematics as an aid for the real world, the other three aspects deal with the opposite direction, the real world as an aid for mathematics, in a broad sense. Instead of "justifications for the inclusion of applications and modelling" we could also say "*aims* of the teaching of applications and modelling".

In order to advance those aims, suitable examples are needed. There is a broad spectrum of real world examples, from small dressed-up word problems to authentic modelling problems or projects that require days or weeks. The justifications or aims just mentioned require certain specific *types of examples*:

- "*pragmatic*": concrete authentic examples (from shopping, newspapers, taxes, traffic flow, wind park planning, air fare calculation, …);
- "*formative*": cognitively rich examples, accompanied by meta-cognitive activities;

- "*cultural*": either authentic examples that show students how strongly mathematics shapes the world (sometimes hidden and invisible, embedded in technology—the famous relevance paradox, see e.g. Niss 1999) or epistemologically rich examples that shed some light on mathematics as a science (including ethno-mathematical examples); in both cases, the role of mathematics and its relations to the real world must be made more conscious;
- "*psychological*": either interesting examples for motivation or illustration purposes, to make mathematics better marketable for students (these examples might quite well be dressed-up or whimsy problems, it is only a matter of honesty), or mathematically rich examples that serve the purpose to make certain mathematical topics better comprehensible.

So, examples are not good or bad per se, it depends on their *purpose*.

It was Gabriele Kaiser's idea, together with colleagues (see Kaiser et al. 2006), to distinguish between various *perspectives* of modelling. On the basis of what I have just presented, I have conceptualized the notion of "perspective" a bit more formally, as a pair (aim | suitable examples), with a slightly different terminology. So we can distinguish between six perspectives.

- (pragmatic | authentic) → "*applied* modelling" (Burghes, Haines, Kaiser, and others; particularly rooted in the Anglo-Saxon tradition)
- (formative | cognitively rich) → "*educational* modelling" (Burkhardt/Swan, Blomhøj, and others)
- (cultural with an emancipatory intention | authentic) → "*socio-critical* modelling" (Keitel/Jablonka, Skovsmose, Julie, Barbosa, and others)
- (cultural concerning mathematics | epistemologically rich) → "*epistemological* modelling" (d'Ambrosio, Garcia, Bosch, and others; more rooted in the Romanic tradition)
- (psychological with marketing intention | motivating) → "*pedagogical* modelling" (by far the most important aspect in school)
- (psychological | mathematically rich) → "*conceptual* modelling" (Freudenthal, de Lange, Gravemeijer, and others)

For each perspective, there is a certain model of the modelling process that is best suitable for that purpose. For instance, for applied modelling, a four step model "Mathematising → Math. Working → Interpreting → Validating" seems most appropriate. There is no space here to elaborate more on this. In effect, it is more appropriate to conceptualise a "perspective" as a tripel (aim | examples | cycle).

All these perspectives also contribute to the question of *sense-making*. Here, I mean by the "sense" of an activity the subjective meaning of this activity to the individual whereby the individual can understand the purpose of this activity. Each perspective offers to learners a specific aspect of sense:

- "applied": sense through understanding and mastering real world situations
- "educational": sense through realizing own competency growth
- "socio-critical": sense through understanding the role of mathematics
- "epistemological": sense through comprehending mathematics as a science

- "pedagogical": sense through enjoying doing mathematics
- "conceptual": sense through understanding mathematical concepts

It is important to offer various aspects of sense since learners will react differently, also according to their beliefs about and attitudes towards mathematics. The hope is that, by offering various aspects of sense, students' beliefs will become broader, and their attitudes will become more positive.

Teaching Modelling

Back from theory to practice. In the first few parts of this paper, the focus was on learning. It is clear that all aims and purposes can only be reached by *high-quality teaching*. Applications and modelling are important, and learning applications and modelling is demanding. This implies that there have to be particularly big efforts to make applications and modelling *accessible* for learners. In fact, there are such efforts in many countries around the world. However, in everyday mathematics teaching practice in most countries, there is still relatively few modelling. Applications in the classroom still occur mostly in the context of dressed-up word problems. We have been deploring this *gap* between the educational debate and classroom practice for decades. Why do we still have this gap? The main reason is that teaching applications and modelling is demanding, too (Freudenthal 1973; Pollak 1979; DeLange 1987; Burkhardt 2004; Ikeda 2007). Also the teachers have to have various competencies available, mathematical and extra-mathematical knowledge, ideas for tasks and for teaching as well as appropriate beliefs. Instruction becomes more open and assessment becomes more complex. This is the main barrier for applications and modelling.

What can we do to improve the situation? What do we know empirically about effective teaching of applications and modelling according to those various aims and purposes? Generally speaking, the well-known findings on quality mathematics teaching of mathematics hold, of course, also for teaching mathematics in the context of relations to the real world. This seems self-evident but is ignored in classrooms around the world every day a million times.

In the following, I will present ten—in my view—important aspects for a *teaching methodology* for applications and modelling, based on empirical findings.

1. A necessary condition is an *effective and learner-oriented classroom management* (see, e.g., Baumert et al. 2004; Hattie 2009; Timperley 2011; Kunter and Voss 2013): using time effectively, separating learning and assessment recognisably, using students' mistakes constructively as learning opportunities (motto: every wrong answer is the right answer to a different question), or varying methods and media flexibly. For modelling, group work is particularly suitable (Ikeda and Stephens 2001). The group is not only a social but also a cognitive environment (co-constructive group work; see Reusser 2001).

2. Just as necessary is to *activate learners cognitively*, to stimulate students' own activities. "Modelling is not a spectator sport" (Schoenfeld, personal communication), one can expect learning effects at most if students engage actively in modelling. This is not a matter of surface structures such as whole-class teaching versus group work versus individualized teaching, which may be dependent on cultural backgrounds. What only counts is that learners are cognitively active (Schoenfeld 1992). We have to distinguish carefully here between students working independently with teacher support, on the one hand, and, on the other hand, students working on their own, alone. Crucial for teaching is a permanent *balance* between students 'independence and teacher's guidance, according to Aebli's famous "Principle of minimal support" (Aebli 1985). I will come back to this aspect in part 6 of this paper.
3. Learners have to be activated not only cognitively but also *meta-cognitively*. All activities ought to be accompanied by reflections and ought to be reflected in retrospective, with the aim to advance appropriate learning *strategies*. Again this is not a matter of lesson surface structures. I will elaborate more on this aspect in part 7 of this paper.
4. There has to be a broad variety of suitable examples as the substance of mathematics lessons since we cannot expect any mystical transfer from one example or context to another. In particular, there has to be a well-aimed variation of real world contexts as well as of mathematical contexts and topics. As I have said in part 4, different kinds of examples may serve different purposes and authenticity is not always required. However, if contexts are made more authentic, the "suspension of sense-making" (see part 3) can be reduced substantially (Palm 2007; Verschaffel et al. 2010). For instance, if the "Army bus" task (see part 3) is embedded in a credible context where students have to write an order form for a bus company, the number of reasonable solutions increases substantially.
   There are a lot of rich teaching/learning environments available for all aims of application and modelling, among many others the following:
   - A wealth of materials from the Shell Centre in Nottingham, the UCSMP project, Roskilde University, the Freudenthal institute (RME) and much more (see Blum et al. 2007, part 6).
   - Dick Lesh's Model Eliciting Activities (Lesh and Doerr 2003); they are primarily meant as a research tool, but they can be used equally well for teaching purposes, together with his Model Exploration Activities and Model Adaptation Activities.
   - "Real objects, contexts and actions" and "local applications" (Alsina 2007); other outdoor activities in the same spirit are "Maths trails" (see, e.g., Shoaf et al. 2004).
   - Materials from the modelling weeks in various cities, in Germany, Singapore or Queensland.

5. Teachers ought to encourage *individual solutions* of modelling tasks. In everyday teaching practice, however, teachers tend to favour strongly their own solution, without even noticing it (Leikin and Levav-Waynberg 2007; Borromeo Ferri and Blum 2009), also because of a limited knowledge of the "task space". There are several reasons for encouraging *multiple solutions* (Schoenfeld 1988; Hiebert and Carpenter 1992; Krainer 1993; Neubrand 2006; Rittle-Johnson and Star 2009; Tsamir et al. 2010): These comply with students' individual preferences, support internal differentiation in the classroom, reflect the genuine spirit of mathematics, and enable comparisons between and reflections on different solutions on a meta-level. In the current project MultiMa (Schukajlow and Krug 2013), two independency-oriented teaching units with modelling tasks are compared where in one unit students are explicitly required to produce multiple solutions. It turned out that those students who developed several solutions had higher learning gains.
6. Competencies such as modelling evolve in *long-term* learning processes, beginning already in primary school with "implicit models" (Greer and Verschaffel 2007; Borromeo Ferri and Lesh 2013) and continuing forever. Necessary and not at all out-of-date are permanent integrated repeating and intelligent practising. It is also important to have a permanent balance between focussing on sub-competencies of modelling and focussing on modelling competency as a whole. It is an open research question what such a balance would look like. What would be needed is a competency development model for modelling, theoretically sound and empirically well-founded, or several such models. This is a big deficit in research.
   An interesting approach to describe competency development comes from the Danish KOM project (Blomhøj and Jensen 2007; Niss and Højgaard Jensen 2011). The authors distinguish between three dimensions in an individual's possession of a given mathematical competency: the "degree of coverage" of aspects of this competency, the "radius of action" that indicates the spectrum of contexts and situations, and the "technical level" that indicates the conceptual and technical level of the involved mathematical entities.
7. Not only teaching but also *assessment* has to reflect the aims of applications and modelling appropriately. Quality criteria such as variation of methods are relevant here, too (Haines and Crouch 2001; Izard et al. 2003; Houston 2007; Antonius et al. 2007; Vos 2007). One method is, of course, to work with *tests*. As we know, tests have several functions, among others to set norms and to illustrate the aspired aims ("What You Test Is What You Get"), but also and particularly to diagnose students' strengths and weaknesses in order to know better how to help.
   An interesting research question is whether and how it is possible to assess modelling sub-competencies and general modelling competency separately. Zöttl et al. (2011) have found that the following model describes their data best (Fig. 8): Some items measure certain sub-competencies and all items measure a general competency.

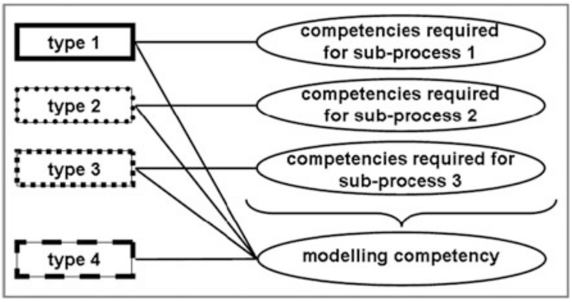

Fig. 8 Model for modelling sub-competencies

8. It is important to care for a parallel development of competencies and appropriate *beliefs* and *attitudes*. Taking into account the remarkable stability of beliefs and attitudes, this also requires long-term learning processes.
9. There are a lot of case studies that show that *digital technologies* can be used as powerful tools for modelling activities, not only in the intra-mathematical phases (see, e.g., Borba and Villarreal 2005; Henn 2007; Geiger 2011; Greefrath et al. 2011). Computers can be used for experiments, investigations, simulations, visualisations or calculations. Greefrath suggests to extend the modelling cycle by adding a third world: the technological world (Fig. 9).
   What we need here are much more controlled studies into the effects of digital technologies on modelling competency development.
10. The best message comes last. Several case studies have shown that mathematical modelling can in fact be learned by secondary school students supposed there is quality teaching (a.o. Kaiser-Meßmer 1987; Galbraith and Clatworthy 1990; Abrantes 1993; Maaß 2007; Biccard and Wessels 2011; Blum and Leiß 2007b; Schukajlow et al. 2012). Some studies have shown that also students' beliefs about mathematics can be broadened by appropriate quality teaching.

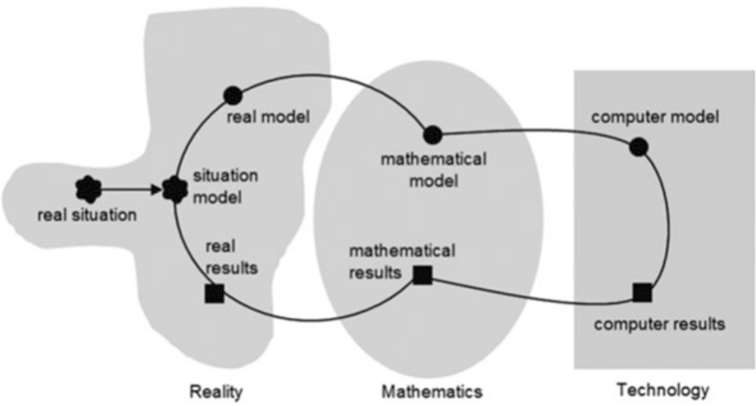

Fig. 9 The extended modelling cycle

However, much more research is needed, especially small-scale studies using a mixture of qualitative and quantitative methods.

In closing part 5, I would like to emphasise that all these efforts will not be sufficient to assign applications and modelling its proper place in curricula and classrooms and to ensure effective and sustainable learning. The implementation of applications and modelling has to take place *systemically*, with all system components collaborating closely: curricula, standards, instruction, assessment and evaluation, and teacher education. I cannot elaborate more on this aspect.

Teacher Support for Modelling Activities

I would like to go more deeply into the second aspect mentioned in part 5: How can the balance between students' independent work and teacher's guidance be put into practice, what does "minimal support" look like? The key concept is *adaptive teacher invention* (see Leiß 2010, for an overview). Such an intervention allows students to continue their work without losing their independence—in the Vygotski terminology: an intervention in the Zone of Proximal Development. Whether an intervention was adaptive or not can, on principle, be only judged afterwards: Is the cognitive barrier really overcome, has the "red flag" vanished? Adaptive interventions can be regarded as a special case of *scaffolding* (Smit et al. 2013). A necessary basis for such a temporary support is a good diagnosis.

In everyday classrooms, teachers tend to strong, content-related interventions, sometimes in order to prevent mistakes or blockages before they occur. According to several studies (see, e.g., Leiß 2007), there are only very few *strategic* interventions, and most interventions seem to be not adaptive. However, especially strategic interventions have the potential of being adaptive (for an impressive example of a successful strategic intervention see Blum and Borromeo Ferri 2009). Here are some examples of strategic interventions:

Read the text carefully! Imagine the situation clearly! Make a sketch! What do you aim at? What is missing? Which data do you need? How far have you got? Does this result make sense for the real situation?

In the DISUM project (see Blum and Leiß 2007b), a ten lesson teaching unit on modelling in 18 grade 9 classes proved to produce significantly higher learning gains in modelling competency in a teaching design oriented towards students' independence with adaptive teacher interventions compared to a design with directive teaching; see Schukajlow et al. (2012) for more details.

Strategies for Learning Modelling

All teacher interventions and support as just discussed will have no long-term effects if they are only applied situationally, transfer cannot be expected. Only accompanying meta-cognitive activities may promise sustainable effects. Students have to be enabled to see the general feature in the concrete step, in the concrete cognitive barrier: How can I help myself in such a difficulty? How can I solve such kind of tasks by myself? For, in assessment situations or in real life contexts, there is no teacher support available.

A promising approach is to teach *learning strategies*, cognitive strategies as well as meta-cognitive strategies such as planning, controlling or regulating. There are a lot of empirical results concerning the effects of using strategies, mostly encouraging, some also disappointing (Tanner and Jones 1993; Schoenfeld 1992, 1994; Matos and Carreira 1997; Stillman and Galbraith 1998; Kramarski et al. 2002; Burkhardt and Pollak 2006; Desoete and Veenman 2006; Stillman 2011; for an overview see Greer and Verschaffel 2007). One of the problems in these empirical studies is: how to measure strategy knowledge, on the one hand, and strategy use, on the other hand, and another problem is how to reliably link students' activities to their strategies.

In particular for novices in modelling there are two strategic instruments that I would like to mention since they turned out to be successful: First, the heuristic worked examples in the KOMMA project, with a three step schema (see Zöttl et al. 2011). Second, the DISUM four step schema ("Understanding task/ Searching mathematics/ Using mathematics/ Explaining result"; see Blum 2011, for more details). This is not meant as a schema that students must follow but as a guiding line, a meta-cognitive aid, particularly in case of difficulties. The problem for students with such strategic devices is: What do these hints mean concretely (for instance in step 2 "Make assumptions": which, how, how many?)? Much more research is needed into the design and use of strategic instruments for modelling.

Teacher Competencies for Modelling

Several empirical studies tell us (recently the comparative study TEDS-M, see Schmidt et al. 2007; Blömeke et al. 2010): The teacher matters most! For quality teaching of applications and modelling, the teacher needs a lot of different competencies. As a theoretical foundation, I would like to use the competence model from the COACTIV project (see Baumert and Kunter 2013). Here, as part of the professional knowledge, five categories are distinguished, especially content knowledge (CK), pedagogical content knowledge (PCK), and pedagogical/psychological knowledge (PK), along the distinction made by Shulman and others. Based on the fundamental assumption about the impact of teaching on learning teacher competencies → quality teaching → student learning,

the COACTIV project has shown, for a representative sample of German secondary mathematics teachers, that subject-related teacher competencies have a strong influence on students' performance (see Baumert et al. 2010). Among the mediators that significantly influence students' performance are classroom management and the cognitive level of tasks set for written class tests. And the TEDS-M study has shown that competencies of beginning teachers vary a lot across different countries, dependent on their learning opportunities. Therefore, *teacher education* is crucial.

What PCK is needed especially for teaching applications and modelling (see Ball et al. 2005, in general and, in particular for modelling, Doerr 2007; Lingefjärd 2013; Kaiser et al. 2010). Borromeo Ferri and Blum (2010) distinguish, in their model, between four dimensions of teachers' PCK for modelling: (1) a *theoretical* dimension (incl. modelling cycles or aims and perspectives of modelling as background knowledge), (2) a *task* dimension (incl. multiple solutions or cognitive analyses of modelling tasks), (3) an *instructional* dimension (incl. interventions, support and feedback), and (4) a *diagnostic* dimension (incl. recognising students' difficulties and mistakes). Also for teachers' learning, no transfer can be expected. Hence, all these elements have to be included as compulsory components in teacher education and professional development. Obviously, in most places where maths teachers are trained, this is not (yet) the case, that means the naïve faith in some mystic transfer is strong here, too. Another myth is that teachers will gain their necessary professional knowledge just by teaching practice. However, in the COACTIV project, there was no correlation between experience and professional knowledge (see Kunter et al. 2013).

One way of providing future teachers with the necessary professional knowledge is to offer specific modelling seminars already at the university, with compulsory own teaching experiences (Borromeo Ferri and Blum 2010). Also the Model Eliciting Activities mentioned in part 5 (see Doerr and Lesh 2011) are very efficient learning environments both for future and for practicing teachers. Nevertheless, a lot has still to be done in research as well, in particular: How will the various teacher competencies play out in teaching practice and how will they influence student learning about applications and modelling?

A Final Real World Example

I would like to come back to the example in part 1, Oldenburg's oversized pick-axe in Kassel. The story that the Kassel Hercules has thrown this pick-axe to the Fulda river is very nice, but we may ask: Is it conceivable?

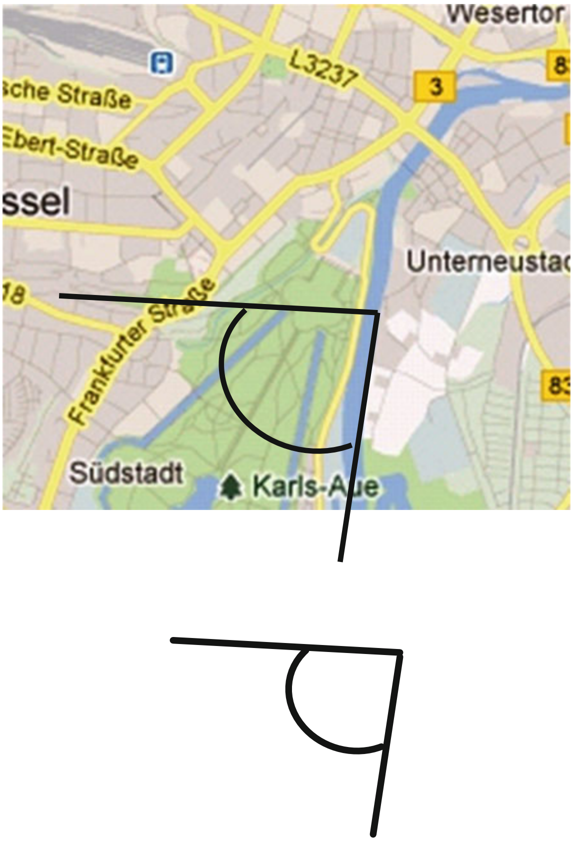

Fig. 10 The angle between axe and river

The first question is: Is the axis correct from the pick-axe to the Hercules? Hercules cannot be seen from the Fulda bank, but we can just measure the angle between the pick-axe and the Fulda River in reality and at the same time the angle between the line Hercules-axe and the river on the map (Fig. 10).

In both cases we find approximately 85°. Since angles are preserved under similarity transformations, this shows that the axis is correct indeed.

The second question is: Can Hercules really throw that far? This depends on a more basic question: Is Hercules able to hold this pick-axe at all? See part 1: Oldenburg's pick-axe is 13 times as long as a normal axe. So, using a cubic model, it weighs more than 2,000 times a normal axe, thus approximately 5 tons. Kassel's Hercules measures 9 m, 5 times a normal man's height, and Hercules is, as one knows from history, much stronger than normal people. The world record in weight-lifting is ¼ ton. Now we can apply two different models. If we assume that the power for weight-lifting only grows proportionally with height, Hercules will be able to hold at most 1.5 tons but not 5 tons, unfortunately. However, if we assume quadratic growth with height, Hercules will be able to hold even 6 tons. I would like to leave this question open: Which model is more appropriate? Personally, I prefer the quadratic model in order not to run down such a nice story about a hero and his pick-axe.

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